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A NOTE ON (a, b)-FIBONACCI SEQUENCES AND SPECIALLY MULTIPLICATIVE ARITHMETIC FUNCTIONS

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Abstract. A specially multiplicative arithmetic function is the Dirichlet convolution of two completely multiplicative arithmetic functions. The aim of this paper is to prove explicitly that two mathematical objects, namely (a, b)-Fibonacci sequences and specially multiplicative prime-independent arithmetic functions, are equivalent in the sense that each can be reconstructed from the other. Replacing one with another, the exploration space of both mathematical objects expands significantly.

Keywords: Fibonacci sequence; multiplicative arithmetic function; Binet's formula; Busche-Ramanujan identities; Möbius inversion

MSC 2020: 11B39, 11A25

1. Introduction and preliminaries

Horadam in [8] and [9] introduced and studied the second-order linear recurrence sequence $\{u_n\}_{n\geqslant 1}$ (u_0 is omitted for later reasons) defined by

$$u_{n+2} = au_{n+1} + bu_n$$

with given u_1 and u_2 and $n > 0$. If $u_1 = u_2 = a = b = 1$ then $\{u_n\}_{n\geqslant 1}$ is the sequence of Fibonacci numbers: $1, 1, 2, 3, 5, 8, 13, \ldots$ The authors of [12] denote by $\mathcal{R}(a, b)$ the set of all second-order linear recurrence sequences for fixed a and b real (even complex) numbers. In every $\mathcal{R}(a, b)$ there is an element $\{u_n\}_{n\geqslant 1}$ that begins with $u_1 = 1$ and $u_2 = a$ and this is the (a, b)-*Fibonacci sequence* for $\mathcal{R}(a, b)$ (see the section Generalized Fibonacci and Lucas numbers in [12]). In mathematics the well known Lucas sequences of the first kind $\{u_n\}_{n\geqslant 1}$ (we again omit $u_0 = 0$) are *integer sequences* and in their definition the same recurrence relations lie: $u_1 = 1$, $u_2 = a$ and $u_{n+2} = au_{n+1} + bu_n$ for $n > 0$, where a and b are fixed integers.

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1.	$a = k,\ b = 1$	$1,\ k,\ k^2 + 1,\ k^3 + 2k, \ldots$	k-Fibonacci sequence (see [6])
1.1.	$a = 1,\ b = 1$	$1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13, \ldots$	Fibonacci sequence
1.2.	$a = 2,\ b = 1$	$1,\ 2,\ 5,\ 12,\ 29,\ 70, \ldots$	Pell sequence
1.3.	$a = -1,\ b = 1$	$1,\ -1,\ 2,\ -3,\ 5,\ -8,\ 13, \ldots$	negaFibonacci sequence
2.	$a = k,\ b = 2$	$1,\ k,\ k^2 + 2,\ k^3 + 4k, \ldots$	k-Jakobsthal sequence (see [11])
2.1.	$a = 1,\ b = 2$	$1,\ 1,\ 3,\ 5,\ 11,\ 21,\ 43, \ldots$	Jakobsthal sequence
2.2.	$a = -1,\ b = 2$	$1,\ -1,\ 3,\ -5,\ 11,\ -21, \ldots$	negaJakobsthal sequence
3.	$a = k,\ b = -1$	$1,\ k,\ k^2 - 1,\ k^3 - 2k, \ldots$	k-Sastry sequence
3.1.	$a = 1,\ b = -1$	$1,\ 1,\ 0,\ -1,\ -1,\ 0,\ 1,\ 1, \ldots$	Sastry sequence (see [16])
3.2.	$a = 2,\ b = -1$	$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8, \ldots$	sequence of positive integers
3.3.	$a = 3,\ b = -1$	$1,\ 3,\ 8,\ 21,\ 55,\ 144, \ldots$	even-numbered Fib. sequence
4.	$a = 3,\ b = -2$	$1,\ 3,\ 7,\ 15,\ 31,\ 63, \ldots$	Mersenne sequence $(2^n - 1)$
5.	$a = 11,\ b = -10$	$1,\ 11,\ 111,\ 1111, \ldots$	sequence of repunits
6.	$a = 2,\ b = k$	$1,\ 2,\ 4 + k,\ 8 + 4k, \ldots$	k-Pell sequence (see [4])

Table 1. $u_1 = 1$, $u_2 = a$, $u_{n+2} = au_{n+1} + bu_n$ $(u_0 = 0)$.

Table 1 shows important sequences (all are Lucas sequences of the first kind only if k is an integer), many of them intensively studied in mathematics over the years. In the following we will keep the real (or even complex) hypothesis for a and b, and the (a, b)-Fibonacci sequence name, as it was used in [12].

Haukkenen in [7], McCarthy and Sivaramakrishnan in [14], and the present authors in [16] pointed out that there exists a connection between specially multiplicative arithmetic functions and Fibonacci numbers. In Section 2 we express explicitly that (a, b)-Fibonacci sequences and specially multiplicative prime-independent arithmetic functions are equivalent mathematical objects in the sense that each can be reconstructed from the other (see Theorems 2.1, 2.2 and 2.3). This purpose and the fact that arithmetic functions are defined as complex valued functions on the set of positive integers have led us to the considerations:

(1) the first two (a, b)-Fibonacci numbers are $u_1 = 1$ and $u_2 = a$ ($u_0 = 0$ being omitted);

(2) a and b are even complex numbers.

An arithmetic function f is called *multiplicative* if $f(1) = 1$ and $f(mn) = f(m)f(n)$ whenever m and n are relatively prime (i.e., $(m, n) = 1$). A multiplicative arithmetic function f is said to be *completely multiplicative* if $f(mn) = f(m)f(n)$ for all m and n. A multiplicative arithmetic function f is uniquely determined by its values $f(p^n)$ at prime power arguments and a completely multiplicative arithmetic function is completely determined by its values at the primes. If f is multiplicative and $f(p^n) = f(q^n)$ for all prime pairs (p, q) and all n, then f is said to be *prime-independent*. If Ω denotes the arithmetic function defined by $\Omega(n)$ (the number of

prime factors of n counted with multiplicity) and c is a nonzero complex number then c^{Ω} is a completely multiplicative prime-independent arithmetic function.

The *Dirichlet convolution* $f * g$ of two arithmetic functions f and g is defined by

$$(f * g)(n) = \sum_{d|n} f(d)g\Big(\frac{n}{d}\Big),$$

where the summation is over positive divisors d of n. It is both commutative and associative. The arithmetic function $\mathbf{1}$ defined by $\mathbf{1}(n) = 1$ if $n = 1$ and $\mathbf{1}(n) = 0$ otherwise, is the identity element for the Dirichlet convolution. The set of all multiplicative arithmetic functions with this operation is a commutative group. We denote this group by $(\mathfrak{M}, *)$.

A *specially multiplicative* arithmetic function is the Dirichlet convolution of two completely multiplicative arithmetic functions.

The *Bell series* of an arithmetic function f modulo a prime p is defined by

$$f(p, x) = \sum_{m=0}^{\infty} f(p^m)x^m.$$

For any prime p, the Bell series of the completely multiplicative function c^{Ω} is the geometric series $\sum\limits_{n=0}^{\infty} c^n x^n$, therefore

$$c^{\Omega}(p, x) = \frac{1}{1 - cx}.$$

Obviously, two multiplicative arithmetic functions are identical if all their Bell series are equal. For any two arithmetic functions f and g, we have (see [2], Theorem 2.25)

$$(f * g)(p, x) = f(p, x) \cdot g(p, x)$$

for every prime p.

2. The main theorems

Theorem 2.1. *Given an (a, b)-Fibonacci sequence $u = \{u_n\}_{n\geqslant 1}$, the multiplicative arithmetic function f_u, defined by*

$$f_u(p^n) = u_{n+1} \quad \text{for all primes } p \text{ and all } n \geqslant 0,$$

is a specially multiplicative prime-independent arithmetic function.

Theorem 2.2. *If f is a specially multiplicative prime-independent arithmetic function and p is a prime, then the sequence $u^f = \{f(p^n)\}_{n\geqslant 0}$ is an (a, b)-Fibonacci sequence with $a = f(p)$ and $b = f(p^2) - f(p)^2$.*

Theorem 2.3. *The constructions of Theorems* 2.1 *and* 2.2 *are inversions of each other. That is, the following statements are true:*

$$f = f_{u^f} \quad \text{and} \quad u = u^{f_u}.$$

P r o o f of Theorem 2.1. If $b = 0$ and $a = 0$ then $f_u = \mathbf{1}$ and $f_u = \mathbf{1} * \mathbf{1}$. If $b = 0$ and $a \neq 0$ then $u = \{1, a, a^2, \ldots\}$ and $f_u = a^\Omega * \mathbf{1}$.

If $b \neq 0$, since

$$f_u(p,x) = \sum_{n=0}^{\infty} u_{n+1}x^n = u_1 + u_2x + \sum_{n=2}^{\infty}(au_n + bu_{n-1})x^n = 1 + axf_u(p,x) + bx^2 f_u(p,x),$$

it follows that

$$f_u(p,x) = \frac{1}{1-ax-bx^2} = \frac{1}{1-c_1x} \cdot \frac{1}{1-c_2x} = c_1^\Omega(p,x) \cdot c_2^\Omega(p,x),$$

that is,

$$f_u = c_1^\Omega * c_2^\Omega,$$

where c_1 and c_2 are the solutions of the quadratic equation $x^2 - ax - b = 0$, i.e.,

$$c_1 = \frac{a + \sqrt{a^2+4b}}{2} \quad \text{and} \quad c_2 = \frac{a - \sqrt{a^2+4b}}{2}.$$

The proof of Theorem 2.1 is complete. □

P r o o f of Theorem 2.2. If f is specially multiplicative prime-independent and $f = g * h$ where g and h are completely multiplicative then

$$f(p^0) = 1, \quad a = f(p) = g(p) + h(p),$$
$$b = f(p^2) - f(p)^2 = g(p^2) + g(p)h(p) + h(p^2) - (g(p) + h(p))^2 = -g(p)h(p)$$

for all primes p, and

$$\begin{aligned} &af(p^{n-1}) + bf(p^{n-2}) \\ &\quad = (g(p) + h(p)) \sum_{i=0}^{n-1} g(p^i)h(p^{n-1-i}) - g(p)h(p) \sum_{i=0}^{n-2} g(p^i)h(p^{n-2-i}) \\ &\quad = \sum_{i=0}^{n-1} g(p^{i+1})h(p^{n-1-i}) + \sum_{i=0}^{n-1} g(p^i)h(p^{n-i}) - \sum_{i=0}^{n-2} g(p^{i+1})h(p^{n-1-i}) \\ &\quad = g(p^n) + \sum_{i=0}^{n-1} g(p^i)h(p^{n-i}) = \sum_{i=0}^{n} g(p^i)h(p^{n-i}) = f(p^n). \end{aligned}$$

Hence $\{f(p^n)\}_{n \geqslant 0}$ is an (a,b)-Fibonacci sequence, where $a = f(p)$ and $b = f(p^2) - f(p)^2$. □

P r o o f of Theorem 2.3. Given a specially multiplicative prime-independent arithmetic function f, the $(n+1)$st term of the (a,b)-Fibonacci sequence u^f (where $a = f(p)$ and $b = f(p^2) - f(p)^2$) is $f(p^n)$. So, $f_{u^f}(p^n) = f(p^n)$ for all primes p and all n. Now, if $u = \{u_n\}_{n\geqslant 1}$ is an (a,b)-Fibonacci sequence then $u^{f_u} = \{f_u(p^n)\}_{n\geqslant 0} = \{u_{n+1}\}_{n\geqslant 0}$, and therefore $u = u^{f_u}$. □

3. Applications. Binet's formula, Busche-Ramanujan identities and Möbius inversion

The purpose of this section is to show how the field of exploration of (a,b)-Fibonacci numbers can be extended by simply applying the results of the previous section. We will get a new picture of them by applying the Dirichlet convolution, Busche-Ramanujan identities, Möbius inversion, etc. to "these numbers". Some of the results can be found in [12] with different proofs. We keep the notations from the previous section.

3.1. Binet's formula.

$$u_n = \frac{c_1^n - c_2^n}{c_1 - c_2}, \quad c_1 \neq c_2.$$

P r o o f.

$$u_n = f_u(p^{n-1}) = (c_1^{\Omega} * c_2^{\Omega})(p^{n-1}) = \sum_{i=0}^{n-1} c_1^i c_2^{n-1-i} = \frac{c_1^n - c_2^n}{c_1 - c_2}.$$

□

Notice that if $c_1 = c_2 = c$ (i.e., $a^2 + 4b = 0$) then the Binet formula becomes $u_n = nc^{n-1}$.

3.2. Busche-Ramanujan's identities. The Busche-Ramanujan identities state that for every positive integers m and n,

$$f(m)f(n) = \sum_{d|(m,n)} f\Big(\frac{mn}{d^2}\Big) g(d)h(d) \tag{1.1}$$

and

$$f(mn) = \sum_{d|(m,n)} f\Big(\frac{m}{d}\Big) f\Big(\frac{n}{d}\Big) \mu(d) g(d)h(d) \tag{1.2}$$

whenever the arithmetic function f is specially multiplicative, that is, $f = g * h$ with g and h completely multiplicative (see [13], Chapter 1). In (1.2), μ denotes the well known multiplicative Möbius arithmetic function: $\mu(p) = -1$ and $\mu(p^n) = 0$ if $n > 1$ for all primes p.

In the next theorem, (1.1′) and (1.2′) are the Busche-Ramanujan identities associated with $f = f_u$.

Theorem 3.1. *Given an (a, b)-Fibonacci sequence $u = \{u_n\}_{n\geqslant 1}$ with $b \neq 0$, we have:*

(i)

$$u_m u_n = \sum_{j=1}^{\min\{m,n\}} (-b)^{j-1} u_{m+n+1-2j}; \tag{1.1′}$$

(i$_1$) *(Cassini's identity)* $u_n^2 - u_{n-1}u_{n+1} = (-b)^{n-1}$;
(i$_2$) *(d'Ocagne's identity) for* $m < n$, $u_{m+1}u_n - u_m u_{n+1} = (-b)^m u_{n-m}$;
(i$_3$) *(Catalan's identity) for* $m < n$, $u_n^2 - u_{n-m}u_{n+m} = (-b)^{n-m}u_m^2$;
(i$_4$) *(Vajda's identity)* $u_{n+i}u_{n+j} - u_n u_{n+i+j} = (-b)^n u_i u_j$.

(ii)

$$u_n = u_{s+1}u_{t+1} + bu_s u_t \tag{1.2′}$$

for any two positive integers s and t such that $s + t = n - 1$;

(ii$_1$) $u_{2m+1} = u_{m+1}^2 + bu_m^2$;
(ii$_2$) $u_{m+n} = u_n u_{m+1} + bu_m u_{n-1}$;
(ii$_3$) $u_{2m} = u_{m+1}u_m + bu_m u_{m-1}$;
(ii$_4$) $u_n = u_m u_{n-m+1} + bu_{m-1}u_{n-m}$.

P r o o f. (i)

$$u_m u_n = f_u(p^{m-1})f_u(p^{n-1}) = \sum_{d|min\{p^{m-1},p^{n-1}\}} f_u\Big(\frac{p^{m+n-2}}{d^2}\Big)(c_1c_2)^{\Omega(d)}$$
$$\underline{\underline{(d=p^i)}} \sum_{i=0}^{\min\{m-1,n-1\}} (-b)^i u_{m+n-1-2i} = \sum_{j=1}^{\min\{m,n\}} (-b)^{j-1} u_{m+n+1-2j}.$$

(ii)

$$u_n = f_u(p^{s+t}) = \sum_{i=0}^{\min\{s,t\}} f_u(p^{s-i})f_u(p^{t-i})(-b)^i \mu(p^i) = u_{s+1}u_{t+1} + bu_s u_t.$$

Now, (i$_1$)–(i$_4$) and (ii$_1$)–(ii$_4$) are specific cases of the Busche-Ramanujan identities (1.1′) and (1.2′), respectively. □

3.3. The generalized Möbius function $\mu_{a,b}$. The classical Möbius function μ,

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^2|n \text{ for some prime } p, \\ (-1)^{\omega(n)} & \text{otherwise} \end{cases}$$

(where $\omega(n)$ is the number of distinct prime factors of the positive integer n), is central in multiplicative number theory.

It is well known that μ is the inverse in the group $(\mathfrak{M}, *)$, $\mu = \zeta^{-1}$, of the zeta function ζ defined by $\zeta(n) = 1$ for all positive integers n. But this ζ sequence

$$\zeta\colon\ 1, 1, 1, 1, 1, 1, 1, \ldots$$

is an (a, b)-Fibonacci sequence, namely with $a = 1$ and $b = 0$. So, the following definition is a two-parameter generalization of the Möbius function.

Definition 3.1. We say that the inverse $\mu_{a,b}$ of f_u in the group $(\mathfrak{M}, *)$,

$$\mu_{a,b} = f_u^{-1},$$

is the generalized (a, b)-Möbius function, where $u = \{u_n\}_{n\geqslant 1}$ is the (a, b)-Fibonacci sequence.

Then $\mu_{1,0} = \mu$ and $\mu_{0,0}$ is the identity element $\mathbf{1}$ of the group $(\mathfrak{M}, *)$. If u is the ordinary Fibonacci sequence (i.e., $a = b = 1$) then

$$\mu_{1,1}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^3|n \text{ for some prime } p, \\ (-1)^{\omega(n)} & \text{otherwise.} \end{cases}$$

This is precisely Cohen's generalized Möbius function of order two (see [5]). But more than that, $\mu_{1,-1}$ is Apostol's generalized Möbius function of order two (see [1]), $\mu_{-1,1}$ is Sastry's generalized Möbius function of order two (see [15]), and $\mu_{2,-1}$ is the generalized Möbius function μ_r of order two ($r = 2$) introduced by Hsu in [10] (see also Brown at al. [3]). All these follow from the statement below.

Theorem 3.2. *If $a \neq 0$ and $b \neq 0$ then*

$$\mu_{a,b}(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } p^3|n \text{ for some prime } p, \\ (-1)^{\omega(n)} a^{\overline{\omega}(n)} b^{\omega(n)-\overline{\omega}(n)} & \text{otherwise,} \end{cases}$$

where $\overline{\omega}(n)$ is the number of square free prime divisors of n.

P r o o f. If u is an (a,b)-Fibonacci sequence then for every prime p,

$$f_u^{-1}(p,x) = 1 - ax - bx^2.$$

Therefore,

$$\mu_{a,b}(p^m) = \begin{cases} 1 & \text{if } m = 0, \\ -a & \text{if } m = 1, \\ -b & \text{if } m = 2, \\ 0 & \text{if } m > 2. \end{cases}$$

Since $\mu_{a,b}$ is multiplicative, the proof is complete. □

Now, $\mu_{a,b}$ being a Dirichlet-convolution inverse, it follows:

Theorem 3.3. *Let g and h be two arithmetic functions and let $u = \{u_n\}_{n\geqslant 1}$ be an (a,b)-Fibonacci sequence. Then,*

$$g = f_u * h \quad \textit{if and only if} \quad h = \mu_{a,b} * g.$$

The full significance of the generalized Möbius function $\mu_{a,b}$ should become with the above (Möbius inversion) theorem. We present below a sample of it.

Theorem 3.4. *Let $u = \{u_n\}_{n\geqslant 1}$ be an (a,b)-Fibonacci sequence with the assumption $b \pm a \neq 1$. Then*

(i) *(the running sum)*

$$\sum_{i=1}^{n} u_i = \frac{u_{n+1} + bu_n - 1}{a + b - 1},$$

(ii) *(the sum of the first n (a,b)-Fibonacci numbers with even indices)*

$$\sum_{i=1}^{n} u_{2i} = \frac{(b-1)u_{2n+2} - abu_{2n+1} + a}{(b-1)^2 - a^2},$$

(iii) *(the sum of the first $n+1$ (a,b)-Fibonacci numbers with odd indices)*

$$\sum_{i=0}^{n} u_{2i+1} = \frac{au_{2n+2} - b(b-1)u_{2n+1} + b - 1}{a^2 - (b-1)^2}.$$

P r o o f. (i) Put $h = \zeta$ in Theorem 3.3. Then $g(p^n) = (f_u * h)(p^n) = \sum\limits_{i=1}^{n+1} u_i$ for all primes p and all $n \geqslant 0$. It follows

$$\begin{aligned}1 = \zeta(p^{n+1}) &= (\mu_{a,b} * g)(p^{n+1}) = \mu_{a,b}(1)g(p^{n+1}) + \mu_{a,b}(p)g(p^n) + \mu_{a,b}(p^2)g(p^{n-1}) \\ &= \sum_{i=1}^{n+2} u_i - a\sum_{i=1}^{n+1} u_i - b\sum_{i=1}^{n} u_i = u_{n+2} + (1-a)u_{n+1} + (1-a-b)\sum_{i=1}^{n} u_i \\ &= u_{n+1} + bu_n + (1-a-b)\sum_{i=1}^{n} u_i.\end{aligned}$$

The proof on running sum is complete.

(ii) and (iii) Let us use h, the multiplicative arithmetic function defined by

$$h(p^m) = \begin{cases} 1 & \text{if } m \text{ is even,} \\ 0 & \text{if } m \text{ is odd} \end{cases}$$

for all primes p, and $g = f_u * h$ again. It follows

$$g(p^{2n}) = \sum_{i=0}^{n} u_{2i+1}, \quad g(p^{2n+1}) = \sum_{i=1}^{n+1} u_{2i} \quad \text{and} \quad g(p^{2n-1}) = \sum_{i=1}^{n} u_{2i}.$$

Hence,

$$0 = h(p^{2n+1}) = (\mu_{a,b} * g)(p^{2n+1}) = u_{2n+2} + (1-b)\sum_{i=1}^{n} u_{2i} - a\sum_{i=0}^{n} u_{2i+1},$$

and together with the running sum:

$$\sum_{i=1}^{n} u_{2i} + \sum_{i=0}^{n} u_{2i+1} = \sum_{j=1}^{2n+1} u_j = \frac{u_{2n+2} + bu_{2n+1} - 1}{a+b-1},$$

the two desired formulas (ii) and (iii) follow immediately by solving the obtained system of equations. □

3.4. Afterword. Many properties of Fibonacci numbers follow directly from the recursive rule. Many properties can be established by induction. But that's not all. The world of Fibonacci numbers is wonderful for its diversity. The general tools presented in [12] are the path to this world. Their origin derives from the fact that $\mathcal{R}(a,b)$ is a two dimensional (if $a^2+4b \neq 0$) subspace of $\mathbb{R}^\infty$ with $\{\{c_1^n\}_{n\geqslant 0}, \{c_2^n\}_{n\geqslant 0}\}$ being a basis and which is the null space of a linear operator. So, difference operators,

Binet formulas and matrix formulation were the three contexts explored extensively in [12]. Applications and examples in [12] (the Binet formula on page 174, the running sum on pages 176–177, Cassini's formula on page 177) led us to the presentation of our three applications: 3.1, 3.2 and 3.3. Our approach differs a lot from the one in [12], but that was our goal in Section 3 after all.

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