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MAXWELL-SCHRÖDINGER EQUATIONS IN SINGULAR ELECTROMAGNETIC FIELD

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Abstract. We investigate the Cauchy problem of the one dimensional Maxwell-Schrödinger (MS) system under the Lorenz gauge condition. Different from the classical case, we consider the electromagnetic and electrostatic potentials which are growing at space infinity. More precisely, the electrostatic potential is allowed to grow linearly, while for the electromagnetic potential the growth is sublinear. Based on the energy estimates and the gauge transformation, we prove the global existence and the uniqueness of the weak solutions to this system.

Keywords: MS system; global solvability; energy space; Lorenz gauge

MSC 2020: 35Q40

1. Introduction and results

Maxwell-Schrödinger (MS) system approximates the dynamics of a charged nonrelativistic quantum mechanical particle interacting with its own electromagnetic field. It occurs as a model in laser physics [4] linked to an extensive empirical material such as atomic spectra, electron diffraction, crystals and their thermal and electric conductivity [14], [12], contributes to the innovative design of photonic devices such as plasmonic antennas and quantum dot laser [25], [3] to detect multi-physics phenomena, and serves as a testing ground for the development of the theory in quantum electrodynamics [9]. Therefore, in recent decades, a possible mathematical description of these physical settings in the framework of this system has been actively studied with growing interests.

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Mathematically, the coupled MS system is known in three-dimensional case to be locally well-posed both in Lorenz gauge and Coulomb gauge in sufficiently regular spaces (Nakamitsu, Tsutsumi [15], [28], Tsutsumi [26], Nakamura, Wada [16], Colin, Watanabe [8]). Guo, Nakamitsu, Strauss [10] constructed a time global weak solution in energy space by the compactness method. The MS system has been shown to be globally well-posed in a space smaller than the energy space by Nakamura, Wada [17], and in the energy space by Bejenaru, Tataru [6]. On the other hand, Wada [29] in two-dimensional case extended the Kato-type smoothing estimates for solutions to the MS system in the Lorenz gauge and proved unique solvability in the energy space. Recently, Scandone [19] and Antonelli et al. [1] focused on a nonlinear MS system derived from a quantum hydrodynamical model [24], established the global wellposedness at fractional Sobolev regularity, and obtained the growth bounds of solutions in H^2 framework based on a modified energy, respectively. For some latest results of such quantum systems, we may also refer to [13], [5], [20], [21], [22].

In one-dimensional case, the MS system seems to have not received much attention. Nakamitsu, Tsutsumi [15] showed the 1d MS system has a unique global solution at H^2 regular level. However, if compatibility conditions on the boundary are taken into account, their results will degenerate into trivial solutions. For this, Tsutsumi [27] considered the initial boundary value problem and proved global well-posedness of solutions in the energy space.

In this paper, we revisit the one-dimensional MS system

$$\partial_t^2 A - \partial_x^2 A = 2\Im(\nabla_A \psi \cdot \overline{\psi}), \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{1}$$

$$\partial_t^2 \varphi - \partial_x^2 \varphi = -|\psi|^2, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{2}$$

$$\mathrm{i}\partial_t \psi + \Delta_A \psi + \varphi\psi = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}, \tag{3}$$

$$A(0,x) = A_0(x),\ \partial_t A(0,x) = A_1(x),\ \psi(0,x) = \psi_0(x),$$

$$\varphi(0,x) = \varphi_0(x),\ \partial_t \varphi(0,x) = \varphi_1(x), \tag{4}$$

$$A(t,x) \to |x|^\sigma,\ \varphi(t,x) \to c_0|x|,\ \psi(t,x) \to 0 \quad (|x| \to \infty), \tag{5}$$

in the Lorenz gauge

$$\partial_x A = \partial_t \varphi, \quad t \in \mathbb{R},\ x \in \mathbb{R}; \tag{6}$$

here, $A(t,x)$ and $\varphi(t,x)$ are the functions from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R}$, which denote the electromagnetic potential, $\psi(t,x)$ is the function from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{C}$, which can be interpreted as the order parameter associated to a nonrelativistic charged quantum plasma [24], [23], and (6) is called the Lorenz gauge condition. In (3), $\Delta_A = \nabla_A^2$ is the magnetic Schrödinger operator with the magnetic derivative $\nabla_A = \partial_x - \mathrm{i}A$. Moreover, c_0 and σ are real positive constants.

In fact, (6) requires the initial data to satisfy the following two compatibility conditions:

$$\partial_x A_0 - \varphi_1 = 0, \tag{7}$$

$$\partial_x A_1 - \partial_x^2 \varphi_0 = -|\psi_0|^2. \tag{8}$$

If $\partial_x \varphi_0$ and A_1 vanish at $x = \pm\infty$, then (8) implies that $c_0 = 0$, which produces only the trivial solution. Therefore, an assumption that the boundary values of $A(t,x)$ at $x = \pm\infty$ are independent of t and the behaviors of $\varphi(t,x)$ near $x = \infty$ and $x = -\infty$ are the same, was proposed in [27], where the sustained evolution of MS system in singular electric field

$$\varphi(t,x) \to c_0|x| \quad (|x| \to \infty) \quad \text{with } c_0 = \frac{1}{2}\int_{\mathbb{R}} |\psi_0|^2 \,\mathrm{d}x \tag{9}$$

is shown to be possible.

If we further simultaneously consider the space-time structure of the function $\varphi(t,x)$, from Lorenz gauge condition (6), it will induce the nontrivial boundary of magnetic field $A(t,x)$, which is similar to the singularity (9) of $\varphi(t,x)$. For instance, $\varphi(t,x) \to |x|^{\sigma-1}t + c_0|x|$ with $0 \leqslant \sigma < 1$ as $|x| \to \infty$. Obviously, for any $t \in \mathbb{R}$, it satisfies (9). From (6), it also may yield $A(t,x) \to |x|^\sigma$ at infinity.

The main goal of this paper is to investigate the well-posedness of the MS system with nontrivial boundary condition (5) associated to electromagnetic field. To this aim, we introduce two time-independent real-valued functions $V(x)$ and $W(x)$ in $C^\infty(\mathbb{R})$ satisfying the following structures: for some $R > 0$,

$$\begin{cases} V(x),\ W(x) > 0, \\ V(x) = c_0|x|,\ W(x) = |x|^\sigma \quad \text{for } |x| \geqslant R. \end{cases} \tag{H}$$

Moreover, set $\mathcal{A}(t,x) = A(t,x) - W(x)$, $\phi(t,x) = \varphi(t,x) - V(x)$, $\mathcal{A}_0(x) = A_0(x) - W(x)$, $\phi_0(x) = \varphi_0(x) - V(x)$ and $\mathcal{A}_1(x) = A_1(x)$, $\phi_1(x) = \varphi_1(x)$. Therefore, the MS system (1)–(5) in the Lorenz gauge can be rewritten as follows:

$$\partial_t^2 \mathcal{A} - \partial_x^2 \mathcal{A} = \partial_x^2 W + 2\Im\{[\partial_x - \mathrm{i}(\mathcal{A} + W)]\psi \cdot \overline{\psi}\}, \tag{10}$$

$$\partial_t^2 \phi - \partial_x^2 \phi = \partial_x^2 V - |\psi|^2, \tag{11}$$

$$\mathrm{i}\partial_t \psi + [\partial_x - \mathrm{i}(\mathcal{A} + W)]^2 \psi + \phi\psi + V\psi = 0, \tag{12}$$

$$\partial_x \mathcal{A} + \partial_x W = \partial_t \phi, \tag{13}$$

$$\mathcal{A}(0,x) = \mathcal{A}_0(x),\ \partial_t \mathcal{A}(0,x) = \mathcal{A}_1(x),\ \psi(0,x) = \psi_0(x),$$

$$\phi(0,x) = \phi_0(x),\ \partial_t \phi(0,x) = \phi_1(x), \tag{14}$$

$$\mathcal{A}(t,x),\ \phi(t,x),\ \psi(t,x) \to 0 \quad (|x| \to \infty). \tag{15}$$

It is not difficult to check that the solution $(\mathcal{A}, \phi, \psi)$ of (10)–(15) formally satisfies the conservation of the charge $\mathcal{Q}(t)$ and the energy $\mathcal{E}(t)$ defined as

$$\mathcal{Q}(t) = \|\psi(t)\|_{L^2} = \|\psi_0\|_{L^2}, \tag{16}$$

$$\begin{aligned}\mathcal{E}(t) = {} & \frac{1}{2}\|\partial_t \mathcal{A} - \partial_x \phi\|_{L^2}^2 + \|\nabla_{\mathcal{A}}\psi\|_{L^2}^2 + 2\int_{\mathbb{R}} W\mathcal{A}|\psi|^2 \,\mathrm{d}x \\ & - \int_{\mathbb{R}} V|\psi|^2 \,\mathrm{d}x + \int_{\mathbb{R}} W^2|\psi|^2 \,\mathrm{d}x - 2\Im \int_{\mathbb{R}} W\overline{\psi}\partial_x\psi \,\mathrm{d}x = \mathcal{E}(0),\end{aligned} \tag{17}$$

which induce a weak energy space $H^1(\mathbb{R}) \times H^1(\mathbb{R}) \times (H^1(\mathbb{R}) \cap L^2(\mathbb{R}; \max\{V, W^2\}\,\mathrm{d}x))$. The third H^1 here can be replaced by magnetic Sobolev space H^1_A. Indeed, if A belongs to $H^1(\mathbb{R})$, H^1_A is actually equivalent to H^1 (see, e.g., Lemma 2.4, [2]). In the following, we are concerned about the unique solvability of (10)–(15) in this space.

Before stating the results, we first introduce several basic notations used throughout this paper. For $m \in \mathbb{N}$, we denote the standard L^2-Sobolev space by $H^m(\mathbb{R})$ and its dual space by $H^{-m}(\mathbb{R})$. We define the function space $\mathcal{H}^m = H^m(\mathbb{R}) \cap L^2(\mathbb{R}; |x|\,\mathrm{d}x)$. In addition, we also use the notation $\|f[t]\|_{H^m} = \|f(t)\|_{H^m} + \|\partial_t f(t)\|_{H^{m-1}}$ to simplify the representation. Now we state our main results in this paper.

Theorem 1. *Let $0 \leqslant \sigma < 1/2$. Assume that $(\mathcal{A}_0, \mathcal{A}_1), (\phi_0, \phi_1) \in H^1(\mathbb{R}) \times L^2(\mathbb{R})$ and $\psi_0 \in \mathcal{H}^1$, where $\mathcal{A}_0$, $\mathcal{A}_1$, ϕ_0 and ϕ_1 are real-valued, and $V(x)$ and $W(x)$ satisfy assumption* (H). *Suppose moreover that $\mathcal{A}_0$, $\mathcal{A}_1$, ϕ_0, ϕ_1, $V(x)$ and $W(x)$ admit the following two compatibility conditions*

$$\begin{aligned}&\partial_x \mathcal{A}_0 + \partial_x W - \phi_1 = 0 \quad \text{in } L^2(\mathbb{R}),\\ &\partial_x \mathcal{A}_1 - \partial_x^2 \phi_0 = -|\psi_0|^2 + \partial_x^2 V \quad \text{in } H^{-1}(\mathbb{R}).\end{aligned}$$

Then there exists a solution $(\mathcal{A}, \phi, \psi)$ of (10)–(15) *in the energy class*

$$\mathcal{A} \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; H^{1-j}), \quad \phi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; H^{1-j}), \quad \psi \in \bigcap_{j=0}^{1} C^j(\mathbb{R}; \mathcal{H}^{1-2j}) \tag{18}$$

satisfying the bound, for any $t \in \mathbb{R}$,

$$\|\mathcal{A}[t]\|_{H^1} + \|\phi[t]\|_{H^1} + \|\psi(t)\|_{\mathcal{H}^1} \leqslant C + C(R)|t|\mathrm{e}^{C(R)|t|},$$

where both C and $C(R)$ also depend on $\mathcal{E}(0)$ and $\mathcal{Q}(0)$.

Theorem 2. *Under the assumptions of Theorem* 1, *suppose moreover that* $\mathcal{A}_1(x)$ *also satisfies*

$$\int_0^x \mathcal{A}_1(y)\,\mathrm{d}y \in L^\infty(\mathbb{R}). \tag{19}$$

Then the solution $(\mathcal{A}, \phi, \psi)$ *of system* (10)–(15) *is unique and depends continuously on the initial data in the weak energy space given in* (18).

Our proof is based on energy estimates and particularly, on the strategies developed in [15], [27]. In this process, we have to ensure that the Lorenz gauge condition (6) is preserved in time. We also encounter the highly singular derivative coupling term $\mathrm{i}A\partial_x\psi$, and nonlinear terms $(V-W^2)\psi$, $AW\psi$ of the Schrödinger part, which can be shown not to be effectively controlled in the energy space $H^1(\mathbb{R})\times H^1(\mathbb{R})\times\mathcal{H}^1$ by a straight computation. To this end, we follow the original idea of Hayashi and Ozawa [11]. By introducing a gauge transformation, we remove these "bad" terms from the nonlinear Schrödinger equation. Combining the L^p-L^q and Strichartz estimates of the linear Schrödinger equation with a magnetic field due to [31], we prove the uniqueness of the energy solutions of (10)–(15), among of which the restriction $0\leqslant\sigma<1/2$ has to be imposed to bound the coupling $W\partial_x W$ in $L^\infty(\mathbb{R})$.

R e m a r k 1. As an example, we can actually choose nontrivial data and V, W satisfying all the assumptions in Theorem 1. We consider $\sigma=0$ and let $W(x)\equiv 1$, and set $V(x)$ satisfying

$$V(x)=(1+x^2)^{1/2}. \tag{20}$$

Correspondingly, we can take $\psi_0=(1+x^2)^{-3/4}$, and the rest initial data as follows:

$$\mathcal{A}_1=0,\quad \phi_0=0,\quad \mathcal{A}_0=\operatorname{sech}(x),\quad \phi_1=-\tanh(x)\operatorname{sech}(x). \tag{21}$$

It is not difficult to check that they satisfy the compatibility conditions of Theorem 1 and also $\int_{\mathbb{R}}|\psi_0(x)|^2\,\mathrm{d}x=2=2c_0$, where c_0 is given in (5) and $c_0=1$ is indicated by (20).

R e m a r k 2. Besides being able to control $W\partial_x W$ in $L^\infty(\mathbb{R})$, the restriction $0\leqslant\sigma<1/2$ is also required in order to bound $\partial_x W$ in the energy space. In fact, due to (18), the equation (13) implies that $\partial_x W$ should belong to $L^2(\mathbb{R})$.

2. Preliminaries

In this section, we collect several results needed for the proof of our main results. The following lemmas will be repeatedly used in the rest of the content.

Lemma 1. *If $u \in H^1$, $\mathcal{A}$ is a real-valued function, then*

$$|\partial_x |u|| \leqslant |(\partial_x - \mathrm{i}\mathcal{A})u| \quad \text{a.e. in } \mathbb{R}.$$

Proof. This inequality can be derived from a direct computation concerning the distributional gradient of $|u|$: $(\partial_x|u|)(x) = \Re(\bar{u}|u|^{-1}(\partial_x - \mathrm{i}\mathcal{A})u)(x)$. □

Next we introduce some time-space estimates of the solutions for the linear magnetic Schrödinger equation [31], [30].

Lemma 2. *Assume $G(t) = (\partial_x + \mathrm{i}t\partial_x V(x))^2$, where $V(x)$ satisfies assumption* (H). *Then the operator $\mathrm{i}G(t)$ generates the evolution operator $U_G(t,s)$, $-\infty < s \leqslant t < \infty$ associated with the linear Schrödinger equation:*

$$\mathrm{i}\partial_t \psi + G(t)\psi = 0, \quad t \in \mathbb{R},\ x \in \mathbb{R}.$$

The evolution operator $U_G(t,s)$ satisfies the following three estimates for any $T > 0$.

(i) *Let p and q be two positive constants such that $1/p + 1/q = 1$ and $2 \leqslant p \leqslant \infty$. Then*

$$\|U_G(t,s)v\|_{L^p} \leqslant C|t-s|^{(2-p)/(2p)}\|v\|_{L^q}, \quad -T \leqslant s < t \leqslant T,\ v \in L^q,$$

where C depends only on p and T.

(ii) *Let $I = (-T,T)$. Assume that $f \in L^1(I;L^2)$, q and r are positive constants such that $2 \leqslant q \leqslant \infty$ and $(1/2 - 1/q)r = 2$. Then*

$$\left\| \int_0^t U_G(t,s)f(s)\,\mathrm{d}s \right\|_{L^r(I;L^q)} \leqslant C\|f\|_{L^1(I;L^2)},$$

where C depends only on q and T.

(iii) *Let $I = (-T,T)$, q and r be two positive constants such that $1 \leqslant q \leqslant 2$ and $1/q + 2/r = 5/2$. Assume that $f \in L^r(I;L^q)$. Then*

$$\left\| \int_0^t U_G(t,s)f(s)\,\mathrm{d}s \right\|_{L^\infty(I;L^2)} \leqslant C\|f\|_{L^r(I;L^q)},$$

where C depends only on q and T.

3. Proof of the main results

3.1. Global existence.

P r o o f of Theorem 1. From the energy equation, one has

$$\frac{1}{2}\|\partial_t\mathcal{A}-\partial_x\phi\|_{L^2}^2+\|\nabla_{\mathcal{A}}\psi\|_{L^2}^2+2\int_{\mathbb{R}}W\mathcal{A}|\psi|^2\,\mathrm{d}x-2\Im\int_{\mathbb{R}}W\overline{\psi}\partial_x\psi\,\mathrm{d}x \leqslant \mathcal{E}_0+\int_{\mathbb{R}}V|\psi|^2\,\mathrm{d}x+\int_{\mathbb{R}}W^2|\psi|^2\,\mathrm{d}x. \tag{22}$$

Note that

$$\int_{\mathbb{R}}W\mathcal{A}|\psi(t)|^2\,\mathrm{d}x-\Im\int_{\mathbb{R}}W\partial_x\psi\overline{\psi}\,\mathrm{d}x=-\Im\int_{\mathbb{R}}W(\partial_x-\mathrm{i}\mathcal{A})\psi\cdot\overline{\psi}\,\mathrm{d}x \leqslant C\|W\psi\|_{L^2}^2+\frac{1}{2}\|\nabla_{\mathcal{A}}\psi\|_{L^2}^2.$$

From (22), it follows that

$$\|\nabla_{\mathcal{A}}\psi\|_{L^2}^2\leqslant C(R)+C\|W\psi\|_{L^2}^2+C\|V^{1/2}\psi\|_{L^2}^2, \tag{23}$$

where $C(R)$ is a positive constant that depends only on R.

Next we estimate $\int_{\mathbb{R}}W^2(x)|\psi(x)|^2\,\mathrm{d}x$. Multiplying (12) by $W^2\overline{\psi}$ and taking L^2-estimate, we have

$$\begin{aligned}\|W\psi\|_{L^2}^2&\leqslant\|W\psi_0\|_{L^2}^2+C\int_0^t\bigg(\int_{\mathbb{R}}\nabla_{\mathcal{A}}\psi W\partial_xW\overline{\psi}\,\mathrm{d}x\\&\quad+\int_{\mathbb{R}}W^2\partial_xW|\psi|^2\,\mathrm{d}x+\Re\int_{\mathbb{R}}W^3\psi\partial_x\overline{\psi}\,\mathrm{d}x\bigg)\,\mathrm{d}s\\&\leqslant C(R)|t|+C(R)\int_0^t(\|W\psi\|_{L^2}^2+\|V^{1/2}\psi\|_{L^2}^2)\,\mathrm{d}s,\end{aligned}$$

where we used (23). In the same way, we also have

$$\|V^{1/2}\psi\|_{L^2}^2\leqslant C(R)|t|+C(R)\int_0^t(\|W\psi\|_{L^2}^2+\|V^{1/2}\psi\|_{L^2}^2)\,\mathrm{d}s.$$

The application of Gronwall's inequality will produce

$$\|V^{1/2}\psi\|_{L^2}^2+\|W\psi\|_{L^2}^2\leqslant C(R)|t|\mathrm{e}^{C(R)|t|}.$$

Combining this with (23), we have

$$\|\nabla_{\mathcal{A}}\psi\|_{L^2}^2\leqslant C(R)|t|\mathrm{e}^{C(R)|t|}. \tag{24}$$

On the other hand, by Lemma 1 and Gagliardo-Nirenberg inequality, it holds that

$$\|\psi\|_{L^\infty} \leqslant C\|\psi\|_{L^2}^{1/2}\|\nabla_{\mathcal{A}}\psi\|_{L^2}^{1/2}. \tag{25}$$

Formulas (23)–(25) and the L^2-estimate of wave equation show that

$$\|\mathcal{A}[t]\|_{H^1} + \|\phi[t]\|_{H^1} \leqslant C(R)|t|\mathrm{e}^{C(R)|t|}, \quad t \in [-T, T]. \tag{26}$$

Inequalities (24), and (26) imply that

$$\|\partial_x\psi\|_{L^2} \leqslant \|(\partial_x - \mathrm{i}\mathcal{A})\psi\|_{L^2} + \|\mathcal{A}\psi\|_{L^2} \leqslant C(R)|t|\mathrm{e}^{C(R)|t|}, \quad t \in [-T, T]. \tag{27}$$

On this basis, a standard compactness argument implies that there exists a global energy solution $(\mathcal{A}(t), \phi(t), \psi(t))$ of (10)–(12) satisfying

$$\mathcal{A}, \quad \phi \in \bigcap_{j=0}^{2} W^{j,\infty}_{\mathrm{loc}}(\mathbb{R}; H^{1-j}), \quad \psi \in \bigcap_{j=0}^{1} W^{j,\infty}_{\mathrm{loc}}(\mathbb{R}; \mathcal{H}^{1-2j}). \tag{28}$$

Moreover, using equation (10)–(12), (28) and the theory of linear hyperbolic equations, we have (18), which completes the proof of the existence. □

3.2. The uniqueness.

P r o o f of Theorem 2. We prove the uniqueness of weak solutions by exploiting a modified evolution operator and the gauge transformation defined below. Assume that $(\mathcal{A}, \phi, \psi)$ are the solutions of (10)–(15). We introduce the gauge transformation

$$u(t,x) = \psi(t,x)\exp\left\{-\mathrm{i}\int_0^x (\mathcal{A}(t,y) + W(y))\,\mathrm{d}y - \mathrm{i}tV(x)\right\}. \tag{29}$$

Then equation (12) becomes

$$\mathrm{i}\partial_t u - u\int_0^x \partial_t\mathcal{A}(t,y)\,\mathrm{d}y + \partial_x^2 u + 2\mathrm{i}t\partial_x V\partial_x u - t^2(\partial_x V)^2 u + \mathrm{i}tu\partial_x^2 V + \phi u = 0, \tag{30}$$

which can be rearranged as

$$\mathrm{i}\partial_t u + (\partial_x + \mathrm{i}t\partial_x V)^2 u = u\int_0^x \partial_t\mathcal{A}(t,y)\,\mathrm{d}y - \phi u.$$

Let $U_G(t,s)$ be the evolution operator generated by $G = (\partial_x + \mathrm{i}t\partial_x V)^2$, see Lemma 2 for details. By the Duhamel principle, we have

$$u(t) = U_G(t,0)u_0 - \mathrm{i}\int_0^t U_G(t,s)\left(u\int_0^x \partial_s\mathcal{A}(s,y)\,\mathrm{d}y - \phi u\right)\mathrm{d}s. \tag{31}$$

Let $(\mathcal{A},\phi,\psi)$ and $(\widetilde{\mathcal{A}},\widetilde{\phi},\widetilde{\psi})$ be two solutions of (10)–(12) with the same initial data such that $(\widetilde{\mathcal{A}},\widetilde{\phi},\widetilde{\psi})$ satisfy (24). And let u, $\widetilde{u}$ be defined as in (29) for ψ, $\widetilde{\psi}$. Then u and $\widetilde{u}$ satisfy (31). Set $\delta_{\mathcal{A}}=\mathcal{A}-\widetilde{\mathcal{A}}$, $\delta_\phi=\phi-\widetilde{\phi}$, $\delta_u=u-\widetilde{u}$. Then we get

$$\delta_u(t)=-\mathrm{i}\int_0^t U_G(t,s)\biggl[\delta_u\int_0^x\partial_s\mathcal{A}(s,y)\,\mathrm{d}y+\widetilde{u}\int_0^x\partial_s\delta_{\mathcal{A}}(s,y)\,\mathrm{d}y-\phi\delta_u-\delta_\phi\widetilde{u}\biggr]\,\mathrm{d}s. \tag{32}$$

Let T be a positive constant with $[0,T]\subset I$ to be determined later. For convenience, we suppose $0<T<1$. Denote $I_T=[0,T]$, and take the $L^\infty(I_T;L^2)$ norm of (32). In view of Lemma 2 (iii) and (24)–(27), we obtain

(33)

$$\begin{aligned}\|\delta_u\|_{L^\infty(I_T;L^2)}&\leqslant CT(\|\phi\|_{L^\infty(I_T\times\mathbb{R})}\|\delta_u\|_{L^\infty(I_T;L^2)}+\|\widetilde{u}\|_{L^\infty(I_T\times\mathbb{R})}\|\delta_\phi\|_{L^\infty(I_T;L^2)})\\&\quad+C\int_0^T\biggl(\|\delta_u\int_0^x\partial_t\mathcal{A}(t,y)\,\mathrm{d}y\|_{L^2}+\|\widetilde{u}\int_0^x\partial_t\delta_{\mathcal{A}}(t,y)\,\mathrm{d}y\|_{L^2}\biggr)\,\mathrm{d}t\\&\leqslant CT(\|\delta_\phi\|_{L^\infty(I_T;L^2)}+\|\delta_u\|_{L^\infty(I_T;L^2)})\\&\quad+C\int_0^T\|\delta_u(t)\int_0^x\partial_t\mathcal{A}(t,y)\,\mathrm{d}y\|_{L^2}\,\mathrm{d}t\\&\quad+C\int_0^T\|\widetilde{u}(t)\int_0^x\partial_t\delta_{\mathcal{A}}(t,y)\,\mathrm{d}y\|_{L^2}\,\mathrm{d}t.\end{aligned}$$

Moreover, D'Alambert formula yields

$$\biggl|\int_0^x\partial_t\delta_{\mathcal{A}}(t,y)\,\mathrm{d}y\biggr|\leqslant C\int_0^t\biggl[\biggl|\int_{t-s}^{x+(t-s)}\delta_F(s,y)\,\mathrm{d}y\biggr|+\biggl|\int_{-(t-s)}^{x-(t-s)}\delta_F(s,y)\,\mathrm{d}y\biggr|\biggr]\,\mathrm{d}s, \tag{34}$$

where

$$\begin{aligned}\delta_F&=2\Im(\partial_x(\psi-\widetilde{\psi})\overline{\psi})+2\Im(\partial_x\widetilde{\psi}(\overline{\psi}-\overline{\widetilde{\psi}}))-2\widetilde{\mathcal{A}}(\psi-\widetilde{\psi})\overline{\psi}\\&\quad-2\widetilde{\mathcal{A}}\widetilde{\psi}(\overline{\psi}-\overline{\widetilde{\psi}})+2W(\psi-\widetilde{\psi})\overline{\psi}+2W\widetilde{\psi}(\overline{\psi}-\overline{\widetilde{\psi}})-2\delta_{\mathcal{A}}|\psi|^2.\end{aligned}$$

Integrating by parts, we can deduce

$$\begin{aligned}&\int_0^t\biggl|\int_{t-s}^{x+(t-s)}\delta_F(s,y)\,\mathrm{d}y\biggr|\,\mathrm{d}s\\&\quad\leqslant C\int_0^t|\Im((\psi-\widetilde{\psi})\overline{\psi})(s,x+(t-s))-\Im((\psi-\widetilde{\psi})\overline{\psi})(s,t-s)|\,\mathrm{d}s\\&\qquad+C\int_0^t\biggl|\int_{t-s}^{x+(t-s)}(-\Im((\psi-\widetilde{\psi})\partial_y\overline{\psi})+\Im(\partial_y\widetilde{\psi}(\overline{\psi}-\overline{\widetilde{\psi}})))\,\mathrm{d}y\biggr|\,\mathrm{d}s\\&\qquad+C\int_0^t\biggl|\int_{t-s}^{x+(t-s)}(\widetilde{\mathcal{A}}\widetilde{\psi}(\overline{\psi}-\overline{\widetilde{\psi}})+W(\psi-\widetilde{\psi})\overline{\psi})\,\mathrm{d}y\biggr|\,\mathrm{d}s\\&\qquad+C\int_0^t\biggl|\int_{t-s}^{x+(t-s)}\delta_{\mathcal{A}}|\psi|^2\,\mathrm{d}y\biggr|\,\mathrm{d}s\end{aligned} \tag{35}$$

$$
\begin{aligned}
&\leqslant C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,x+(t-s))|\,\mathrm{d}s + C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,t-s)|\,\mathrm{d}s\\
&\quad + C\int_0^t \|\psi-\widetilde{\psi}\|_{L^2}(\|\partial_x\psi\|_{L^2}+\|\partial_x\widetilde{\psi}\|_{L^2}+\|\mathcal{A}\|_{L^\infty}\|\psi\|_{L^2}+\|W\psi\|_{L^2})\,\mathrm{d}s\\
&\quad + C\int_0^t \|\delta_{\mathcal{A}}\|_{L^2}\|\psi\|_{L^2}\|\psi\|_{L^\infty}\,\mathrm{d}s\\
&\leqslant C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,x+(t-s))|\,\mathrm{d}s + C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,t-s)|\,\mathrm{d}s\\
&\quad + C\int_0^t (\|\psi(s)-\widetilde{\psi}(s)\|_{L^2}+\|\delta_{\mathcal{A}}(s)\|_{L^2})\,\mathrm{d}s, \quad t\in I_T,\ x\in\mathbb{R}.
\end{aligned}
$$

We can treat the second term of (34) analogously:

$$
\begin{aligned}
&\int_0^t \Big|\int_{-(t-s)}^{x-(t-s)} \delta_F(s,y)\,\mathrm{d}y\Big|\,\mathrm{d}s\\
&\leqslant C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,x-(t-s))|\,\mathrm{d}s + C\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,-(t-s))|\,\mathrm{d}s\\
&\quad + C\int_0^t (\|\psi(s)-\widetilde{\psi}(s)\|_{L^2}+\|\delta_{\mathcal{A}}(s)\|_{L^2})\,\mathrm{d}s, \quad t\in I_T, x\in\mathbb{R}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
(36)\quad &\int_0^T \Big\|\Big(\int_0^x \partial_t\delta_{\mathcal{A}}(t,y)\,\mathrm{d}y\Big)\widetilde{u}(t)\Big\|_{L^2}\,\mathrm{d}t\\
&\leqslant C\int_0^T\int_0^t \|((\psi-\widetilde{\psi})\overline{\psi})(s,x+(t-s))\|_{L^2}\,\mathrm{d}s\,\mathrm{d}t\|\widetilde{u}\|_{L^\infty(I_T\times\mathbb{R})}\\
&\quad + C\int_0^T\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,t-s)|\,\mathrm{d}s\,\mathrm{d}t\|\widetilde{u}\|_{L^\infty(I_T;L^2)}\\
&\quad + C\int_0^T\int_0^t \|((\psi-\widetilde{\psi})\overline{\psi})(s,x-(t-s))\|_{L^2}\,\mathrm{d}s\,\mathrm{d}t\|\widetilde{u}\|_{L^\infty(I_T\times\mathbb{R})}\\
&\quad + C\int_0^T\int_0^t |((\psi-\widetilde{\psi})\overline{\psi})(s,-(t-s))|\,\mathrm{d}s\,\mathrm{d}t\|\widetilde{u}\|_{L^\infty(I_T;L^2)}\\
&\quad + C\int_0^T\int_0^t (\|\psi(s)-\widetilde{\psi}(s)\|_{L^2}+\|\delta_{\mathcal{A}}(s)\|_{L^2})\,\mathrm{d}s\,\mathrm{d}t\|\widetilde{u}\|_{L^\infty(I_T;L^2)}\\
&\leqslant CT^2\|\psi-\widetilde{\psi}\|_{L^\infty(I_T;L^2)} + CT\|\psi-\widetilde{\psi}\|_{L^\infty(I_T;L^2)} + CT^2\|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)}.
\end{aligned}
$$

In order to deal with $\|\psi(t)-\widetilde{\psi}(t)\|_{L^2}$, we need the following auxiliary estimate. We claim: Let $1\leqslant p\leqslant\infty$. For any $t\in I_T$, it holds that

$$
(37)\qquad \|\psi-\widetilde{\psi}\|_{L^p} \leqslant \|\delta_u\|_{L^p} + \int_0^t \|\widetilde{u}(t)\int_0^x \partial_s\delta_{\mathcal{A}}(s,y)\,\mathrm{d}y\|_{L^p}\,\mathrm{d}s.
$$

Indeed, according to relationship (29) between ψ and u, we get

$$\begin{aligned}\|\psi-\widetilde{\psi}\|_{L^p} &\leqslant \|u-\widetilde{u}\|_{L^p}+\Big\|\widetilde{u}(t)\Big[\exp\Big(-\mathrm{i}\int_0^x(\mathcal{A}(t,y)+W(y))\,\mathrm{d}y-\mathrm{i}tV(x)\Big)\\ &\qquad-\exp\Big(-\mathrm{i}\int_0^x(\widetilde{\mathcal{A}}(t,y)+W(y))\,\mathrm{d}y-\mathrm{i}tV(x)\Big)\Big]\Big\|_{L^p}\\ &\leqslant \|\delta_u\|_{L^p}+\Big\|\widetilde{u}(t)\int_0^x\delta_{\mathcal{A}}(t,y)\,\mathrm{d}y\Big\|_{L^p}\\ &\leqslant \|\delta_u\|_{L^p}+\int_0^t\Big\|\widetilde{u}(t)\int_0^x\partial_s\delta_{\mathcal{A}}(s,y)\,\mathrm{d}y\Big\|_{L^p}\mathrm{d}s.\end{aligned}$$

Applying claim (37) with $p=2$ and running the same proof as (36), we deduce

$$\begin{aligned}\|\psi-\widetilde{\psi}\|_{L^2} &\leqslant \|\delta_u\|_{L^2}+\int_0^t\Big\|\widetilde{u}(t)\int_0^x\partial_s\delta_{\mathcal{A}}(s,y)\,\mathrm{d}y\Big\|_{L^2}\mathrm{d}s\\ &\leqslant \|\delta_u\|_{L^2}+C(T^2+T)\|\psi-\widetilde{\psi}\|_{L^\infty(I_T;L^2)}+CT^2\|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)}.\end{aligned}$$

By choosing $T>0$ sufficiently small such that $C(T^2+T)<1/4$, we have

$$\|\psi-\widetilde{\psi}\|_{L^\infty(I_T;L^2)}\leqslant\|\delta_u\|_{L^\infty(I_T;L^2)}+CT^2\|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)}. \tag{38}$$

In addition, note that the right-hand side of equation (10) belongs to $L^1(\mathbb{R})$. By (19) and the D'Alembert formula, a direct calculation yields

$$\int_0^x\partial_t\mathcal{A}(t,y)\,\mathrm{d}y\in L^\infty(I_T\times\mathbb{R}).$$

Therefore,

$$\begin{aligned}\int_0^T\Big\|\Big(\int_0^x\partial_t\mathcal{A}(t,y)\,\mathrm{d}y\Big)\delta_u(t)\Big\|_{L^2}\mathrm{d}t &\leqslant CT\|\int_0^x\partial_t\mathcal{A}(t,y)\,\mathrm{d}y\|_{L^\infty(I_T\times\mathbb{R})}\|\delta_u\|_{L^\infty(I_T;L^2)}\\ &\leqslant CT\|\delta_u\|_{L^\infty(I_T;L^2)}.\end{aligned} \tag{39}$$

These estimates (33), (36)–(39) conclude

$$\begin{aligned}\|\delta_u\|_{L^\infty(I_T;L^2)}&\leqslant CT(\|\delta_\phi\|_{L^\infty(I_T;L^2)}+\|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)}+\|\delta_u\|_{L^\infty(I_T;L^2)})\\ &\quad+C(T^2+T)\|\delta_u\|_{L^\infty(I_T;L^2)}+C(T^2+T+1)T^2\|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)}.\end{aligned}$$

Finally, we deal with $\|\delta_{\mathcal{A}}\|_{L^2}$ and $\|\delta_\phi\|_{L^2}$. From the equation of $\mathcal{A}$, it follows that $\delta_{\mathcal{A}}(t)$ satisfies

$$\begin{aligned}\partial_t^2\delta_{\mathcal{A}}-\partial_x^2\delta_{\mathcal{A}}+\delta_{\mathcal{A}}=&-\mathrm{i}\{(\partial_x\psi\overline{\psi}-\psi\partial_x\overline{\psi})-(\partial_x\widetilde{\psi}\overline{\widetilde{\psi}}-\widetilde{\psi}\partial_x\overline{\widetilde{\psi}})\}\\ &-2(\mathcal{A}|\psi|^2-\widetilde{\mathcal{A}}|\widetilde{\psi}|^2)-2W(|\psi|^2-|\widetilde{\psi}|^2)+\delta_{\mathcal{A}}.\end{aligned}$$

Suppose that $\omega = (1-\partial_x^2)^{1/2}$. Then we get

$$\delta_{\mathcal{A}}(t) = \int_0^t \omega^{-1}\sin[(t-s)\omega][-\mathrm{i}\{(\partial_x\psi\overline{\psi} - \psi\partial_x\overline{\psi}) - (\partial_x\widetilde{\psi}\overline{\widetilde{\psi}} - \widetilde{\psi}\partial_x\overline{\widetilde{\psi}})\} - 2W(|\psi|^2 - |\widetilde{\psi}|^2) - 2(\mathcal{A}|\psi|^2 - \widetilde{\mathcal{A}}|\widetilde{\psi}|^2) + \delta_{\mathcal{A}}]\,\mathrm{d}s. \tag{40}$$

From this, by the duality, we have

$$\|\delta_{\mathcal{A}}\|_{L^2} \leqslant C(T)\int_0^t (\|\delta_u\|_{L^2} + \|\delta_{\mathcal{A}}\|_{L^2})\,\mathrm{d}s.$$

Applying the same method as above, we obtain

$$\|\delta_\phi\|_{L^2} \leqslant C(T)\int_0^t (\|\delta_u\|_{L^2} + \|\delta_{\mathcal{A}}\|_{L^2} + \|\delta_\phi\|_{L^2})\,\mathrm{d}s.$$

Up to now, all of the terms on the right-hand side of (33) have been bounded. We can conclude that

$$\|\delta_u\|_{L^\infty(I_T;L^2)} + \|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)} + \|\delta_\phi\|_{L^\infty(I_T;L^2)} \leqslant C(T)T(\|\delta_u\|_{L^\infty(I_T;L^2)} + \|\delta_{\mathcal{A}}\|_{L^\infty(I_T;L^2)} + \|\delta_\phi\|_{L^\infty(I_T;L^2)}). \tag{41}$$

If we choose $T > 0$ so small that $C(T)T < 1$, then (41) and (38) imply that

$$\mathcal{A}(t) = \widetilde{\mathcal{A}}(t), \quad \phi(t) = \widetilde{\phi}(t), \quad \psi(t) = \widetilde{\psi}(t), \quad t \in [0,T]. \tag{42}$$

Repeating the above procedure, we can obtain the desired conclusion (42) for all $t \in \mathbb{R}$. □

Let $(\mathcal{A},\phi,\psi), (\mathcal{A}',\phi',\psi')$ be the solutions of the MS system (10)–(13) with initial data $(\mathcal{A}_0,\mathcal{A}_1,\phi_0,\phi_1,\psi_0)$, $(\mathcal{A}_0',\mathcal{A}_1',\phi_0',\phi_1',\psi_0') \in H^1\times L^2\times H^1\times L^2\times\mathcal{H}^1$, respectively. Based on the gauge transformation (29) and Duhamel's integral (31) and (40), continuous dependence of the solution on the data can be also treated analogously as above for $T > 0$ sufficiently small. We omit the details here.

R e m a r k 3. Coincidentally, transformation (29) actually eliminated both terms $V\psi$ and $W^2\psi$ from (12) at the same time, which indicates that the introduction of W does not affect the uniqueness.

R e m a r k 4. The gauge transformation (29) can be replaced by a more general form

$$u(t,x) = \psi(t,x)\exp\bigg(-\mathrm{i}\int_0^x (\mathcal{A}(t,y) + \lambda W(y))\,\mathrm{d}y - \mathrm{i}tV(x) + \mathrm{i}\beta tW^2(x)\bigg),$$

where β, λ are real constants satisfying $\beta + (\lambda-1)^2 = 0$.

R e m a r k 5. The Strichartz estimates constructed by Oh [18] and Carles [7] for the operator $G = \Delta + F(x)$ with real-valued measurable smooth function $F(x)$, can also be applied to prove the uniqueness, but restriction (19) still seems to be unable to be weakened.

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References

[1] *P. Antonelli, P. Marcati, R. Scandone*: Global well-posedness for the non-linear Maxwell-Schrödinger system. Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) *23* (2022), 1293–1324. zbl MR doi
[2] *P. Antonelli, A. Michelangeli, R. Scandone*: Global, finite energy, weak solutions for the NLS with rough, time-dependent magnetic potentials. Z. Angew. Math. Phys. *69* (2018), Article ID 46, 30 pages. zbl MR doi
[3] *K. Aoki, D. Guimard, M. Nishioka, M. Nomura, S. Iwamoto, Y. Arakawa*: Coupling of quantum-dot light emission with a three-dimensional photonic-crystal nanocavity. Nature Photonics *2* (2008), 688–692. doi
[4] *L. E. Ballentine*: Quantum Mechanics: A Modern Development. World Scientific, Singapore, 2014. zbl MR doi
[5] *W. Bao, Y. Cai*: Uniform error estimates of finite difference methods for the nonlinear Schrödinger equation with wave operator. SIAM J. Numer. Anal. *50* (2012), 492–521. zbl MR doi
[6] *I. Bejenaru, D. Tataru*: Global wellposedness in the energy space for the Maxwell-Schrödinger system. Commun. Math. Phys. *288* (2009), 145–198. zbl MR doi
[7] *R. Carles*: Nonlinear Schrödinger equations with repulsive harmonic potential and applications. SIAM J. Math. Anal. *35* (2003), 823–843. zbl MR doi
[8] *M. Colin, T. Watanabe*: Cauchy problem for the nonlinear Schrödinger equation coupled with the Maxwell equation. Ann. Henri Lebesgue *3* (2020), 67–85. zbl MR doi
[9] *W. Greiner, J. Reinhardt*: Quantum Electrodynamics. Springer, Berlin, 2003. zbl MR doi
[10] *Y. Guo, K. Nakamitsu, W. Strauss*: Global finite-energy solutions of the Maxwell-Schrödinger system. Commun. Math. Phys. *170* (1995), 181–196. zbl MR doi
[11] *N. Hayashi, T. Ozawa*: Remarks on nonlinear Schrödinger equations in one space dimension. Differ. Integral Equ. *7* (1994), 453–461. zbl MR doi
[12] *A. I. Komech*: On quantum jumps and attractors of the Maxwell-Schrödinger equations. Ann. Math. Qué. *46* (2022), 139–159. zbl MR doi
[13] *Y. Liu, T. Wada*: Long range scattering for the Maxwell-Schrödinger system in the Lorenz gauge without any restriction on the size of data. J. Differ. Equations *269* (2020), 2798–2852. zbl MR doi
[14] *H. Lourenço-Martins*: A brief introduction to nano-optics with fast electrons. Plasmon Coupling Physics. Advances in Imaging and Electron Physics 222. Elsevier, Amsterdam, 2022, pp. 1–82. doi
[15] *K. Nakamitsu, M. Tsutsumi*: The Cauchy problem for the coupled Maxwell-Schrödinger equations. J. Math. Phys. *27* (1986), 211–216. zbl MR doi
[16] *M. Nakamura, T. Wada*: Local well-posedness for the Maxwell-Schrödinger equation. Math. Ann. *332* (2005), 565–604. zbl MR doi
[17] *M. Nakamura, T. Wada*: Global existence and uniqueness of solutions to the Maxwell-Schrödinger equations. Commun. Math. Phys. *276* (2007), 315–339. zbl MR doi

[18] *Y.-G. Oh*: Cauchy problem and Ehrenfest's law of nonlinear Schrödinger equations with potentials. J. Differ. Equations *81* (1989), 255–274. zbl MR doi
[19] *R. Scandone*: Global solutions to the nonlinear Maxwell-Schrödinger system. Harmonic Analysis and Partial Differential Equations. Trends in Mathematics. Birkhäuser, Cham, 2022, pp. 91–96. MR doi
[20] *Q. Shi*: Global boundedness for the nonlinear Klein-Gordon-Schrödinger system with power nonlinearity. Differ. Integral Equ. *36* (2023), 837–858. zbl MR doi
[21] *Q. Shi, Y. Jia, J. Cao*: Spatially singular solutions for Klein-Gordon-Schrödinger system. Appl. Math. Lett. *131* (2022), Article ID 108038, 7 pages. zbl MR doi
[22] *Q. Shi, C. Peng, Q. Wang*: Blowup results for the fractional Schrödinger equation without gauge invariance. Discrete Contin. Dyn. Syst., Ser. B *27* (2022), 6009–6022. zbl MR doi
[23] *P. K. Shukla, B. Eliasson*: Nonlinear aspects of quantum plasma physics. Phys. Usp. *53* (2010), 51–76. doi
[24] *P. K. Shukla, B. Eliasson*: Colloquium: Nonlinear collective interactions in quantum plasmas with degenerate electron fluids. Rev. Mod. Phys. *83* (2011), 885–906. doi
[25] *M. Sugawara, N. Hatori, M. Ishida, H. Ebe, Y. Arakawa, T. Akiyama, K. Otsubo, T. Yamamoto, Y. Nakata*: Recent progress in self-assembled quantum-dot optical devices for optical telecommunication: Temperature-insensitive 10 $\mathrm{Gb\,s^{-1}}$ directly modulated lasers and 40 $\mathrm{Gb\,s^{-1}}$ signal-regenerative amplifiers. J. Phys. D, Appl. Phys. *38* (2005), 2126–2134. doi
[26] *Y. Tsutsumi*: Global existence and asymptotic behavior of solutions for the Maxwell-Schrödinger equations in three space dimensions. Commun. Math. Phys. *151* (1993), 543–576. zbl MR doi
[27] *Y. Tsutsumi*: Global existence and uniqueness of energy solutions for the Maxwell-Schrödinger equations in one space dimension. Hokkaido Math. J. *24* (1995), 617–639. zbl MR doi
[28] *M. Tsutsumi, K. Nakamitsu*: Global existence of solutions to the Cauchy problem for coupled Maxwell-Schrödinger equations in two space dimensions. Physical Mathematics and Nonlinear Partial Differential Equations. Lecture Notes in Pure and Applied Mathematics 102. Marcel Dekker, New York, 1985, pp. 139–155. zbl MR doi
[29] *T. Wada*: Smoothing effects for Schrödinger equations with electro-magnetic potentials and applications to the Maxwell-Schrödinger equations. J. Funct. Anal. *263* (2012), 1–24. zbl MR doi
[30] *K. Yajima*: Existence of solutions for Schrödinger evolution equations. Commun. Math. Phys. *110* (1987), 415–426. zbl MR doi
[31] *K. Yajima*: Schrödinger evolution equations with magnetic fields. J. Anal. Math. *56* (1991), 29–76. zbl MR doi

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