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WEAK SOLVABILITY AND NUMERICAL ANALYSIS OF A CLASS OF TIME-FRACTIONAL HEMIVARIATIONAL INEQUALITIES WITH APPLICATION TO FRICTIONAL CONTACT PROBLEMS

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Abstract. We investigate a generalized class of fractional hemivariational inequalities involving the time-fractional aspect. The existence result is established by employing the Rothe method in conjunction with the surjectivity of multivalued pseudomonotone operators and the properties of the Clarke generalized gradient. We are also exploring a numerical approach to address the problem, utilizing both spatially semi-discrete and fully discrete finite elements, along with a discrete approximation of the fractional derivative. All these results are applied to the analysis and numerical approximations of a frictional contact model that describes the quasi-static contact between a viscoelastic body and a solid foundation. The constitutive relation is modeled using the fractional Kelvin-Voigt law. The contact and friction are described by the subdifferential boundary conditions. The variational formulation of this problem leads to a fractional hemivariational inequality. The error estimates for this problem are derived. Finally, here's a second concrete example to illustrate the application. We propose a frictional contact model that incorporates normal compliance and Coulomb friction to describe the quasi-static contact between a viscoelastic body and a solid foundation.

Keywords: hemivariational inequality; Rothe method; Clarke subdifferential; Caputo derivative; fractional viscoelastic constitutive law; contact with friction; numerical scheme; finite element method; convergence analysis; error estimation

MSC 2020: 35L15, 35L86, 35L87, 74Hxx, 74M10, 74D10, 74S05

1. Introduction

Variational inequalities are a significant concept in solid mechanics, a branch of mechanics that examines the behavior of solid materials under the influence of forces and deformations. They also play a critical role in various problems involving equations in mechanics, control, optimization, and mathematical physics. Variational

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inequalities allow for modeling and analyzing situations where stresses or deformations are not uniform within the solid material. Specifically, they are used to describe boundary conditions, boundary stresses, and internal stresses within materials.

In 1980, Panagiotopoulos [35] was the first to introduce hemivariational inequalities to address mechanical systems involving nonconvex and nonsmooth superpotentials.

Recently, mixed variational-hemivariational inequalities have been investigated in [1], [28]. The application of homogenization to linear elasticity, specifically for pure boundary hemivariational inequalities, can be found in [24].

The mathematical study of various contact models gives rise to variational inequalities related to convex potentials and convex unilateral constraints [2], [9], [17], [21], and more recently, in [18], [5]. For additional instances and mathematical findings concerning hemivariational and variational-hemivariational inequalities as well as their mechanical applications, one can consult references [16], [27], [29], [30], [32], [47].

Fractional calculus, as a generalization of traditional calculus, has gained popularity in recent decades due to its effectiveness in modeling and analyzing various real phenomena, such as anomalous convection and diffusion [23], [26], viscoelastic rheology [6], control theory [3], and more.

When we incorporate the concepts of fractional calculus into the hemivariational inequality, it gives rise to what we term a "fractional hemivariational inequality". This inequality serves as a potent tool for comprehending and investigating real frictional interactions between a fractional viscoelastic material and a solid surface. In prior studies [48], [11], researchers explored two varieties of viscoelastic friction models employing mathematical approaches grounded in Caputo derivatives.

In this category, we use the Rothe method, demonstrated in studies like [4], [19], [47], [33], along with the theory of multivalued pseudomonotone operators as explained in references like [13], [44], [45], to confirm that there is a solution for the abstract operator inclusion.

The authors of [48], [11], [46] have examined a category of hemivariational inequalities that incorporate the time-fractional order integral operator. They employ the fractional Kelvin-Voigt law, wherein the contact is characterized by the Clarke subdifferential of a nonconvex and nonsmooth functional. A comparable model featuring a frictionless contact problem is presented in [16], [8]. In 2021, Weng et al. [42] introduced and examined a novel category of fractional differential hemivariational inequalities, which are defined by an evolutionary hemivariational inequality and a fractional differential equation. The authors applied the Rothe method and leveraged the surjectivity result to establish the existence of a unique solution. Additionally, a fully discrete scheme was employed to approximate the problem, and the authors illustrated this approach by presenting an example involving a viscoelastic frictional contact problem with adhesion.

Just recently, in 2021, Bouallala et al. [7] initiated the exploration of a novel model for a fractional contact problem involving normal compliance and Coulomb's friction in thermo-viscoelasticity.

A lot of research has been done to create numerical and analytical methods for dealing with fractional hemicontact variational inequalities, which are important in physics. A few important methods in this area are the finite difference technique [4], [25], as well as other numerical plans [14], [43]. Several studies exploring the numerical study of fractional differential equations have been recently released, such as [36], [33].

Given the crucial importance of fractional calculus, whether in generalizing the laws governing the behavior of rheological models used in linear viscoelasticity (composed of springs and dampers [38]) or in fractional models that describe viscoelastic behavior over a wide frequency range with a limited number of parameters [34], [22], we will focus on the mathematical study and numerical analysis of a specific class of hemi-variant inequalities. These inequalities take into account the constitutive law of Kelvin-Voight with the Caputo fractional derivative, as well as the generalized directional derivative (Clarke).

We demonstrate the existence and regularity of a weak solution to the abstract hemi-variational inequality using the Rothe method in combination with a surjectivity result for a pseudomonotone perturbation of a maximal monotone operator.

In the second part, the paper introduces semi-discretized finite element schemes in space and a fully discretized scheme in time, effectively approximating the fractional derivative using the Caputo approach. Additionally, the study presents error estimates to validate the proposed methods.

In the third part of the article, we apply the abstract findings to a frictional contact model for fractional viscoelastic materials. The process is outlined using multivalued contact and friction conditions. We present the variational formulation of the contact problem as a hemivariational inequality and provide a result regarding its solvability. We address the spatial and spatiotemporal discretization of the weak problem using the Euler scheme and finite element method. Subsequently, we demonstrate an error assessment.

The rest of paper is structured as follows: In Section 2 we show the existence of solution to a class of abstract hemivariational inequality studied in the framwork. A spatially semi-discrete scheme to approximate the hemivariational problem using the finite element method is presented in Section 3. In Section 4, we propose and analyze a fully discrete scheme using the approximation of the time-fractional derivative. In Section 5, we explore a time-fractional viscoelastic contact model controlled by a quasistatic hemivariational inequality, to which we apply our abstract outcome. Furthermore, we offer another specific instance of a contact problem that encom-

passes normal compliance and Coulomb friction within the domain of viscoelasticity, utilizing the conventional derivative. Finally, in Appendix, we review specific findings related to fractional calculus theory, nonlinear analysis, and discrete analysis.

2. Time-fractional hemivariational inequality

In this section, we introduce a general time-fractional hemivariational inequality, while also outlining the functional framework of the problem. Following that, we present a result concerning the existence of solutions using Rothe's method.

Let V be a separable and reflexive Banach space. Denote V^* as the dual space of V. By identifying H with its own dual, we establish the inclusion $V \subset H = H^* \subset V^*$. The duality pairing between V^* and V is denoted as $\langle \cdot, \cdot \rangle_{V^* \times V}$, and $\|\cdot\|_V$ represents the norm in V.

Given $0 < T < \infty$, we consider the function spaces

$$\mathcal{V} = L^p(0,T;V),\ \mathcal{V}^* = L^p(0,T;V),\ \text{and}\ X = \{x \in \mathcal{V},\ {}^C_0D^\alpha_t x \in \mathcal{V}\}, \tag{2.1}$$

where $\alpha \in]0,\alpha]$, $1/\alpha < p < \infty$ and $q = p/(p-1)$.

Given bounded linear operators $A, B\colon V \to V^*$, a locally Lipschitz function $J\colon V \to \mathbb{R}$, $f \in V^*$, and $x_0 \in \mathcal{V}$, the main problem under consideration is stated as:

Problem (P1). Find $x \in X$ such that for all $y \in V$, a.e. $t \in (0,T)$

$$\begin{aligned} A({}^C_0D^\alpha_t x(t), y - x(t)) + B(x(t), y - x(t)) + J^\circ(x(t), y) \\ - J^\circ(x(t), x(t)) \geqslant (f(t), y - x(t))_{V^* \times V}, \\ x(0) = x_0. \end{aligned} \tag{2.2}$$

Let $w(t) = {}^C_0D^\alpha_t x(t)$ for almost every $t \in (0,T)$. Applying Proposition 6.3 (b) below, we have $x(t) = {}_0I^\alpha_t w(t) + x_0$. Then Problem (P1) can be written as:

Problem (P2). Find $w \in \mathcal{V}$ such that for all $y \in V$, a.e. $t \in (0,T)$

$$\begin{aligned} A(w(t), y - x_0 - {}_0I^\alpha_t w(t)) + B(u_0 + {}_0I^\alpha_t w(t), y - x_0 - {}_0I^\alpha_t w(t)) \\ + J^\circ({}_0I^\alpha_t w(t) + x_0, y) - J^\circ({}_0I^\alpha_t w(t) + x_0, {}_0I^\alpha_t w(t) + x_0) \\ \geqslant (f(t), y - x_0 - {}_0I^\alpha_t w(t))_{V^* \times V}. \end{aligned} \tag{2.3}$$

In the study of Problem (P2), we impose certain hypotheses.

(H1) The operators A and B satisfy the property of ellipticity, i.e., there exist positive constants m_A and m_B such that

$$A(v,v) \geqslant m_A \|v\|^2_V \text{ and } B(v,v) \geqslant m_B \|v\|^2_V \quad \forall\, v \in V.$$

(H2) The operators A and B satisfy the standard property of boundedness

$$|A(u,v)| \leqslant M_A \|u\|_V \|v\|_V \text{ and } |B(u,v)| \leqslant M_B \|u\|_V \|v\|_V,$$

where M_A, $M_B > 0$.

(H3) $J\colon V \to \mathbb{R}$ is locally Lipschitz continuous and there exists $m_J > 0$ such that

$$\|\partial J(v)\|_{V^*} = m_J(1 + \|v\|_V) \quad \forall\, v \in V.$$

(H4)

$$f \in C(0,T;V^*) \text{ and } x_0 \in V.$$

(H5) There exists $\alpha_J > 0$ such that

$$|J^\circ(x_1,y_2) - J^\circ(x_1,y_1) + J^\circ(x_2,y_1) - J^\circ(x_2,y_2)| \leqslant \alpha_J \|x_1 - x_2\|_V \cdot \|y_1 - y_2\|_V$$

for all x_1, x_2, y_1, y_2 in V.

We obtain the existence result.

Theorem 2.1. *Assuming that assumptions* (H1)–(H5) *are satisfied, then Problem* (P1) *has at least one solution* $x \in X$.

Let $N \in \mathbb{N}$ be fixed, $\tau = T/N$, and $\{t_n\}_{n=0}^N = \{n\tau\}_{n=0}^N$ represent the equidistant time points in the interval $[0,T]$.

We consider the integral operator ${}_0I_{t_n}^\alpha w(t_n)$, given by

$$\begin{aligned} {}_0I_{t_n}^\alpha w(t_n) &= \frac{1}{\Gamma(\alpha)} \sum_{j=1}^n \int_{t_{j-1}}^{t_j} (t_n - s)^{\alpha-1} w(t_j)\,\mathrm{d}s \\ &= \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^n w(t_j)[(n-j+1)^\alpha - (n-j)^\alpha]. \end{aligned} \tag{2.4}$$

Additionally, we put

$$f_\tau^j = \frac{1}{\tau} \int_{t_{j-1}}^{t_j} f(s)\,\mathrm{d}s \quad \forall\, j = 1,\dots,N. \tag{2.5}$$

Now, we apply Rothe's method to Problem (P2), and consider the auxiliary time-discretization problem:

Problem (PTD). Find $\{w_\tau^n\}_{n=1}^N \in V$ for each $n = 1, \dots, N$ such that

$$A(w_\tau^n, y - z_\tau^n) + B(z_\tau^n, y - z_\tau^n) + J^\circ(z_\tau^n, y) - J^\circ(z_\tau^n, z_\tau^n) \geqslant (f_\tau^n, z_\tau^n)_{V^* \times V}, \tag{2.6}$$

where z_τ^n is defined by

$$z_\tau^n = x_0 + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^n w_\tau^j [(n-j+1)^\alpha - (n-j)^\alpha]. \tag{2.7}$$

Note that Problem (PTD) can be reformulated equivalently as

Problem (PTDE). Find $\{w_\tau^n\}_{n=1}^N \in V$ for each $n = 1, \dots, N$ such that

$$A(w_\tau^n) + B(z_\tau^n) + \varphi_\tau^n = f_\tau^n, \text{ with } \varphi_\tau^n \in \partial J(z_\tau^n). \tag{2.8}$$

We can establish the following existence result for problem (2.8).

Lemma 2.2. *Assuming that* (H1)–(H5) *are satisfied, problem* (2.8) *possesses at least one solution* $w_\tau^n \in V$ *for all* $n = 1, \dots, N$, *provided that* $\tau > 0$ *is sufficiently small.*

P r o o f. The proof relies on a surjectivity result and mathematical induction carried out in several steps.

In the first step, we analyze the multivalued operator $L\colon V \to 2^{V^*}$, defined as

$$L(w) := A(w) + B\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\Big) + \partial J\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\Big) \tag{2.9}$$

for all $w \in V$.

We will demonstrate that L is a surjective operator. According to Theorem 6.5, it suffices to establish that L is pseudomonotone and coercive. To begin, we prove the coerciveness of the operator L. Let

$$\varphi \in \partial J\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(\alpha+1)} w\Big).$$

Using (H3), we have

$$\begin{aligned} \|\varphi\|_{V^*} &\leqslant m_J\Big(1 + \|x_0\| + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} \|w_n^j\| [(n-j-1)^\alpha - (n-j)^\alpha] \\ &\qquad + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \|w\|\Big) \\ &= m_J\Big(1 + c_\tau + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \|w\|\Big), \end{aligned} \tag{2.10}$$

where

$$c_\tau = \|x_0\| + \frac{\tau^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} \|w_n^j\| [(n-j-1)^\alpha - (n-j)^\alpha]. \tag{2.11}$$

Next, using (H1) and (H2), we can derive:

(2.12)
$$\begin{aligned}
&\Big\langle A(w) + B\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(\alpha+1)}\Big), w\Big\rangle_{V^*\times V} + \langle \varphi, w\rangle_{V^*\times V} \\
&\quad \geqslant m_A\|w\|_V^2 - (M_B c_\tau + m_J + m_J c_\tau)\|w\|_V - \frac{\tau^\alpha M_B}{\Gamma(\alpha+1)}\|w\|_V^2 - \frac{m_J\tau^\alpha}{\Gamma(\alpha+1)}\|w\|_V^2 \\
&\quad = \Big(m_A - \frac{\tau^\alpha}{\Gamma(\alpha+1)}(M_B + m_J)\Big)\|w\|_V^2 - (M_B c_\tau + m_J(1+c_\tau))\|w\|_V.
\end{aligned}$$

By choosing
$$\tau_0 = \Big(\frac{m_A\Gamma(\alpha+1)}{M_B + m_J}\Big)^{1/\alpha},$$
we establish that L is a coercive operator.

Now, we proceed to demonstrate that L is a pseudomonotone operator. To achieve this, let $L_1\colon V \to V^*$ be defined as

$$L_1(w) = A(w) + B\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(\alpha+1)}w\Big) \quad \forall\, w \in V. \tag{2.13}$$

It is evident that, in accordance with (H1), T_1 is both bounded and continuous.

By virtue of the linearity and coercivity of A, and the boundedness of B, it follows that

(2.14)
$$\begin{aligned}
\langle L_1(v_1) - L_1(v_2), v_1 - v_2\rangle &= \Big\langle A(v_1 - v_2) + B\Big(\frac{\tau^\alpha}{\Gamma(\alpha+1)}(v_1 - v_2)\Big), v_1 - v_2\Big\rangle_{V^*\times V} \\
&\geqslant \Big(m_A - \frac{M_B\tau^\alpha}{\Gamma(\alpha+1)}\Big)\|v_1 - v_2\|_V^2
\end{aligned}$$

for all v_1, v_2 in V. Then L_1 is a monotone operator.

Therefore, according to [32], Theorem 3.69 (ii), and considering the boundedness, monotonicity, and continuity of the operator L_1, we conclude that L_1 is pseudomonotone.

Now, let us utilize Proposition 6.4 to establish that the operator $L_2\colon V \to 2^{V^*}$, defined as

$$L_2(w) = \partial J\Big(z_\tau^{n-1} + \frac{\tau^\alpha}{\Gamma(1+\alpha)}w\Big) \quad \forall\, w \in V, \tag{2.15}$$

is pseudomonotone.

Leveraging the properties of the generalized gradient as outlined in [12], Proposition 2.1.2, along with the reflexivity of V, it can be deduced that L_2 is nonempty, convex, and weakly compact for all $w \in V$. Additionally, due to (H3), L_2 is bounded.

Consider $\{w_m\} \in V$ such that $w_m \to w$ weakly in V as $m \to \infty$, and let $\varphi_m \in L_2(w_m)$. Given that L_2 is bounded, the sequence $\{\varphi_m\}$ is bounded in V^*. Consequently, by considering a subsequence if needed, we ensure that $\varphi_m \to \varphi$ in V^* as $m \to \infty$, thus leading to the conclusion that $\varphi \in L_2(w)$.

In addition, it is clear that

$$\langle \varphi_m, w_n \rangle_{V^* \times V} \to \langle \varphi, w \rangle_{V^* \times V}, \text{ as } m \to \infty. \tag{2.16}$$

Therefore, by employing Proposition 6.4(c), we establish the pseudomonotonicity of the operator L_2, which consequently implies the pseudomonotonicity of L.

As a result, problem (2.8) possesses at least one solution. □

In the second step, we derive estimates for the sequence of solutions of problem (2.8).

Lemma 2.3. *Assuming that $\{w_\tau^n\}_{n=1}^N$ constitutes a sequence of solutions to problem* (2.8)*, then*

$$\|w_\tau^n\|_V + \|z_\tau^n\|_V + \|\varphi_\tau^n\|_{V^*} \leqslant c \quad \forall\, \tau \in (0, \tau^0), \tag{2.17}$$

where c is a positive constant independent of τ and $\varphi_\tau^n \in \partial J(z_\tau^n)$ for $n = 1, \dots, N$.

P r o o f. For all $n = 1, 2, \dots, N$, we have:

$$A(w_\tau^n) + B(z_\tau^n) + \varphi_\tau^n = f_\tau^n, \text{ where } \varphi_\tau^n \in \partial J(z_\tau^n). \tag{2.18}$$

We test equation (2.18) with w_τ^n, and utilizing (H1)–(H3), we arrive at

(2.19)

$$\begin{aligned}
&\langle f_\tau^n, w_\tau^n \rangle_{V^* \times V} \\
&\quad \geqslant m_A \|w_\tau^n\|_V^2 - \frac{\tau^\alpha M_B}{\Gamma(\alpha+1)} \|w_\tau^n\|_V^2 - (M_B \|x_0\| + m_J \|x_0\|) \|w_\tau^n\|_V \\
&\qquad - \frac{\tau^\alpha M_B}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} \|w_\tau^j\|_V \cdot \|w_\tau^n\|_V [(n-j+1)^\alpha - (n-j)^\alpha] - \frac{\tau^\alpha m_J}{\Gamma(\alpha+1)} \|w_\tau^n\|_V^2 \\
&\qquad - m_J \|w_\tau^n\|_V - \frac{\tau^\alpha m_J}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} \|w_\tau^j\|_V \cdot \|w_\tau^n\|_V [(n-j+1)^\alpha - (n-j)^\alpha].
\end{aligned}$$

Consequently, we deduce that

$$\begin{aligned}
&\|f_\tau^n\| + M_B \|x_0\| \\
&\qquad + \frac{\tau^\alpha (m_J + M_B)}{\Gamma(\alpha+1)} \sum_{j=1}^{n-1} \|w_\tau^j\|_V [(n-j+1)^\alpha - (n-j)^\alpha] + m_J + m_J \|x_0\| \\
&\quad \geqslant \Big(m_A - \frac{\tau^\alpha (m_J + M_B)}{\Gamma(\alpha+1)} \Big) \|w_\tau^n\|_V.
\end{aligned} \tag{2.20}$$

Choosing

$$\tau^0 = \Big(\frac{m_A\Gamma(\alpha+1)}{2(M_B+m_J)}\Big)^{1/\alpha}$$

which gives

$$m_A - \frac{\tau^\alpha(M_b+m_J)}{\Gamma(\alpha+1)} \geqslant \frac{m_A}{2}.$$

Therefore,

$$\begin{aligned}(2.21)\qquad &\frac{2\|f_\tau^n\|_{V^*}}{m_A} + \frac{2M_B\|x_0\|}{m_A} + \frac{2(m_J+m_J\|x_0\|)}{m_A} \\ &\quad + \frac{2\tau^\alpha(M_B+m_J)}{m_A\Gamma(1+\alpha)}\sum_{j=1}^{n-1}[(n-j+1)^\alpha - (n-j)^\alpha] \geqslant \|w_\tau^n\|_V.\end{aligned}$$

Given (H4), there exists a positive constant $m_f > 0$ such that $\|f_\tau^n\|_{V^*} \leqslant m_f$ for all $\tau > 0$ and $\tau \in \mathbb{N}$. Observing this, we have:

$$(2.22)\qquad m_0 = \frac{2m_f}{m_A} = \frac{2M_B\|x_0\|}{m_A} + \frac{2(m_J+m_J\|x_0\|)}{m_A}.$$

By utilizing the generalized discrete Grönwall inequality, we conclude that:

$$(2.23)\qquad \|w_\tau^n\|_V = m_0\exp\Big(\frac{2(M_B+m_J)T^\alpha}{m_A\Gamma(\alpha+1)}\Big) \leqslant C.$$

Considering (2.23) and (2.7), we arrive at the estimate

$$\begin{aligned}(2.24)\qquad \|w_\tau^n\|_V &\leqslant \|x_0\| + \frac{C}{\Gamma(1+\alpha)}\sum_{j=1}^{n}(t_{n-j+1}^\alpha - t_{n-j}^\alpha) + \|x_0\| + \frac{C}{\Gamma(1+\alpha)}t_n^\alpha \\ &\leqslant \|x_0\| + \frac{CT^\alpha}{\Gamma(1+\alpha)}.\end{aligned}$$

Finally, assumption (H3) implies that:

$$(2.25)\qquad \|\varphi_\tau^n\|_{V^*} \leqslant m_J(1+\|w_\tau^n\|_V) \leqslant m_J\Big(1+\|x_0\| + \frac{CT^\alpha}{\Gamma(1+\alpha)}\Big).$$

Thus, Lemma 2.3 is established. □

In the final step, with all the necessary components in place, we are prepared to present the proof of Theorem 2.1.

P r o o f of Theorem 2.1. Let $\{\tau_n\}$ be a sequence such that $\tau_n \to 0$ as $n \to \infty$. Based on the estimate (2.17), the sequences $\{\overline{w}_\tau\}$, $\{\overline{z}_\tau\}$ and $\{\overline{\varphi}_\tau\}$ interpolate $\{w_\tau\}$, $\{z_\tau\}$ and $\{\varphi_\tau\}$, respectively, and are bounded for $n = 1, \ldots, N$.

Subsequently, there exist $w \in V$, $z \in V$, and $\varphi \in V^*$ such that

$$\begin{aligned} \overline{w}_\tau &\rightharpoonup w \text{ weakly in } V \quad \text{as } \tau \to 0, \\ \overline{z}_\tau &\rightharpoonup z \text{ weakly in } V \quad \text{as } \tau \to 0, \\ \overline{\varphi}_\tau &\rightharpoonup \varphi \text{ weakly in } V^* \text{ as } \tau \to 0. \end{aligned} \tag{2.26}$$

Hence, we obtain

$$ {}_0I_t^\alpha \overline{w}_\tau \to {}_0I_t^\alpha w \text{ weakly in } C(0,T;V) \text{ as } \tau \to 0, \tag{2.27}$$

and utilizing [30], Lemma 4 (a), we derive

$$ {}_0I_t^\alpha \overline{w}_\tau \to {}_0I_t^\alpha w \text{ weakly in } V \text{ as } \tau \to 0. \tag{2.28}$$

Considering (2.17) for all $t \in (0,T)$, we can conclude that

$$ \|\overline{w}_\tau(t) - \overline{z}_\tau(t)\| \leqslant \frac{C}{\Gamma(\alpha)}[2(t_n - t)^\alpha + t^\alpha - t_n^\alpha]. \tag{2.29}$$

This implies that

$$ \overline{w}_\tau(t) \to \overline{z}_\tau(t) \text{ strongly in } V \text{ as } \tau \to 0. \tag{2.30}$$

By combining the preceding result with (2.28), we arrive at

$$ \overline{w}_\tau(t) \to z_\tau(t) \text{ weakly in } V \text{ as } \tau \to 0. \tag{2.31}$$

Given that the mapping $w \to \partial J(w)$ exhibits upper-semi continuity from V to V^*, and with the reference to [32], Theorem 3.13, we can deduce

$$ \varphi(t) \in \partial J(z(t)) \quad \text{for a.e. } t \in (0,T). \tag{2.32}$$

Now, we introduce the Nemytskii operators $\overline{A}$ and $\overline{B}$, which correspond to the operators A and B, and are defined as

$$ (\overline{A}w)(t) = Aw(t) \text{ and } (\overline{B}w)(t) = Bz(t) \quad \forall\, w \in V \text{ and a.e. } t \in (0,T). \tag{2.33}$$

Applying (H1)–(H2) and utilizing (2.26), we can derive for all $t \in (0,T)$ that

$$ \overline{A}\overline{w}_\tau \to \overline{A}w \text{ weakly in } V \text{ as } \tau \to 0, \tag{2.34}$$

and

$$ B(\overline{z}_\tau) \to Bz \text{ weakly in } V \text{ as } \tau \to 0. \tag{2.35}$$

By employing (H2) and utilizing (2.26), we get

$$\int_0^T \|B(z_\tau(t))\|_V \,\mathrm{d}t \leqslant \frac{M_B C T^{\alpha+1}}{\Gamma(\alpha+2)} + M_B T\|x_0\|. \tag{2.36}$$

By utilizing the Lebesgue dominated convergence theorem, we can express

$$\begin{aligned} \lim_{\tau\to 0}\langle \overline{B}\overline{w}_\tau, y\rangle &= \lim_{\tau\to 0}\int_0^T \langle B(\overline{z}_\tau(t)), y(t)\rangle \,\mathrm{d}t = \int_0^T \lim_{\tau\to 0}\langle B(\overline{z}_\tau(t)), y(t)\rangle \,\mathrm{d}t \\ &= \int_0^T \langle B(z_\tau(t)), y(t)\rangle \,\mathrm{d}t = \langle \overline{B}w, y\rangle. \end{aligned} \tag{2.37}$$

On the other hand, as suggested by [10], Lemma 3.3, we can infer that

$$f_\tau \to f \text{ strongly in } V \text{ as } \tau \to 0. \tag{2.38}$$

Finally, by combining (2.32), (2.34), and (2.36), and passing to the limit in (2.8), we deduce that $w \in L^2(0,T;V)$ is a solution to Problem (P2).

Consequently, we can infer that $x \in W^{1,2}(0,T;X)$ and, with $x(t) = x_0 + {}_0I_t^\alpha w(t)$ for almost every $t \in (0,T)$, it follows that x is a solution to Problem (P1). □

3. Semidiscrete approximation

In this section, we explore an approximation of Problem (P1) by discretizing the space X. We define $X^h \subset X$ as a finite-dimensional subspace, where $h > 0$ is the spatial discretization parameter.

A spatially semi-discrete scheme can be formulated as the problem:

Problem (PH). Find $x^h \in X^h$ for all $\alpha \in]0,1]$ and $y^h \in X^h$ such that

$$\begin{aligned} A({}_0^C D_t^\alpha x^h(t), y^h - x^h(t)) + B(x^h(t), y^h - x^h(t)) + J^\circ(x^h(t), y^h) \\ - J^\circ(x^h(t), x^h(t)) \geqslant (f(t), y^h - x^h(t))_{X^*\times X}, \end{aligned} \tag{3.1}$$

$$x^h(0) = x_0^h. \tag{3.2}$$

Here, $x_0^h \in X^h$ represents a suitable approximation of x_0.

Throughout the remainder of the paper, we denote by c a generic positive constant. Its value may vary in different inequalities, but it remains independent of both h and k.

Building upon the reasoning in the previous section, we can demonstrate that Problem (PH) possesses at least one solution, denoted as $x^h \in W^{1,2}(0,T;X)$. The ensuing result is as follows.

Theorem 3.1. *Suppose that assumptions* (H1)–(H5) *hold. Let* x *and* x^h *represent the solutions to Problem* (P1) *and Problem* (PH)*, respectively. We can then establish the estimate*

$$\begin{aligned}(3.3)\quad &\|x(t)-x^h(t)\|_{L^2(0,T;X)}\\ &\leqslant c\inf_{y^h\in X^h}\{\|y^h-x(t)\|_{L^2(0,T;X)}+\|x_0-x_0^h\|_X+\|R(x(t),y^h)\|^{1/2}_{L^2(0,T;X)}\},\end{aligned}$$

where

$$\begin{aligned}(3.4)\quad R(x(t),y^h)&=A({}_0^C D_t^\alpha x(t),y^h-x(t))+B(x(t),y^h-x(t))+J^\circ(x(t),y^h)\\ &\quad -J^\circ(x(t),x(t))-(f(t),y^h-x(t))_{X^*\times X}.\end{aligned}$$

P r o o f. Substituting $y=x^h$ into (2.2), we obtain

$$\begin{aligned}(3.5)\quad B(x(t),x(t)-x^h(t))&\leqslant A({}_0^C D_t^\alpha x(t),x^h(t)-x(t))+J^\circ(x(t),x^h(t))\\ &\quad -J^\circ(x(t),x(t))-(f(t),x^h(t)-x(t))_{X^*\times X}.\end{aligned}$$

Utilizing the relationship described in (3.1), we get

$$\begin{aligned}(3.6)\quad B(-x^h(t),x(t)-x^h(t))&\leqslant B(x^h(t),y^h-x(t))+A({}_0^C D_t^\alpha x^h(t),y^h-x^h(t))\\ &\quad +J^\circ(x^h(t),y^h)-J^\circ(x^h(t),x^h(t))\\ &\quad -(f(t),y^h-x(t))-(f(t),x(t)-x^h(t)).\end{aligned}$$

Adding together (3.5) and (3.6), we acquire

$$(3.7)\qquad B(x(t)-x^h(t),x(t)-x^h(t))\leqslant R_1+R_2+R(x(t),y^h),$$

where

$$(3.8)\qquad R_1=B(x^h(t)-x(t),y^h-x(t))+A({}_0^C D_t^\alpha x^h(t)-{}_0^C D_t^\alpha x(t),y^h-x(t)),$$

and

$$(3.9)\qquad R_2=J^\circ(x(t),x^h(t))-J^\circ(x(t),y^h)+J^\circ(x^h(t),y^h)-J^\circ(x^h(t),x^h(t)).$$

Based on (H2), we can deduce that

$$(3.10)\qquad |R_1|\leqslant c\{\|x(t)-x^h(t)\|_X^2+\|{}_0^C D_t^\alpha x(t)-{}_0^C D_t^\alpha x^h(t)\|_X^2+\|y^h-x(t)\|_X^2\}.$$

Consulting Definition 6.2, we acquire

$$\|{}_0^C D_t^\alpha x(t) - {}_0^C D_t^\alpha x^h(t)\|_X \leqslant \frac{T^{1-\alpha}}{\Gamma(2-\alpha)} \|\dot{x}(t) - \dot{x}^h(t)\|_X. \tag{3.11}$$

By employing (H5), we derive

$$|R_2| \leqslant \alpha_J \|x(t) - x^h(t)\|_X \cdot \|y^h - x^h(t)\|_X. \tag{3.12}$$

In order to advance with the proof, we need the inequalities

$$\|y^h - x^h(t)\|_X \leqslant \|y^h - x(t)\|_X + \|x(t) - x^h(t)\|_X, \tag{3.13}$$

and

$$\|x(t) - x^h(t)\|_X \leqslant c\left\{ \|x_0 - x_0^h\|_X + \int_0^t \|\dot{x}(s) - \dot{x}^h(s)\|_X \, \mathrm{d}s \right\}. \tag{3.14}$$

Through a combination and application of the coercivity property exhibited by the operator B, we establish the existence of a positive constant c, dependent on T, α, α_J, M_A, m_A, M_B, and m_B. This constant is such that

(3.15)
$$\|x(t) - x^h(t)\|_X^2 \leqslant c\{\|y^h - x(t)\|_X^2 + \|x_0 - x_0^h\|_X^2 + \|\dot{x}(t) - \dot{x}^h(t)\|_X^2 + |R(x(t), y^h)|\}.$$

Upon integrating this inequality from 0 to T, we derive the outcome expressed in (3.3). □

4. Fully discrete approximation

In this section, we explore a fully discrete approximation of Problem (P1). We utilize the finite element space X^h introduced in the preceding section.

We introduce a uniform partition of the time interval into N subintervals. Let $\Delta t = k_n = t_n - t_{n-1}$, where k_n is the time step-size for $n = 1, \ldots, N$. We define $k = \max\limits_{1 \leqslant n \leqslant N} \{k_n\}$ as the maximal step size. For a sequence $\{x_n\}_{n=0}^N$, we put $\delta x = (x_n - x_{n-1})/k_n$. The fully discrete approximation method takes the following form:

Problem (PHK). Find $\{x_n^{hk}\}_{n=0}^N \in X^h$ for all $\alpha \in]0, 1]$ and $y^h \in X^h$ such that

$$\begin{aligned} A({}_0^C D_{t_n}^\alpha x_n^{hk}, y^h - x_n^{hk}) + B(x_n^{hk}, y^h - x_n^{hk}) + J^\circ(x_n^{hk}, y^h) \\ - J^\circ(x_n^{hk}, x_n^{hk}) \geqslant (f_n, y^h - x_n^{hk})_{X^* \times X}, \end{aligned} \tag{4.1}$$

$$x_0^{hk} = x_0^h. \tag{4.2}$$

This problem has at least one solution. The proof is similar to that one used in Theorem 2.1.

For the sake of simplification, we put

$$\delta x_n^{hk} = \frac{1}{k_n}(x_n^{hk} - x_{n-1}^{hk}), \quad n = 1, \dots, N, \tag{4.3}$$

and

$$x_n^{hk} = k\sum_{j=1}^{n} \delta x_n^{hk} + x_0^h. \tag{4.4}$$

The following estimate result holds.

Theorem 4.1. *Under assumptions* (H1)–(H5), *we can establish the subsequent error estimates*

$$\begin{aligned}\max_{1\leqslant n\leqslant N} \|x_n - x_n^{hk}\|_X \leqslant{}& \max_{1\leqslant n\leqslant N} \inf_{y_n^h \in X^h} c\{\|y_n^h - x_n\|_X + |R_n(x_n, y_n^h)|^{1/2}\} \\ &+ c\|x_0 - x_0^h\|_X,\end{aligned} \tag{4.5}$$

where

$$\begin{aligned}R_n(x_n, y_n^h) = {}& B(x_n, y_n^h - x_n) + A({}_0^C D_{t_n}^\alpha x_n, y_n^h - x_n) \\ &+ J^\circ(x_n, y_n^h) - J^\circ(x_n, x_n) - (f_n, y_n^h - x_n)_X.\end{aligned} \tag{4.6}$$

P r o o f. Substituting $y = x_n^{hk}$ into (2.2) at $t = t_n$ yields

$$\begin{aligned}B(x_n, x_n - x_n^{hk}) \leqslant{}& A({}_0^C D_{t_n}^\alpha x_n, x_n^{hk} - x_n) + J^\circ(x_n, x_n^{hk}) \\ &- J^\circ(x_n, x_n) - (f_n, x_n^{hk} - x_n)_{X^*\times X}.\end{aligned} \tag{4.7}$$

Substituting y^h with y_n^h into (4.1), we obtain

$$\begin{aligned}B(x_n^{hk}, x_n^{hk} - y_n^h) \leqslant{}& A({}_0^C D_{t_n}^\alpha x_n^{hk}, y_n^h - x_n^{hk}) + J^\circ(x_n^{hk}, y_n^h) \\ &- J^\circ(x_n^{hk}, x_n^{hk}) - (f_n, y_n^h - x_n^{hk})_{X^*\times X}.\end{aligned} \tag{4.8}$$

Taking the most basic inequalities into consideration, we achieve

$$\begin{aligned}&B(x_n, x_n - x_n^{hk}) + B(x_n^{hk}, x_n^{hk} - y_n^h) \\ &\quad\leqslant A({}_0^C D_{t_n}^\alpha x_n, x_n^{hk} - x_n) + A({}_0^C D_{t_n}^\alpha x_n^{hk}, y_n^h - x_n^{hk}) + J^\circ(x_n, x_n^{hk}) \\ &\qquad - J^\circ(x_n, x_n) + J^\circ(x_n^{hk}, y_n^h) - J^\circ(x_n^{hk}, x_n^{hk}) \\ &\qquad - (f_n, x_n^{hk} - x_n)_{X^*\times X} - (f_n, y_n^h - x_n^{hk})_{X^*\times X}.\end{aligned} \tag{4.9}$$

Employing the subsequent equality

$$B(x_n, x_n - x_n^{hk}) + B(x_n^{hk}, x_n^{hk} - y_n^h) = B(x_n - x_n^{hk}, x_n - x_n^{hk}) - B(x_n^{hk}, y_n^h - x_n), \tag{4.10}$$

we deduce that

$$B(x_n - x_n^{hk}, x_n - x_n^{hk}) \leqslant R_n(x_n, y_n^h) + R_3 + R_4, \tag{4.11}$$

where

$$R_3 = J^\circ(x_n, x_n^{hk}) - J^\circ(x_n, y_n^h) + J^\circ(x_n^{hk}, y_n^h) - J^\circ(x_n^{hk}, x_n^{hk}) \tag{4.12}$$

and

$$R_4 = B(x_n^{hk} - x_n, y_n^h - x_n) + A({}_0^C D_{t_n}^\alpha x_n^{hk} - {}_0^C D_{t_n}^\alpha x_n, y_n^h - x_n). \tag{4.13}$$

Analogously to (3.12), we find

$$|R_3| \leqslant \alpha_J \|x_n - x_n^{hk}\|_X \cdot \|y_n^h - x_n^{hk}\|_X. \tag{4.14}$$

Exploiting (H2), there exists a positive constant c that is dependent on M_A and M_B such that

$$|R_4| \leqslant \{\|x_n - x_n^{hk}\|_X^2 + \|y_n^h - x_n^{hk}\|_X^2 + \|{}_0^C D_{t_n}^\alpha x_n^{hk} - {}_0^C D_{t_n}^\alpha x_n\|_X^2\}. \tag{4.15}$$

Now by employing Definition 6.2, we can deduce the discrete approximation for the fractional derivative in the following manner:

(4.16)

$$\begin{aligned}
{}_0^C D_{t_n}^\alpha x_n^{hk} &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_n} (t_n - s)^{-\alpha} \frac{x_n^{hk} - x_{n-1}^{hk}}{k_n} \,\mathrm{d}s \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=1}^{n} \int_{(j-1)k}^{jk} (nk - s)^{-\alpha} \frac{x_j^{hk} - x_{j-1}^{hk}}{k_n} \,\mathrm{d}s \\
&= \frac{1}{(1-\alpha)\cdot\Gamma(1-\alpha)} \sum_{j=1}^{n} \frac{x_j^{hk} - x_{j-1}^{hk}}{k_n} [(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}] k_n^{1-\alpha} \\
&= \frac{1}{(1-\alpha)\cdot\Gamma(1-\alpha)\cdot k_n^\alpha} \sum_{j=1}^{n} (x_j^{hk} - x_{j-1}^{hk}) [(n-j+1)^{1-\alpha} - (n-j)^{1-\alpha}] \\
&\quad + \frac{1}{(1-\alpha)\cdot\Gamma(1-\alpha)} \sum_{j=1}^{n} [(n-j+1)^{1-\alpha} (n-j)^{1-\alpha}] k_n^{2-\alpha}.
\end{aligned}$$

We shall introduce the definitions

$$m_\alpha = ((1-\alpha)\cdot\Gamma(1-\alpha)\cdot k_n^\alpha)^{-1} \text{ and } c_\alpha = j^{1-\alpha} - (j-1)^{1-\alpha}.$$

With these established, we are now prepared to articulate

$${}_0^C D_{t_n}^\alpha x_n^{hk} = m_\alpha \sum_{j=0}^{n} c_N (x_{n-j-1}^{hk} - x_{n-j}^{hk}) + \frac{N^{1-\alpha} \cdot k_n^{2-\alpha}}{(1-\alpha)\cdot\Gamma(1-\alpha)}. \tag{4.17}$$

Then, we deduce that

$${}_0^C D_{t_n}^\alpha x_n^{hk} - {}_0^C D_{t_n}^\alpha x_n = m_\alpha \sum_{j=0}^{n} c_\alpha ((x_{n-j}^{hk} - x_{n-j}) + (x_{n-j-1}^{hk} - x_{n-j-1})). \tag{4.18}$$

For the estimation of the term $\|x_{n-j} - x_{n-j}^{hk}\|_X$, it can be observed that:

(4.19)
$$\begin{aligned} x_{n-j} - x_{n-j}^{hk} &= \int_0^{t_{n-j}} \delta x(s)\,\mathrm{d}s + x_0 - \sum_{l=1}^{n-j} \delta x_l^{hk} k_l - x_0^h \\ &= (x_0 - x_0^h) + \sum_{l=1}^{n-j} (\delta x_l - \delta x_l^{hk}) k_l + \sum_{l=1}^{n-j} \left(\int_{t_{l-1}}^{t_l} \delta x(s)\,\mathrm{d}s - \delta x_l k_l \right). \end{aligned}$$

Also, we find

$$\begin{aligned} \left\| \sum_{l=1}^{n-j} \left(\int_{t_{l-1}}^{t_l} \delta x(s)\,\mathrm{d}s - \delta x_l k_l \right) \right\|_X &= \left\| \sum_{l=1}^{n-j} \int_{t_{l-1}}^{t_l} (\delta x(s) - \delta x_l)\,\mathrm{d}s \right\|_X \\ &\leqslant \sum_{l=1}^{n-j} \int_{t_{l-1}}^{t_l} \|\delta x(s) - \delta x_l\|_X\,\mathrm{d}s = I_k(\delta x). \end{aligned} \tag{4.20}$$

Then

$$\|x_{n-j} - x_{n-j}^{hk}\|_X \leqslant \sum_{l=1}^{n-j} \|\delta x_l - \delta x_l^{hk}\|_X k_l + \|x_0 - x_0^h\|_X + I_k(\delta x). \tag{4.21}$$

We use the inequality $j \leqslant n \leqslant N$ and $Nk = T$, we deduce that

$$\sum_{j=0}^{n} k\|x_{n-j} - x_{n-j}^{hk}\|_X \leqslant cT(\|x_0 - x_0^{hk}\|_X + I_k(\delta x)) + T \sum_{j=0}^{n} k \sum_{l=1}^{n-j} \|\delta x_l - \delta x_l^{hk}\|_X. \tag{4.22}$$

In a similar way, we have that

(4.23)
$$\sum_{j=0}^{n} k\|x_{n-j-1} - x_{n-j-1}^{hk}\|_X \leqslant cT(\|x_0 - x_0^{hk}\|_X + I_k(\delta x)) + T \sum_{j=0}^{n} k \sum_{l=1}^{n-j-1} \|\delta x_l - \delta x_l^{hk}\|_X.$$

Consider this inequality for the tenant account

$$\|y_n^h - x_n^{hk}\|_X \leqslant \|y_n^h - x_n\|_X + \|x_n - x_n^{hk}\|_X. \tag{4.24}$$

When combined with all previous relationships, this results in:

$$\begin{aligned} \|x_n^h - x_n\|_X^2 \leqslant c \Bigg\{ &\|y_n^h - x_n\|_X^2 + \|x_0 - x_0^h\|_X^2 + \sum_{j=0}^{n} k^2 \sum_{l=1}^{n-j} \|\delta x_l - \delta x_l^{hk}\|_X^2 \\ &+ I_k(\delta x) + |R_n(x_n, y_n^h)|^2 \Bigg\}. \end{aligned} \tag{4.25}$$

Conclusively, through utilization of the generalized discrete variant of the Gronwall inequality from Lemma 6.7, in conjunction with the implications derived from Lemma 6.8, we establish the outcome denoted as equation (4.5). □

5. An application of the fractional Kelvin-Voigt law to a viscoelastic frictional contact problem

In this section, we employ the abstract theoretical findings from the preceding sections to investigate a category of time-fractional hemivariational inequalities, which will subsequently be utilized to analyze a frictional contact problem.

5.1. The physical setting. The physical formulation of the contact problem is as follows: We examine a viscoelastic body exhibiting fractional behavior, situated within an open and connected domain $\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$, where the boundary $\partial\Omega = \Gamma$ is Lipschitz continuous. The boundary Γ is partitioned into three distinct and measurable components: Γ_D, Γ_N, and Γ_C, where meas $(\Gamma_D) > 0$.

We denote as $\mathbb{S}^d$ the space of second-order tensors that are symmetric over $\mathbb{R}^d$. The inner products and corresponding norms on $\mathbb{S}^d$ and $\mathbb{R}^d$ are given by

$$\begin{cases} \boldsymbol{u} \cdot \boldsymbol{v} = u_i v_i, \ \|\boldsymbol{v}\|_{\mathbb{R}^d} = (\boldsymbol{v} \cdot \boldsymbol{v})^{1/2} & \forall\, \boldsymbol{u} = (u_i), \ \boldsymbol{v} = (v_i) \in \mathbb{R}^d, \\ \boldsymbol{\sigma} \colon \boldsymbol{\tau} = \sigma_{ii}\tau_{ij}, \ \|\boldsymbol{\tau}\|_{\mathbb{S}^d} = (\boldsymbol{\tau} \colon \boldsymbol{\tau})^{1/2} & \forall\, \boldsymbol{\sigma} = (\sigma_{ij}), \ \boldsymbol{\tau} = (\tau_{ij}) \in \mathbb{S}^d. \end{cases}$$

The normal and tangential components of a displacement field vector u and the stress tensor σ are defined as

$$u_\nu = \boldsymbol{u} \cdot \nu, \ \boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \nu \quad \text{and} \quad \sigma_\nu = (\boldsymbol{\sigma}\nu) \cdot \nu, \ \boldsymbol{\sigma}_\tau = \boldsymbol{\sigma}\nu - \sigma_\nu \nu.$$

The operator Div signifies the divergence operator for stress, that is, $\operatorname{Div} \boldsymbol{\sigma} = (\sigma_{ij,j})$.

We now introduce the following notation: $\boldsymbol{u} \colon \Omega \times [0, T] \to \mathbb{R}^d$ represents the displacement field, $\boldsymbol{\sigma} = (\sigma_{ij}) \colon \Omega \times [0, T] \to \mathbb{S}^d$ denotes the stress tensor, and $\varepsilon(\boldsymbol{u}) = (\varepsilon_{ij}(\boldsymbol{u}))$ signifies the linearized strain tensor. This tensor is defined as $\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}(u_{i,j} + u_{j,i})$.

We examine the time evolution of the mechanical state of the body over the time interval $[0, T]$ with $T > 0$.

The body is assumed to be clamped on $\Gamma_D \times (0, T)$ and subjected to a volume force $\boldsymbol{f}_0$ in $\Omega \times (0, T)$. Additionally, we consider a density of traction forces denoted by $\boldsymbol{f}_N$, acting on the boundary portion $\Gamma_N \times (0, T)$.

The classical formulation of the mechanical contact problem is presented as follows.

Problem (P). Find a displacement field $\boldsymbol{u} \colon \Omega \times (0, T) \to \mathbb{R}^d$ and a stress tensor $\boldsymbol{\sigma} \colon \Omega \times (0, T) \to \mathbb{S}^d$ for all $\alpha \in]0, 1]$ and a.e. $t \in (0, T)$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{V}\varepsilon({}_0^C D_t^\alpha \boldsymbol{u}(t)) + \mathcal{E}\varepsilon(\boldsymbol{u}(t)) \quad \text{in } \Omega \times (0, T), \tag{5.1}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \quad \text{in } \Omega \times (0, T), \tag{5.2}$$

$$\boldsymbol{u}(t) = 0 \quad \text{on } \Gamma_D \times (0, T), \tag{5.3}$$

$$\boldsymbol{\sigma}(t)\nu = \boldsymbol{f}_N(t) \quad \text{on } \Gamma_N \times (0, T), \tag{5.4}$$

$$-\sigma_\nu(t) \in \partial\pi(u_\nu(t)) \quad \text{on } \Gamma_C \times (0,T), \tag{5.5}$$

$$-\boldsymbol{\sigma}_\tau(t) \in \partial\varpi(\boldsymbol{u}_\tau(t)) \quad \text{on } \Gamma_C \times (0,T), \tag{5.6}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{5.7}$$

We remind that (5.1) represents the Kelvin-Voigt constitutive law for a viscoelastic body of the Caputo type, as detailed in [48], where $\mathcal{V}$ and $\mathcal{E}$ denote the viscosity and elasticity operators. Equation (5.2) corresponds to the equilibrium equation. Furthermore, conditions (5.3) and (5.4) specify the displacement and traction boundary conditions on Γ_D and Γ_N, respectively. The contact condition (5.5) is referred to as a multivalued normal compliance condition [17], [41], where $\pi\colon\ \Omega \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ is a convex function. This leads to the relations

$$\pi(r) = I_{(-\infty,0]}(r) = \begin{cases} 0, & r \leqslant 0, \\ \infty, & r > 0. \end{cases} \tag{5.8}$$

The equation (5.5) is streamlined to align with Signorini's well-established contact condition as described in the referenced works [40], [41],

$$u_\nu(t) \leqslant 0,\ \sigma_\nu(\boldsymbol{u}(t)) \leqslant 0,\ \sigma_\nu(\boldsymbol{u}(t))u_\nu(t) = 0 \quad \text{on } \Gamma_C \times (0,T).$$

The multivalued condition (5.6) characterizes friction, where $\varpi\colon\ \Omega \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ serves as a convex potential [32]. By encompassing several friction laws found in the literature, this formulation encompasses a range of possibilities. One of the simplest choices is a finite-valued function

$$\varpi(\boldsymbol{x}, \boldsymbol{\xi}) = \mu(\boldsymbol{x})\|\boldsymbol{\xi}\| \quad \forall\, \boldsymbol{\xi} \in \mathbb{R}^d,\ \text{a.e. } \boldsymbol{x} \in \Gamma_C, \tag{5.9}$$

where μ is the friction bound. Adopting this approach leads us to Coulomb's renowned law that governs dry friction within this particular configuration. Lastly, the initial condition is expressed through equation (5.7).

To formulate the variational approach for the contact problem (5.1)–(5.7), we introduce the subsequent function spaces

$$H = L^2(\Omega)^d, \quad H_1 = H^1(\Omega)^d, \quad \mathcal{H} = \{\boldsymbol{\sigma} \in \mathbb{S}^d,\ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega)\},$$
$$V = \{\boldsymbol{v} \in H_1,\ \boldsymbol{v} = 0 \quad \text{on } \Gamma_D\}.$$

These spaces are Hilbert spaces equipped with inner products

$$(\boldsymbol{u}, \boldsymbol{v})_H = \int_\Omega u_i v_i \,\mathrm{d}x, \quad (\boldsymbol{u}, \boldsymbol{v})_{H_1} = (\boldsymbol{u}, \boldsymbol{v})_H + (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v})),$$
$$(\boldsymbol{\sigma}, \boldsymbol{\tau})_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} \,\mathrm{d}x, \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))_{\mathcal{H}},$$

respectively, along with their associated norms $\|\cdot\|_H$, $\|\cdot\|_{H_1}$, $\|\cdot\|_{\mathcal{H}}$ and $\|\cdot\|_V$. Because $\operatorname{meas}(\Gamma_D) > 0$, Korn's inequality

$$\|\varepsilon(\boldsymbol{v})\|_{\mathcal{H}} \geqslant c_k \|\boldsymbol{v}\|_{H_1} \quad \forall\, \boldsymbol{v} \in V \tag{5.10}$$

is valid. Furthermore, employing the Sobolev trace theorem, there exists a positive constant c_s that depends solely on Ω, Γ_C, and Γ_D, such that

$$\|\boldsymbol{v}\|_{L^2(\Gamma_C)^d} \leqslant c_s \|\boldsymbol{v}\|_V \quad \forall\, \boldsymbol{v} \in V. \tag{5.11}$$

In the analysis of Problem (P), it is essential to account for the ensuing assumptions:

(HP1) The viscosity operator adheres to the subsequent hypotheses:
- (i) $v_{ijkl} = v_{jikl} = v_{lkij} \in L^\infty(\Omega)$ for all i, j, k, l;
- (ii) $\mathcal{V}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\sigma} = \boldsymbol{\xi} \cdot \mathcal{V}(\boldsymbol{x}, \boldsymbol{\sigma})$ for all $\boldsymbol{\xi}, \boldsymbol{\sigma} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
- (iii) $v_{ijkl}(\boldsymbol{x})\xi_{ij}\xi_{kl} \geqslant m_{\mathcal{V}} \|\boldsymbol{\xi}\|^2$ with $m_{\mathcal{V}} > 0$ for all $\boldsymbol{\xi} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
- (iv) $\|(\mathcal{V}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))\|_{\mathcal{H}} \leqslant M_{\mathcal{V}} \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V$.

(HP2) The elasticity operator fulfills the subsequent conditions:
- (i) $e_{ijkl} = e_{jikl} = e_{lkij} \in L^\infty(\Omega)$ for all i, j, k, l;
- (ii) $\mathcal{E}(\boldsymbol{x}, \boldsymbol{\xi}) \cdot \boldsymbol{\sigma} = \boldsymbol{\xi} \cdot \mathcal{E}(\boldsymbol{x}, \boldsymbol{\sigma})$ for all $\boldsymbol{\xi}, \boldsymbol{\sigma} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
- (iii) $e_{ijkl}(x)\xi_{ij}\xi_{kl} \geqslant m_{\mathcal{E}} \|\boldsymbol{\xi}\|^2$ with $m_{\mathcal{E}} > 0$ for all $\boldsymbol{\xi} \in \mathbb{S}^d$, a.e. $\boldsymbol{x} \in \Omega$;
- (iv) $\|(\mathcal{E}\varepsilon(\boldsymbol{u}), \varepsilon(\boldsymbol{v}))\|_{\mathcal{H}} \leqslant M_{\mathcal{E}} \|\boldsymbol{u}\|_V \|\boldsymbol{v}\|_V$.

(HP3) The density of volume force, surface traction, and initial condition meet the subsequent regularity hypotheses:

$$\boldsymbol{f}_0 \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{f}_N \in C(0, T; L^2(\Omega; \mathbb{R}^d)), \quad \boldsymbol{u}_0 \in V.$$

(HP4) The contact function π satisfies
- (i) $\pi\colon\ \Gamma_C \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$;
- (ii) $\pi(\cdot, \boldsymbol{x})$ is measurable on Γ_C for all $r \in \mathbb{R}$;
- (iii) $\pi(\boldsymbol{x}, \cdot)$ is a proper, convex and lower semi-continuous for a.e. $\boldsymbol{x} \in \Gamma_C$;
- (iv) there exists $m_\pi > 0$ such that

$$|\partial\pi(\boldsymbol{x}, r)| \leqslant c_\pi(1 + |r|) \quad \forall\, r \in \mathbb{R};$$

- (v) there exists a constant $L_\pi \geqslant 0$ such that

$$|\pi(\boldsymbol{x}, r_1) - \pi(\boldsymbol{x}, r_2)| \leqslant L_\pi |r_1 - r_2| \quad \forall\, r_1, r_2 \in \mathbb{R}.$$

(HP5) The function ϖ of friction satisfies
- (i) $\varpi\colon\ \Gamma_C \times \mathbb{R} \to \mathbb{R} \cup \{\infty\}$;
- (ii) $\varpi(\cdot, \boldsymbol{x})$ is measurable on Γ_C for all $r \in \mathbb{R}$;

(iii) $\varpi(\boldsymbol{x},\cdot)$ is a proper, convex and lower semi-continuous for a.e. $\boldsymbol{x}\in\Gamma_C$;
(iv) there exists $m_\varpi>0$ such that

$$|\partial\varpi(\boldsymbol{x},r)|\leqslant c_\varpi(1+|r|)\quad\forall\, r\in\mathbb{R};$$

(v) there exists a constant $L_\varpi\geqslant 0$ such that

$$|\varpi(\boldsymbol{x},r_1)-\varpi(\boldsymbol{x},r_2)|\leqslant L_\varpi|r_1-r_2|\quad\forall\, r_1,r_2\in\mathbb{R}.$$

We introduce the operators

$$\begin{aligned} &a\colon V\times V\to\mathbb{R},\ a(\boldsymbol{u},\boldsymbol{v}):=(\mathcal{V}\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}},\\ &b\colon V\times V\to\mathbb{R},\ b(\boldsymbol{u},\boldsymbol{v}):=(\mathcal{E}\varepsilon(\boldsymbol{u}),\varepsilon(\boldsymbol{v}))_{\mathcal{H}}. \end{aligned}\tag{5.12}$$

Applying Riesz's representation theorem, there exists an element $\boldsymbol{f}\in V$ such that

$$(\boldsymbol{f}(t),\boldsymbol{v})_V:=\int_\Omega \boldsymbol{f}_0(t)\cdot\boldsymbol{v}\,\mathrm{d}x+\int_\Gamma \boldsymbol{f}_N(t)\cdot\boldsymbol{v}\,\mathrm{d}\Gamma\quad\forall\,\boldsymbol{v}\in V.\tag{5.13}$$

Furthermore, we establish the mapping $j\colon V\to\mathbb{R}\cup\{\infty\}$ as follows

$$j(\boldsymbol{u}(t),\boldsymbol{v}):=\int_{\Gamma_C}\pi(u_\nu(t))v_\nu\,\mathrm{d}\Gamma+\int_{\Gamma_C}\varpi(u_\nu(t))\|\boldsymbol{v}_\tau\|\,\mathrm{d}\Gamma\quad\forall\,\boldsymbol{v}\in V.\tag{5.14}$$

We turn to the variational formulation of the problem (5.1)–(5.7). First, we assume that $(\boldsymbol{u},\boldsymbol{\sigma})$ are smooth functions defined on $\Omega\times(0,T)$ which satisfy (5.1)–(5.7).

Let $\boldsymbol{v}\in V$. By multiplying the equilibrium equation (5.1) by $(\boldsymbol{v}-\boldsymbol{u}(t))$ and applying Green's formula (refer to [32], Theorem 2.25), we obtain

$$(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v})-\varepsilon(\boldsymbol{u}(t)))_{\mathcal{H}}=(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_H+\int_{\Gamma_C}\sigma(t)\nu\cdot(\boldsymbol{v}-\boldsymbol{u}(t))\,\mathrm{d}\Gamma\tag{5.15}$$

for a.e. $t\in(0,T)$. Employing the boundary conditions (5.3)–(5.4) combined with the decomposition formula, cf. (6.33) in [32] and (5.13),

$$(\boldsymbol{\sigma}(t),\varepsilon(\boldsymbol{v})-\varepsilon(\boldsymbol{u}(t)))_{\mathcal{H}}=(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_H+\int_{\Gamma_C}(\sigma_\nu(t)(v_\nu-u_\nu(t))+\boldsymbol{\sigma}_\tau(t)(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))).\tag{5.16}$$

On the other hand, based on (5.5) and (5.6), and utilizing the definition of the subdifferential, we deduce that

$$\begin{aligned} &-\sigma_\nu(t)(v_\nu-u_\nu(t))\leqslant\pi(u_\nu(t),v_\nu)-\pi(u_\nu(t),u_\nu(t))\quad\text{on }\Gamma_C\times(0,T),\\ &-\boldsymbol{\sigma}_\tau(t)(\boldsymbol{v}_\tau-\boldsymbol{u}_\tau(t))\leqslant\varpi(\boldsymbol{u}_\tau(t),\boldsymbol{v}_\tau)-\varpi(\boldsymbol{u}_\tau(t),\boldsymbol{u}_\tau(t))\quad\text{on }\Gamma_C\times(0,T). \end{aligned}\tag{5.17}$$

By combining (5.16)–(5.17) with (5.12) and (5.14), we derive the inequality

$$\begin{aligned} a({}_0^C D_t^\alpha\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+b(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+j(\boldsymbol{u}(t),\boldsymbol{v})-j(\boldsymbol{u}(t),\boldsymbol{u}(t))\\ \geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V. \end{aligned}\tag{5.18}$$

Finally, we arrive at the following variational formulation of Problem (P).

Problem (PV). Find a displacement field $\boldsymbol{u}\colon\ \Omega\times(0,T)\to\mathbb{R}^d$ for all $\alpha\in\,]0,1]$, a.e. $t\in(0,T)$ and for all $\boldsymbol{v}\in V$ such that

$$a({}_0^C D_t^\alpha \boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+b(\boldsymbol{u}(t),\boldsymbol{v}-\boldsymbol{u}(t))+j(\boldsymbol{u}(t),\boldsymbol{v}) -j(\boldsymbol{u}(t),\boldsymbol{u}(t))\geqslant(\boldsymbol{f}(t),\boldsymbol{v}-\boldsymbol{u}(t))_V, \tag{5.19}$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0. \tag{5.20}$$

We have the following existence result.

Theorem 5.1. *Assuming that assumptions* (HP1)–(HP5) *hold, we conclude that Problem* (PV) *has at least one solution* $\boldsymbol{u}\in W^{1,2}(0,T;V)$.

P r o o f. It is easy to see that under hypotheses (HP1) and (HP2), the operators a and b satisfy (H1) and (H2) with $m_A=m_{\mathcal{V}}$, $m_B=m_{\mathcal{E}}$, $M_A=M_{\mathcal{V}}$, and $M_B=M_{\mathcal{E}}$.

Using (HP4)–(HP5), we can infer that j is proper, convex, and lower semicontinuous. Therefore, (H3) is satisfied with $m_J=\sup(m_\pi,m_\varpi)$.

Next, due to the regularity hypotheses (HP3), we establish that (H4) holds.

Having confirmed the hypotheses of Theorem 5.1, we conclude that Problem (PV) has a solution $\boldsymbol{u}\in W^{1,2}(0,T;V)$. This completes the proof. □

5.2. Error estimates for the approximate solutions. In this section, we state some results on error estimates for the numerical approximation of Problem (PV).

We consider a finite element triangulation $\mathcal{T}^h=\{\mathcal{T}r\}r$ of $\overline{\Omega}$ that is compatible with the boundary partitions, where h represents the spatial discretization parameter. Let $\mathbb{P}1(\mathcal{T}r)$ be the space of polynomials of global degree less than or equal to 1 in $\mathcal{T}_r$. We define the following finite-dimensional space $V^h\subset V$ as

$$V^h=\{\boldsymbol{v}^h\in[C(\overline{\Omega})]^d;\ \boldsymbol{v}^h_{|\mathcal{T}_r}\in[\mathbb{P}_1(\mathcal{T}_r)]^d;\ \boldsymbol{v}^h=0\text{ on }\Gamma_D\}.$$

Next, we consider a spatially semidiscrete approximation of Problem (PV).

Problem (PEH). Find a displacement field $\boldsymbol{u}^h\colon\ (0,T)\to V^h$ for a.e. $t\in(0,T)$, all $\boldsymbol{v}^h\in V^h$ and $\alpha\in\,]0,1]$ such that

$$a({}_0^C D_t^\alpha \boldsymbol{u}^h(t),\boldsymbol{v}^h-\boldsymbol{u}^h(t))+b(\boldsymbol{u}^h(t),\boldsymbol{v}^h-\boldsymbol{u}^h(t)) +j(\boldsymbol{u}^h(t),\boldsymbol{v}^h)-j(\boldsymbol{u}^h(t),\boldsymbol{u}^h(t))\geqslant(\boldsymbol{f}(t),\boldsymbol{v}^h-\boldsymbol{u}^h(t))_V, \tag{5.21}$$

$$\boldsymbol{u}^h(0)=\boldsymbol{u}_0^h, \tag{5.22}$$

where $\boldsymbol{u}_0^h\in V^h$ is a suitable approximation of $\boldsymbol{u}_0$.

Using the same arguments as presented in the previous section, we can deduce that Problem (PEH). has at least one solution $\boldsymbol{u}^h \in V^h$, and we obtain the next result.

Theorem 5.2. *Keep the assumptions of Theorem* 5.1. *Let* $\boldsymbol{u}$ *and* $\boldsymbol{u}^h$ *denote the solutions to Problem* (PV) *and Problem* (PEH), *respectively. If*

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \to 0 \text{ as } h \to 0, \tag{5.23}$$

the semidiscrete solution converges as

$$\|\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\|_{L^2(0,T;V)} \to 0 \text{ as } h \to 0. \tag{5.24}$$

P r o o f. From (3.3), we have the estimate

$$\begin{aligned} &\|\boldsymbol{u}(t) - \boldsymbol{u}^h(t)\|_{L^2(0,T;V)} \\ &\quad \leqslant c \inf_{\boldsymbol{v}^h \in V^h} \{\|\boldsymbol{v}^h - \boldsymbol{u}(t)\|_{L^2(0,T;V)} + \|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V + \|R(\boldsymbol{u}(t), \boldsymbol{v}^h)\|^{1/2}_{L^2(0,T;V)}\}, \end{aligned} \tag{5.25}$$

where

$$\begin{aligned} R(\boldsymbol{u}(t), \boldsymbol{v}^h) &= a({}_0^C D_t^\alpha \boldsymbol{u}(t), \boldsymbol{v}^h - \boldsymbol{u}(t)) + b(\boldsymbol{u}(t), \boldsymbol{v}^h - \boldsymbol{u}(t)) \\ &\quad + j(\boldsymbol{u}(t), \boldsymbol{v}^h) - j(\boldsymbol{u}(t), \boldsymbol{u}(t)) - (\boldsymbol{f}(t), \boldsymbol{v}^h - \boldsymbol{u}(t))_V \\ &= (\boldsymbol{\sigma}(t), \varepsilon(\boldsymbol{v}^h) - \varepsilon(\boldsymbol{u}(t))) + j(\boldsymbol{u}(t), \boldsymbol{v}^h) - j(\boldsymbol{u}(t), \boldsymbol{u}(t)) \\ &\quad - (\boldsymbol{f}(t), \boldsymbol{v}^h - \boldsymbol{u}(t))_V. \end{aligned} \tag{5.26}$$

Subsequently, the expression $R(\boldsymbol{u}(t), \boldsymbol{v}^h)$ is further simplified to

$$\begin{aligned} |R(\boldsymbol{u}(t), \boldsymbol{v}^h)| \leqslant &\int_{\Gamma_C} \boldsymbol{\sigma}_\tau(t) \cdot (\boldsymbol{v}_\tau^h - \boldsymbol{u}_\tau^h(t))\,\mathrm{d}\Gamma + \int_{\Gamma_C} \pi(u_\nu(t))(v_\nu^h - u_\nu(t))\,\mathrm{d}\Gamma \\ &+ \int_{\Gamma_C} \varpi(u_\nu(t))(\|\boldsymbol{v}_\tau\| - \|\boldsymbol{u}_\tau(t)\|)\,\mathrm{d}\Gamma. \end{aligned} \tag{5.27}$$

Considering the regularity of $\boldsymbol{u}$ along with the properties of π and ϖ, and taking into account (5.11), we obtain

$$\|R(\boldsymbol{u}(t), \boldsymbol{v}^h)\| \leqslant c\|\boldsymbol{v}^h - \boldsymbol{u}(t)\|_V. \tag{5.28}$$

Furthermore, based on (HP4)–(HP5) and utilizing the definition (5.14) of j, there exists a positive constant c that depends on c_s, L_π, and M_ϖ, ensuring that

$$|j(\boldsymbol{u}_1, \boldsymbol{v}_2) - j(\boldsymbol{u}_1, \boldsymbol{v}_1) + j(\boldsymbol{u}_2, \boldsymbol{v}_1) - j(\boldsymbol{u}_2, \boldsymbol{v}_2)| \leqslant c\|\boldsymbol{u}_1 - \boldsymbol{u}_2\|_V \|\boldsymbol{v}_1 - \boldsymbol{v}_2\|_V \tag{5.29}$$

for all $\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{v}_1, \boldsymbol{v}_2$ in V.

Next, we employ $\Pi^h \boldsymbol{u}$ as the standard finite element interpolation operator for $\boldsymbol{u}$. Subsequently, we derive the interpolation error estimate as shown in [15]:

$$\|\boldsymbol{u} - \Pi^h \boldsymbol{u}\|_V \leqslant ch\|\boldsymbol{u}\|_{L^2(0,T;V)}, \quad \|\boldsymbol{u}_0 - \Pi^h \boldsymbol{u}_0\|_V \leqslant ch\|\boldsymbol{u}_0\|_{L^2(0,T;V)}. \tag{5.30}$$

By combining equations (5.25), (5.28)–(5.30), along with the condition (5.23), we arrive at the order estimate, thereby concluding the proof of Theorem 5.2. □

For the fully discrete approximation, we employ a uniform partition $t_0 = 0 < t_1 < \ldots < t_N = T$ of the interval $[0, T]$. Additionally, we utilize notations introduced in the previous section. The numerical scheme under consideration is as follows.

Problem (PEHK). Find a displacement field $\{\boldsymbol{u}_n^{hk}\}_{n=0}^N \in V^h$ such that for a.e. $t \in (0, T)$, $\boldsymbol{v}^h \in V^h$ and $\alpha \in]0, 1]$,

$$\begin{aligned} a({}_0^C D_{t_n}^\alpha \boldsymbol{u}_n^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}) + b(\boldsymbol{u}_n^{hk}, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk}) \\ + j(\boldsymbol{u}_n^{hk}, \boldsymbol{v}^h) - j(\boldsymbol{u}_n^{hk}, \boldsymbol{u}_n^{hk}) \geqslant (\boldsymbol{f}_n, \boldsymbol{v}^h - \boldsymbol{u}_n^{hk})_V, \end{aligned} \tag{5.31}$$

$$\boldsymbol{u}_0^{hk} = \boldsymbol{u}_0^h. \tag{5.32}$$

Using the same argument as presented in the previous section, it can be shown that Problem (PEHK) has at least one solution $\boldsymbol{u}_n^{hk} \in V^h$.

Our interest lies in estimating the numerical error $\|\boldsymbol{u} - \boldsymbol{u}_n^{hk}\|_V$. We present the main error estimate result.

Theorem 5.3. *Let $\boldsymbol{u}$ and $\boldsymbol{u}_n^{hk}$ be solutions to Problems* (PV) *and* (PEHK)*, respectively, assuming the presumptions stated in Theorem* 5.2*. Let*

$$\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V \to 0 \text{ as } h \to 0. \tag{5.33}$$

The fully discrete solution converges,

$$\max_{1 \leqslant n \leqslant N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \to 0 \text{ as } h, k \to 0. \tag{5.34}$$

P r o o f. By utilizing (4.5), we derive the estimate

$$\begin{aligned} &\max_{1 \leqslant n \leqslant N} \|\boldsymbol{u}_n - \boldsymbol{u}_n^{hk}\|_V \\ &\quad \leqslant \max_{1 \leqslant n \leqslant N} \inf_{\boldsymbol{v}_n^h \in V^h} c\{\|\boldsymbol{v}_n^h - \boldsymbol{u}_n\|_V + |R_n(\boldsymbol{u}_n, \boldsymbol{v}_n^h)|^{1/2}\} + c\|\boldsymbol{u}_0 - \boldsymbol{u}_0^h\|_V, \end{aligned} \tag{5.35}$$

where

$$\begin{aligned} R_n(\boldsymbol{u}_n, \boldsymbol{v}_n^h) = a({}_0^C D_{t_n}^\alpha \boldsymbol{u}_n, \boldsymbol{v}^h - \boldsymbol{u}_n) + b(\boldsymbol{u}_n, \boldsymbol{v}^h - \boldsymbol{u}_n) \\ + j(\boldsymbol{u}_n, \boldsymbol{v}_n^h) - j(\boldsymbol{u}_n, \boldsymbol{v}_n^h) - (\boldsymbol{f}_n, \boldsymbol{v}_n^h - \boldsymbol{u}_n)_V. \end{aligned} \tag{5.36}$$

In a manner analogous to (5.28), we observe that

$$\|R_n(\boldsymbol{u}_n, \boldsymbol{v}_n^h)\| \leqslant c\|\boldsymbol{v}_n^h - \boldsymbol{u}_n\|_V. \tag{5.37}$$

Subject to the conditions (5.32), (5.30), and (5.37), we obtain the error estimate of order (5.34). □

5.3. Example to contact problem with $\alpha = 1$. In the following, we present a twelfth specific example of a contact problem with normal compliance and Coulomb's friction.

Problem (PP). Find a displacement field $\boldsymbol{u}\colon \Omega \times (0,T) \to \mathbb{R}^d$ and a stress tensor $\boldsymbol{\sigma}\colon \Omega \times (0,T) \to \mathbb{S}^d$ for a.e. $t \in (0,T)$ such that

$$\boldsymbol{\sigma}(t) = \mathcal{V}\varepsilon(\dot{\boldsymbol{u}}(t)) + \mathcal{E}\varepsilon(\boldsymbol{u}(t)) \quad \text{in } \Omega \times (0,T), \tag{5.38}$$

$$\operatorname{Div} \boldsymbol{\sigma}(t) + \boldsymbol{f}_0(t) = 0 \quad \text{in } \Omega \times (0,T), \tag{5.39}$$

$$\boldsymbol{u}(t) = 0 \quad \text{on } \Gamma_D \times (0,T), \tag{5.40}$$

$$\sigma(t)\nu = \boldsymbol{f}_N(t) \quad \text{on } \Gamma_N \times (0,T), \tag{5.41}$$

$$-\sigma_\nu(t) = p_\nu(u_\nu(t)) \quad \text{on } \Gamma_C \times (0,T), \tag{5.42}$$

$$\left.\begin{aligned} &\|\boldsymbol{\sigma}_\tau(t)\| \leqslant p_\tau(u_\nu(t)), \\ &\|\boldsymbol{\sigma}_\tau(t)\| \leqslant p_\tau(u_\nu(t)) \Rightarrow \boldsymbol{u}_\tau(t) = 0, \\ &\|\boldsymbol{\sigma}_\tau(t)\| = p_\tau(u_\nu(t)) \Rightarrow \boldsymbol{\sigma}_\tau(t) = -\lambda \boldsymbol{u}_\tau(t), \text{ with } \lambda \neq 0 \end{aligned}\right\} \quad \text{on } \Gamma_C \times (0,T), \tag{5.43}$$

$$\boldsymbol{u}(0,x) = \boldsymbol{u}_0 \quad \text{in } \Omega. \tag{5.44}$$

Evidently, the variational formulation of Problem (PP) is as follows:

Problem (PPV). Find a displacement field $\boldsymbol{u}\colon \Omega \times (0,T) \to \mathbb{R}^d$ for all $\boldsymbol{v} \in V$ and a.e. $t \in (0,T)$ such that

$$\begin{aligned} a(\dot{\boldsymbol{u}}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + b(\boldsymbol{u}(t), \boldsymbol{v} - \boldsymbol{u}(t)) + \ell(\boldsymbol{u}(t), \boldsymbol{v}) - \ell(\boldsymbol{u}(t), \boldsymbol{u}(t)) \\ + (\boldsymbol{f}(t), \boldsymbol{v} - \boldsymbol{u}(t))_V, \end{aligned} \tag{5.52}$$

$$\boldsymbol{u}(0) = \boldsymbol{u}_0, \tag{5.53}$$

where

$$\ell(\boldsymbol{u}(t), \boldsymbol{v}) = \int_{\Gamma_C} p_\nu(u_\nu(t)) v_\nu \,\mathrm{d}\Gamma + \int_{\Gamma_C} p_\tau(u_\nu(t)) |\boldsymbol{v}_\tau| \,\mathrm{d}\Gamma \quad \forall\, \boldsymbol{u}, \boldsymbol{v} \in V. \tag{5.54}$$

As a consequence of Theorem 5.1, and if p_ν and p_τ satisfy the hypotheses (HP4) and (HP5), we conclude the following result.

Corollary 5.4. *Assume that hypotheses* (HP1)–(HP5) *are satisfied. Then Problem* (PP) *has a solution* $\boldsymbol{u} \in W^{1,2}(0,T;V)$.

6. Appendix

Now, we revisit the established definition in fractional calculus theory and nonlinear analysis, as seen in references [12], [20], [32], [37], [39], [31].

Definition 6.1 (Riemann-Liouville fractional integral)**.** Let X be a Banach space and $(0,T)$ be a finite time interval. The Riemann-Liouville fractional integral of order $\alpha > 0$ for a given function $f \in L^1(0,T;X)$ is defined by

$$ {}_0I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s)\,\mathrm{d}s \quad \forall\, t \in (0,T), $$

where $\Gamma(\cdot)$ stands for the gamma function defined by

$$ \Gamma(\alpha) = \int_0^\infty t^{\alpha-1}\mathrm{e}^{-t}\,\mathrm{d}t. $$

To further elaborate on the definition, we introduce $0It^0 = I$, where I represents the identity operator. This implies that $0It^0 f(t) = f(t)$ for almost every $t \in (0,T)$.

Definition 6.2 (Caputo derivative of order $0 < \alpha \leqslant 1$)**.** Let X be a Banach space, $0 < \alpha \leqslant 1$ and $(0,T)$ be a finite time interval. For a given function $f \in AC(0,T;W)$, the Caputo fractional derivative of f is defined by

$$ {}_0^C D_t^\alpha f(t) = {}_0\, I_t^{1-\alpha} f'(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f'(s)\,\mathrm{d}s \quad \forall\, t \in (0,T). $$

The notation $AC(0,T;X)$ refers to the space of all absolutely continuous functions from $(0,T)$ into X.

It is obvious that if $\alpha = 1$, the Caputo derivative reduces to the classical first-order derivative, that is, we have

$$ {}_0^C D_t^1 f(t) = If'(t) = f'(t) \quad \text{for a.e. } t \in (0,T). $$

Proposition 6.3. *Let X be a Banach space and $\alpha, \beta > 0$. Then, the following statements hold:*

(a) *for $y \in L^1(0,T;X)$, we have ${}_0I_t^\alpha\, {}_0I_t^\beta y(t) = {}_0I_t^{\alpha+\beta} y(t)$ for a.e. $t \in (0,T)$,*

(b) *for $y \in AC(0,T;X)$ and $\alpha \in (0,\alpha]$, we have*

$$ {}_0I_t^\alpha\, {}_0^C D_t^\alpha y(t) = y(t) - y(0) \quad \text{for a.e. } t \in (0,T), $$

(c) *for $y \in L^1(0,T;X)$, we have ${}_0^C D_t^\alpha\, {}_0I_t^\alpha y(t) = y(t)$ for a.e. $t \in (0,T)$.*

Proposition 6.4. *Let X be a reflexive Banach space and let that $T\colon X \to 2^{X^*}$ satisfy the conditions*

(a) *for every $v \in X$, Tv is a nonempty, closed and convex subset of X^*,*
(b) *the operator T is bounded,*
(c) *if $v_n \to v$ weakly in X and $v_n^* \to v^*$ weakly in X^* with $v_n^* \in Tv_n$, and if*

$$\limsup_{n\to\infty} \langle v_n^*, v_n - v\rangle \leqslant 0,$$

then $v^* \in Tv$ and $\langle v_n^*, v_n\rangle \to \langle v^*, v\rangle$.

Then the operator A is pseudomonotone.

Theorem 6.5. *Let X be a reflexive Banach space and $T\colon X \to 2^{X^*}$ be pseudomonotone and coercive. Then T is surjective, i.e., for every $f \in X^*$, there is $u \in X$ such that $Tu \ni f$.*

Definition 6.6. Given a locally Lipschitz function $J\colon E \to \mathbb{R}$ on a Banach space E, we denote by $J^0(u, v)$ the generalized (Clarke) directional derivative of J at the point $u \in E$ in the direction $v \in E$ defined by

$$J^0(u, v) = \lim_{\lambda\to 0^+} \sup_{w\to u} \frac{J(w + \lambda v) - J(w)}{\lambda}.$$

The generalized gradient of $J\colon E \to \mathbb{R}$ at $u \in E$ is defined by

$$\partial J(u) = \{\xi \in E^*;\ J^0(u, v) \geqslant \langle \xi, v\rangle\ \forall\, v \in E\}.$$

Lemma 6.7. *Let $\{u_n\}$, $\{v_n\}$, and $\{w_n\}$ be nonnegative sequences satisfying*

$$u_n \leqslant v_n + \sum_{k=1}^{n-1} w_k u_k \quad \text{for } n \geqslant 1.$$

Then, we have

$$u_n \leqslant v_n + \sum_{k=1}^{n-1} v_k w_k \exp\Bigg(\sum_{j=k+1}^{n-1} w_j\Bigg) \quad \text{for } n \geqslant 1.$$

Moreover, if $\{u_n\}$ and $\{w_n\}$ are such that

$$u_n \leqslant \alpha + \sum_{k=1}^{n-1} w_k u_k \quad \text{for } n \geqslant 1,$$

where $\alpha > 0$ is a constant, then for all $n \geqslant 1$, it holds

$$u_n \leqslant \alpha \exp\Bigg(\sum_{k=1}^{n-1} w_k\Bigg).$$

Lemma 6.8. *The quantities $I_k(\delta x)$ defined by*

$$I_k(\delta x) = \sum_{j=1}^{n} \int_{t_{j-1}}^{t_j} \|\delta x_j - \delta x(s)\|_V \, \mathrm{d}s$$

converge to zero as $k \to 0$.

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