Applications of Mathematics

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Applications of Mathematics, Vol. 69 (2024), No. 4, 481–499

Persistent URL: http://dml.cz/dmlcz/152530

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A MODIFIED FLETCHER-REEVES CONJUGATE GRADIENT METHOD FOR UNCONSTRAINED OPTIMIZATION WITH APPLICATIONS IN IMAGE RESTORATION

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Received January 11, 2024. Published online June 7, 2024.

Abstract. The Fletcher-Reeves (FR) method is widely recognized for its drawbacks, such as generating unfavorable directions and taking small steps, which can lead to subsequent poor directions and steps. To address this issue, we propose a modification to the FR method, and then we develop it into the three-term conjugate gradient method in this paper. The suggested methods, named "HZF" and "THZF", preserve the descent property of the FR method while mitigating the drawbacks. The algorithms incorporate strong Wolfe line search conditions to ensure effective convergence. Through numerical comparisons with other conjugate gradient algorithms, our modified approach demonstrates superior performance. The results highlight the improved efficacy of the HZF algorithm compared to the FR and three-term FR conjugate gradient methods. The new algorithm was applied to the problem of image restoration and proved to be highly effective in image restoration compared to other algorithms.

Keywords: unconstrained optimization; decreasing feature; global convergence; image restoration; conjugate gradient technique

MSC 2020: 35Qxx, 49Q15, 49M41, 47N10

1. Introduction

The conjugate gradient method (CGM) is widely used to tackle unconstrained optimization problems. This approach aims to find the minimum value of a function without any constraints on the variables. It is particularly useful for solving large-scale problems, where computing the Hessian matrix can be prohibitive [7], [22], [38], [24], [37]. We take into account the following unconstrained optimization problem in

DOI: 10.21136/AM.2024.0009-24

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this work:

$$\min f(x), \quad x \in \mathbb{R}^n, \tag{1.1}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function and $g(x)$ represents the gradient at x, i.e., $g(x) := \nabla f(x)$.

The nonlinear conjugate gradient (CG) methods have the form [16], [23], [29], [25], [30]

$$x_{k+1} = x_k + \alpha_k d_k, \tag{1.2}$$

where x_1 is an initial point, $\alpha_k > 0$ is a step-length and the directions d_k are generated as $d_1 = -g_1$ and

$$d_{k+1} = \begin{cases} -g_{k+1} & \text{if } k = 0, \\ -g_{k+1} + \beta_k d_k & \text{if } k \geqslant 1. \end{cases} \tag{1.3}$$

In (1.3), β_k is known as the CG parameter. The line search in (CG) algorithms is oftentimes based on the standard Wolfe line search:

$$f(x_k + \alpha_k d_k) - f(x_k) \leqslant \varrho \alpha_k g_k^\top d_k, \quad g_{k+1}^\top d_k \geqslant \sigma g_k^\top d_k \tag{1.4}$$

or by the strong Wolfe line search conditions [13], [3], [33], given by

$$f(x_k + \alpha_k d_k) - f(x_k) \leqslant \varrho \alpha_k g_k^\top d_k, \qquad |g_{k+1}^\top d_k| \leqslant -\sigma g_k^T d_k, \tag{1.5}$$

where $0 < \varrho \leqslant \sigma < 1$. The choice of conjugate parameter β_k plays a critical role in the convergence and numerical performance of the CGMs, which is a well-known fact in the field of numerical optimization [39], [19], [27], [1], [26]. Fletcher and Reeves (FR) introduced the first nonlinear conjugate gradient method in the 1960s [12]. This method has been recognized as one of the earliest techniques for addressing large-scale nonlinear optimization problems. Throughout the years, numerous variants of the original scheme have been proposed in [15], [9], [36], [32], [31], [21], [20], [2], some of which have gained widespread usage. For instance, the method proposed by Dai and Yuan (DY) [10], as well as Conjugate Descent (CD) suggested by Fletcher [41], Hestenes-Stiefel (HS) [18], Polak-Ribere (PR) [34], and Liu-Storey (LS) [28] are among the notable examples as

$$\beta_k^{\mathrm{DY}} = \frac{\|g_{k+1}\|^2}{d_k^\top y_k}, \quad \beta_k^{\mathrm{FR}} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2}, \quad \beta_k^{\mathrm{CD}} = \frac{\|g_{k+1}\|^2}{-d_k^\top g_k}, \tag{1.6}$$

$$\beta_k^{\mathrm{HS}} = \frac{y_k^\top g_{k+1}}{d_k^\top y_k}, \quad \beta_k^{\mathrm{PR}} = \frac{y_k^\top g_{k+1}}{\|g_k\|^2}, \quad \beta_k^{\mathrm{LS}} = \frac{y_k^\top g_{k+1}}{-d_k^\top g_k}, \tag{1.7}$$

where $\|\cdot\|$ represents the Euclidean norm and $y_k = g_{k+1} - g_k$. The methods presented in equation (1.6) exhibit strong convergence properties, but their practical performance is limited due to potential interferences. Conversely, the methods outlined in equation (1.7) may not always guarantee convergence, but they often offer improved computational efficiency [16]. These algorithms exhibit significant features: they do not necessitate matrix storage and demonstrate improved speed compared to the steepest descent method. When dealing with strictly convex quadratic functions and utilizing exact line search, all nonlinear conjugate gradient algorithms are equivalent. However, the behavior of these algorithms can vary considerably when applied to general functions [35].

The crucial requirement for function minimization is that the search direction d_k at iteration k must be a descent direction, meaning that

$$g_k^\top d_k < 0, \quad k = 0, 1, 2, \ldots \tag{1.8}$$

This criterion is widely recognized as essential for the effectiveness of an algorithm [6]. In certain cases, it might be necessary for the search direction d_k to occasionally satisfy the sufficient descent criterion to ensure global convergence, as expressed by

$$g_k^\top d_k \leqslant -c\|g_k\|^2, \quad k = 0, 1, 2, \ldots, \tag{1.9}$$

where c denotes a positive constant.

In general, the convergence theory for methods that employ $\|g_{k+1}\|^2$ in the numerator of β_k is more well-established compared to the theory for methods that utilize $g_{k+1}^\top y_k$, in the numerator. However, in practice, the methods with $g_{k+1}^\top y_k$ in the numerator of β_k tend to outperform the methods with $\|g_{k+1}\|^2$ in the numerator [17].

There are trinomial conjugated gradient modalities proposed by Zhang [42]. A trinomial conjugate gradient modality was proposed for FR, PR and HS parameters. These three approaches consistently accomplish the descent property. The three-term FR method proposed by Zhang et al. is defined as it is dented by (TFR):

$$d_{k+1}^{\mathrm{TFR}} = -g_{k+1} + \beta^{\mathrm{FR}} d_k - \theta_k^{(1)} g_{k+1}, \tag{1.10}$$

where $\theta_k^{(1)} = d_k^\top g_{k+1}/(g_k^\top g_k)$. We see that (1.10) can be written as spectral conjugate gradient form, that is,

$$d_{k+1}^{\mathrm{TFR}} = -(1 + \theta^{(1)}) g_{k+1} + \beta^{\mathrm{FR}} d_k. \tag{1.11}$$

Zhang et al. show in [36] that (TFR) satisfies the sufficient descent condition $g_{k+1}^\top d_{k+1}^{\mathrm{TTFR}} = -\|g_{k+1}\|^2$, which is independent of line search used. Under suitable conditions, this method is globally convergent.

2. The Fletcher-Reeves method and its modification

As mentioned in the previous paragraph, the first CG method which was given by Fletcher and Reeves, has some weaknesses in some problems (for more details see [13]). For example, $\beta_k^{\mathrm{FR}} \approx 1$ leads to $d_{k+1} \approx d_k$, so the new search direction will improve little (if at all) on the previous. In order to solve this problem, we should return to Gilbert and Nocedal in [14] as he studied and gave the convergence scope $\mu \in [-1, 1]$ for the form CGMs with

$$\beta_k = \mu_k \beta_k^{\mathrm{FR}} \leqslant \beta_k^{\mathrm{FR}}. \tag{2.1}$$

Based on (2.1), we suggest the following modification to the Fletcher-Reeves method, i.e.

$$\beta_k^{\mathrm{HZF}} = \begin{cases} \dfrac{\|g_{k+1}\|^2}{\|g_k\|^2} - \mu \dfrac{|d_k^\top g_{k+1}|}{\|g_k\|^2} & \text{if } 0 < \mu \dfrac{|d_k^\top g_{k+1}|}{\|g_k\|^2} < 1, \\ \dfrac{\|g_{k+1}\|^2}{\|g_k\|^2} & \text{otherwise,} \end{cases} \tag{2.2}$$

where $\mu \in (0, 1/2)$, and set $\vartheta = \mu |d_k^\top g_{k+1}| / \|g_k\|^2$. Therefore, the search direction can be defined as

$$d_{k+1} = -g_{k+1} + \beta_k^{\mathrm{HZF}} d_k. \tag{2.3}$$

For simplicity, we call the method defined by (2.2) and (2.3) as HZF method. Note that for exact line search $g_{k+1}^T d_k = 0$ we see that (2.2) reduces to the standard Fletcher-Reeves method. On the other hand we can generalize HZF to the three-term conjugate gradient (say THZF) or spectral conjugate gradient method as in (1.11), that is

$$d_{k+1} = -g_{k+1} + \beta^{\mathrm{HZF}} d_k + \vartheta g_{k+1} \tag{2.4}$$

or

$$d_{k+1} = -(1 + \vartheta) g_{k+1} + \beta^{\mathrm{HZF}} d_k. \tag{2.5}$$

In the following, we present a high-level description of our proposed algorithms for minimizing unconstrained optimization problems as follows.

Algorithm (A).

Step 1: Pick a starting point $x_1 \in \mathbb{R}^n$, $\varepsilon \geqslant 0$ sufficiently small. Compute

$$f_1 = f(x_1),\ g_1 = g(x_1).$$ Set $d_1 = -g_1$ and $\alpha_1 = 1/\|g_1\|$. Set $k = 0$.

Step 2: A stop-iteration criterion is a test: for example, if $\|g_{k+1}\| \leqslant \varepsilon$, then stop, otherwise continue with Step 3.

Step 3: Compute the step size α_k satisfying the strong Wole line search conditions.

Step 4: Compute $x_{k+1} = x_k + \alpha_k d_k$. Compute f_{k+1} and g_{k+1}. Set $s_k = x_{k+1} - x_k$ $y_k = g_{k+1} - g_k$.

Step 5: Compute d_{k+1} from equations (2.2) and (2.3) or (2.4) or (2.5).

Step 6: If the Powell restart criterion $|g_{k+1}^\top g_k| \geqslant 0.2\|g_{k+1}\|^2$ is satisfied, then set $d_{k+1} = -g_{k+1}$, otherwise set d_{k+1} as in Step 5, and compute the initial guess $\alpha_k = \alpha_{k-1}\|d_{k-1}\|/\|d_k\|$.

Step 7: Set $k = k + 1$ and go to Step 2.

Al-Baali in [5] demonstrated that the FR method generates sufficient descent direction for each iteration according to the following theorem.

Theorem 2.1. *If the step size α_k is determined using the strong Wolfe line search (1.4) and (1.6), with the constraint that $\sigma \in (0, 1/2)$, then the following inequalities hold for the FR method*

$$-\sum_{j=0}^{k} \sigma^j \leqslant \frac{g_k^\top d_k}{\|g_k\|^2} \leqslant -2 + \sum_{j=0}^{k} \sigma^j \tag{2.6}$$

for each k. Whenever $g_k \neq 0$ in every k, all characteristics of d_k are satisfied.

According to the above theorem our first algorithm (HZF) generates descent directions, since $0 < |g_{k+1}^\top d_k|/\|g_k\|^2 < 1$ by strong Wolfe condition, Theorem 2.1 and $0 \leqslant |g_{k+1}^\top d_k|/\|g_k\|^2 < 1$, therefore we have

$$g_{k+1}^\top d_{k+1} = -\|g_{k+1}\|^2 + \beta_k^{\mathrm{HZF}} d_k^\top g_{k+1} \leqslant -\|g_{k+1}\|^2 + \beta_k^{FR} d_k^\top g_{k+1} < 0. \tag{2.7}$$

For the THZF method we can show that the descent condition holds as follows:

$$\begin{aligned} g_{k+1}^\top d_{k+1} &= -\Big(1 + \frac{d_k^\top g_{k+1}}{\|g_k\|^2}\Big)\|g_{k+1}\|^2 + \Big(\frac{\|g_{k+1}\|^2}{\|g_k\|^2} - \mu\frac{|d_k^\top g_{k+1}|}{\|g_k\|^2}\Big) d_k^\top g_{k+1} \\ &= -\Big(1 + \frac{d_k^\top g_{k+1}}{\|g_k\|^2}\Big)\|g_{k+1}\|^2 + \Big(\frac{\|g_{k+1}\|^2}{\|g_k\|^2} d_k^\top g_{k+1} - \mu\frac{(d_k^\top g_{k+1})^2}{\|g_k\|^2}\Big) \\ &\leqslant -\|g_{k+1}\|^2 - \mu\frac{(d_k^\top g_{k+1})^2}{\|g_k\|^2} < 0. \end{aligned} \tag{2.8}$$

3. Convergence analysis

In the global convergence analysis of many iterative methods, it is commonly required to make the following assumption:

Assumption (CG). (i) The level set $\Omega = \{x \in \mathbb{R}^n \colon f(x) \leqslant f(x_1)\}$ is bounded.

(ii) Within a certain neighborhood N of Ω, f is continuously differentiable and its gradient satisfies Lipschitz continuity. Specifically, there exists a positive constant L

such that

$$\|g(x) - g(y)\| \leqslant L\|x - y\| \quad \forall\, x, y \in \mathrm{N}. \tag{3.1}$$

Gilbert and Nocedal in [14] show that any method with $|\beta_k| \leqslant \beta^{\mathrm{FR}}$ is descent and globally convergent according to the following theorem.

Theorem 3.1. *Assuming that assumption* (CG) *is satisfied, let us consider any conjugate gradient method* (1.2) *and* (1.3), *where the condition holds that is* $|\beta_k| \leqslant \beta^{FR}$. *Additionally, let the step size be determined through the strong Wolfe line search* (1.4) *and* (1.6) *with* $0 < \varrho < \sigma < 1/2$. *Under these conditions, we can conclude that*

$$g_{k+1}^{\top} d_{k+1} < 0 \quad \textit{and} \quad \liminf_{k\to\infty} \|g_k\| = 0. \tag{3.2}$$

P r o o f. For the proof see [7]. □

Since $-\mu_k |d_k^{\top} g_{k+1}|/\|g_k\|^2 \leqslant 0$ for any $\mu \in [0.1)$, therefore

$$\beta_k^{\mathrm{HZF}} = \frac{\|g_{k+1}\|^2}{\|g_k\|^2} - \mu \frac{|d_k^{\top} g_{k+1}|}{\|g_k\|^2} \leqslant \beta^{\mathrm{FR}}. \tag{3.3}$$

Hence, according to Theorem 3.1 our HZF algorithm (2.1), (2.2) and (2.3) is globally convergent.

For a general method for unconstrained optimization, the following result originally given by Zoutendijk [43] is well known and useful in analyzing the property of convergence.

Lemma 3.1 ([16])**.** *Suppose that assumption CG holds. Consider any iteration of the form* (1.2), *where* d_k *is a descent direction and* α_k *satisfies the Wolfe line search conditions. Then*

$$\sum_{k=0}^{\infty} \frac{(g_k^{\top} d_k)^2}{\|d_k\|^2} < \infty. \tag{3.4}$$

For simplicity, we denote $\omega = 1 + \theta_k^{(1)}$ (it can be seen that there exists $\varphi_1, \varphi_2 \in \mathbb{R}$ such that $\varphi_1 \leqslant \omega \leqslant \varphi_2$). Therefore, (2.5) can be written as

$$d_{k+1} = -\omega g_{k+1} + \beta_k^{\mathrm{HZF}} d_k. \tag{3.5}$$

Based on the descent condition and Lemma 3.1, we can establish the global convergence of the THZF with the strong Wolfe line search (1.5).

Theorem 3.2. *Suppose that the assumption CG holds, and let the sequence* $\{x_k\}$ *be generated by the THZF method with strong line search* (1.5). *Then*

$$\liminf_{k\to\infty} \|g_k\| = 0. \tag{3.6}$$

P r o o f. We use the contradiction to prove the theorem and write $\beta_k^{\mathrm{HZF}} = \beta_k$. Let the conclusion be not true. Then there exists $\delta > 0$ such that $\|g_k\| \geqslant \delta$ for all k. We can write (3.5) as

$$\begin{aligned} d_{k+1} + \omega g_{k+1} &= \beta_k d_k, \\ \|d_{k+1}\|^2 &= \beta^2 \|d_k\|^2 - 2\omega_k g_{k+1}^\top d_{k+1} - \omega_k^2 \|g_{k+1}\|^2 \\ &= \frac{\|g_{k+1}\|^2}{\|g_k\|^2} \|d_k\|^2 - 2\omega_k g_{k+1}^\top d_{k+1} - \omega_k^2 \|g_{k+1}\|^2 \\ &\leqslant \|d_k\|^2 - 2\omega_k g_{k+1}^\top d_{k+1} - \omega_k^2 \|g_{k+1}\|^2. \end{aligned} \tag{3.7}$$

Dividing (3.7) of the above inequality by $(g_{k+1}^\top d_{k+1})^2$, we get

$$\begin{aligned} \frac{\|d_{k+1}\|^2}{(g_{k+1}^\top d_{k+1})^2} &\leqslant \frac{\|d_k\|^2}{(g_k^\top d_k)^2} - \frac{2\omega_k}{g_{k+1}^\top d_{k+1}} - \frac{\omega_k^2 \|g_{k+1}\|^2}{(g_{k+1}^\top d_{k+1})^2} \\ &= \frac{\|d_k\|^2}{(g_k^\top d_k)^2} + \frac{1}{\|g_{k+1}\|^2} - \Big(\frac{1}{\|g_{k+1}\|} + \frac{\omega_k}{(g_{k+1}^\top d_{k+1})} \Big)^2 \\ &\leqslant \frac{\|d_k\|^2}{(g_k^\top d_k)^2} + \frac{1}{\|g_{k+1}\|^2}. \end{aligned}$$

In terms of $\|d_1\|^2/(g_1^\top d_1)^2 = 1/\|g_1\|^2$, together with the above relation and $\|g_{k+1}\|^2 \geqslant \delta$ we have

$$\begin{aligned} \frac{\|d_{k+1}\|^2}{(g_{k+1}^\top d_{k+1})^2} &\leqslant \frac{\|d_k\|^2}{(g_k^\top d_k)^2} + \frac{1}{\|g_{k+1}\|^2} \leqslant \frac{\|d_{k-1}\|^2}{(g_{k-1}^\top d_{k-1})^2} + \frac{1}{\|g_k\|^2} + \frac{1}{\|g_{k-1}\|^2} \\ &\leqslant \ldots \leqslant \sum_{j=1}^{k} \frac{1}{\|g_j\|^2} \leqslant \frac{k}{\delta}. \end{aligned}$$

That is, $(g_{k+1}^\top d_{k+1})^2/\|d_{k+1}\|^2 \geqslant \delta/k$, therefore $(g_{k+1}^\top d_{k+1})^2/\|d_{k+1}\|^2 = \infty$, which contradicts Lemma 3.1. Therefore, the proof is complete. □

4. Numerical examples

In this section, we present the numerical results obtained using an implementation of the HZF and THZF algorithms. The code used for the implementation is sourced from Andrei's Homepage (`https://camo.ici.ro/neculai/nandrei.htm`) and written in Fortran, compiled with the default settings of the f77 compiler. The

computations were conducted on an Intel Pentium 4 Workstation with a clock speed of 1.8GHz. We selected a set of 40 large-scale unconstrained optimization test functions in generalized or extended form [11], including some from the CUTE library. For each test function, we conducted 10 numerical experiments with varying numbers of variables, ranging from $n = 100$ to $n = 1000$. The algorithm incorporates the strong Wolfe line search conditions with $\varrho = 0.0001$, $\sigma = 0.8$, and $\mu = 0.6$, while the stopping criterion is set as $\|g_k\|_2 \leqslant 10^{-6}$, where $\|g_k\|_2$ represents the Euclidean norm of vectors. The performance of the algorithms is compared within the subsequent context. Suppose that f_i^{ALG1} and f_i^{ALG2} represent the optimal values obtained by ALG1 and ALG2, respectively, for problem $i = 1, \ldots, 400$. We consider that ALG1 outperforms ALG2 in a specific problem i if the following condition is satisfied:

$$|f_i^{\mathrm{ALG1}} - f_i^{\mathrm{ALG2}}| < 10^{-3}.$$

1.	Freudenstein & Roth FREUROTH (CUTE)	21.	Generalized PSC1
2.	Extended Trigonometric	22.	Extended PSC1
3.	Extended Rosenbrock SROSENBR	23.	Extended Powell
4.	(CUTE).Extended White & Holst	24.	Extended Block-Diagonal BD1
5.	Extended Beale BEALE (CUTE)	25.	Extended Maratos
6.	Extended Penalty	26.	Extended Cliff CLIFF (CUTE)
7.	Perturbed Quadratic	27.	Quadratic Diagonal Perturbed
8.	Raydan 1	28.	Extended Wood WOODS (CUTE)
9.	Raydan 2	29.	Extended Hiebert
10.	Diagonal 1	30.	Quadratic QF1
11.	Diagonal 2	31.	Extended Quadratic Penalty QP1
12.	Diagonal 3	32.	Extended Quadratic Penalty QP2
13.	Hager	33.	Quadratic QF2
14.	Generalized Tridiagonal 1	34.	Extended EP1
15.	Extended Tridiagonal 1	35.	Extended Tridiagonal 2
16.	Extended Three Expo Terms	36.	BDQRTIC (CUTE)
17.	Generalized Tridiagonal 2	37.	TRIDIA (CUTE)
18.	Diagonal 4	38.	ARWHEAD (CUTE)
19.	Diagonal 5	39.	NONDIA (CUTE)
20.	Extended Himmelblau HIMMELBC (CUTE)	40.	NONDQUAR (CUTE)

Table 1. List of the name of the test problems.

We determine that ALG1 performs better than ALG2 in a particular problem i if any of the following conditions are met: the number of iterations (#iter), the number of function-gradient evaluations (#fg), or the CPU time required by ALG1 is less than the corresponding values for ALG2. We conducted a comparison between HZF, THZF, FR and TFR methods. Table 1 gives the description of these test problems. The numerical results concerning the number of iterations and the number of function-gradient evaluations can be found in Table 2.

	FR Method			HKZ Method		TFR Method		THZF Method	
N.Pr	n	Ite	Nfg	Ite	Nfg	Ite	Nfg	Ite	Nfg
	100	133	3136	777	20323	29	58	X	x
1	1000	367	8286	x	x	200	4856	431	11475
	100	234	292	56	78	326	359	70	94
2	1000	342	440	69	100	315	344	61	88
	100	116	204	46	83	86	129	66	120
3	1000	91	151	53	98	111	175	55	101
	100	35	78	48	90	67	128	54	102
4	1000	67	125	61	115	100	177	63	127
	100	41	78	14	29	29	53	18	35
5	1000	24	45	38	73	38	70	23	42
	100	17	30	12	23	22	35	13	25
6	1000	16	33	20	76	17	35	12	27
	100	202	237	93	138	151	190	103	147
7	1000	x	x	287	419	420	472	385	569
	100	150	187	84	117	98	134	79	112
8	1000	799	902	394	614	348	389	437	726
	100	4	9	4	9	4	9	4	9
9	1000	4	9	4	9	4	9	4	9
	100	795	19625	142	1722	586	14782	274	5787
10	1000	x	x	x	x	x	x	X	x
	100	130	161	75	116	68	92	73	115
11	1000	335	403	266	408	211	266	238	363
	100	426	8210	176	3047	539	12789	97	681
12	1000	x	x	x	x	x	x	X	x

Table 2. Numerical results of the methods.

	FR Method			HKZ Method		TFR Method		THZF Method	
N.Pr	n	Ite	Nfg	Ite	Nfg	Ite	Nfg	Ite	Nfg
	100	62	241	56	985	58	515	28	51
13	1000	x	x	362	10330	x	x	385	10636
	100	65	94	25	49	80	693	23	47
14	1000	87	559	47	788	63	1060	45	670
	100	11	22	13	27	10	21	16	31
15	1000	86	155	24	46	6	14	20	39
	100	13	23	26	43	18	31	17	31
16	1000	23	122	40	828	53	1200	10	18
	100	203	334	38	67	115	152	56	85
17	1000	234	274	52	83	93	133	57	90
	100	6	12	7	14	4	8	7	14
18	1000	6	12	9	18	4	8	9	18
	100	4	9	4	9	4	9	4	9
19	1000	4	9	4	9	4	9	4	9
	100	16	32	16	33	9	18	16	30
20	1000	17	32	16	33	8	17	19	36
	100	494	883	360	611	515	786	424	689
21	1000	757	3171	641	1870	x	x	668	1467
	100	11	23	12	24	12	25	17	29
22	1000	17	115	92	2543	12	25	26	45
	100	126	226	119	209	209	366	137	252
23	1000	157	280	141	252	239	414	199	365
	100	33	59	26	48	16	32	46	85
24	1000	42	79	32	61	24	47	35	65
	100	602	796	101	166	x	x	216	265
25	1000	634	861	152	319	x	x	133	163
	100	24	51	70	138	26	50	45	84
26	1000	26	377	13	26	197	4890	49	492
	100	114	213	119	232	101	187	74	139
27	1000	499	916	352	669	353	667	410	795
	100	170	269	164	268	195	343	177	303
28	1000	96	180	171	282	115	202	176	285

Table 2. Continuation of Table 2.

		FR Method		HKZ Method		TFR Method		THZF Method	
N.Pr	n	Ite	Nfg	Ite	Nfg	Ite	Nfg	Ite	Nfg
	100	101	193	118	231	41	87	91	188
29	1000	118	220	102	189	84	162	71	144
	100	216	253	104	147	150	191	88	129
30	1000	949	1019	371	531	442	493	410	614
	100	838	1090	15	26	234	686	15	33
31	1000	x	x	x	x	985	32432	121	3224
	100	129	207	53	88	112	173	37	62
32	1000	x	x	76	140	x	x	78	117
	100	367	409	111	154	207	250	109	157
33	1000	x	x	385	544	697	750	475	697
	100	2	5	2	5	2	5	2	5
34	1000	2	5	2	5	2	5	1002	1003
	100	61	85	35	54	53	76	38	55
35	1000	78	343	36	54	75	942	36	56
	100	943	21845	704	15295	x	x	281	3503
36	1000	x	x	x	x	x	x	x	x
	100	x	x	405	574	393	436	430	630
37	1000	x	x	x	x	x	x	x	x
	100	21	40	18	31	20	55	19	74
38	1000	44	188	10	20	19	76	22	200
	100	31	61	56	105	34	65	56	107
39	1000	50	97	51	98	55	110	54	105
	100	x	x	x	x	x	x	x	x
40	1000	x	x	x	x	x	x	x	x

Table 2. Continuation of Table 2.

Figures 1 and 2 depict the Dolan and More's performance profile, showcasing the comparison of performance of HZF, THZF with FR and TFR algorithms based on iterations (Ite) and function gradient evaluations (Nfg). In the performance profile plot, the top curve represents the method that successfully solved the highest number of problems within a factor close to the best time. The left axis of the plot displays the percentage of test problems for which a method was the fastest. On the right-hand side, the plot indicates the percentage of test problems successfully solved by each

algorithm, serving as a measure of their robustness. Based on Figures 1 and 2 it is evident that HZF and THZFR outperform the FR and TFR algorithms, respectively.

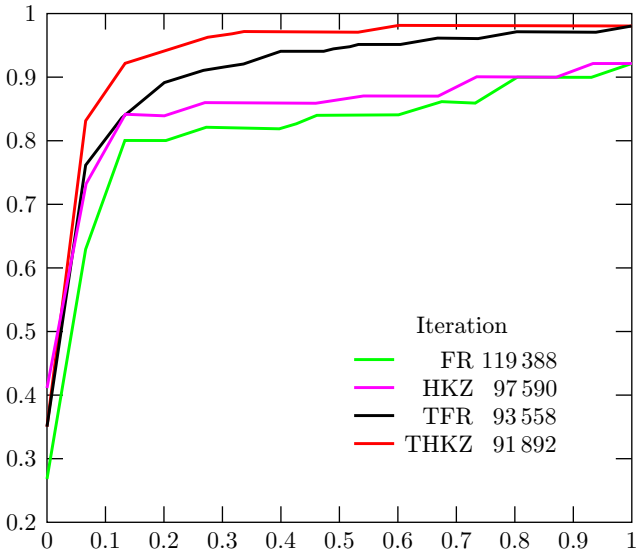


Figure 1. Performance profiles on iteration

4.1. Image restoration problems. Here, we apply the presented FH (Fast Heuristic) approach to address image restoration problems. A two-phase scheme was utilized by Toumi, Raymond [40] to restore images corrupted by impulse noise. During the first phase, a median filter was employed to identify noise pixels. Let X denote an image of size M by N, and $A = \{1, 2, \ldots, M\} \times \{1, 2, \ldots, N\}$ represent the index set of image $X \cdot N \subset A$ represents the set of indices corresponding to the detected noise pixels in the first phase, and $|N|$ indicates the number of elements in N. The set ${}_{V}i,j$ consists of the four closest neighbors of the pixel located at $(i, j) \in A$, i.e., $V_{i,j} = \{(i, j-1), (i, j+1), (i-1, j), (i+1, j)\}$, and $y_{i,j}$ represents the observed pixel value of the image at (i, j). In the second phase, the nonsmooth minimization problem is solved to remove the noise from the detected pixels:

$$\min \sum_{(i \cdot j) \in \mathbb{N}} \Big[|u_{i,j} - y_{i,j}| + \frac{\beta}{2}(2s^1_{i,j} + s^2_{i,j}) \Big], \tag{4.1}$$

where

$$s^1_{i,j} = \sum_{(m,n) \in v_{i,j}/N} \vartheta_\alpha(u_{i,j} - y_{m,n}), \quad s^2_{i,j} = \sum_{(m,n) \in v_{i,j}/N} \vartheta_\alpha(u_{i,j} - u_{m,n}).$$

In this context, the function $\phi\alpha(t) = \sqrt{t_{2+\alpha}}$ is employed as an edge-preserving function with $\alpha > 0$ as its parameter. Additionally, the column vector $u = [u_{i,j}]$, $(i, j) \in \mathbb{N}$ represents a lexicographically ordered vector of length $|N|$. However, solving the nonsmooth minimization problem (4.1) precisely is both time-consuming and costly. To overcome this challenge, Cai et al. [8] eliminated the nonsmooth term and derived the subsequent smooth unconstrained optimization problem instead:

$$\text{(4.2)}\quad \min \mathrm{F}_\alpha(u) := \sum_{i\cdot j \varepsilon N} \left(2 \sum_{i\cdot j \varepsilon v_{i\cdot j}/N} \vartheta_\alpha(u_{i\cdot j} - y_{m\cdot n}) + \sum_{m\cdot n \varepsilon v_{i\cdot j}/N} \vartheta_\alpha(u_{i\cdot j} - y_{m\cdot n}) \right).$$

Obviously, the scale of the problem increases with the noise ratio. It is evident that as the noise ratio increases, the magnitude of (4.1) also grows. Researchers in [8], [4] have identified that the CGM (Conjugate Gradient Method) can effectively restore contaminated images by addressing the aforementioned problem (4.1), even in cases where the noise ratio is high. The numerical results obtained from our study are summarized in Table 3. Additionally, the performance comparisons are visually represented in Figures 3, 4, and 5.

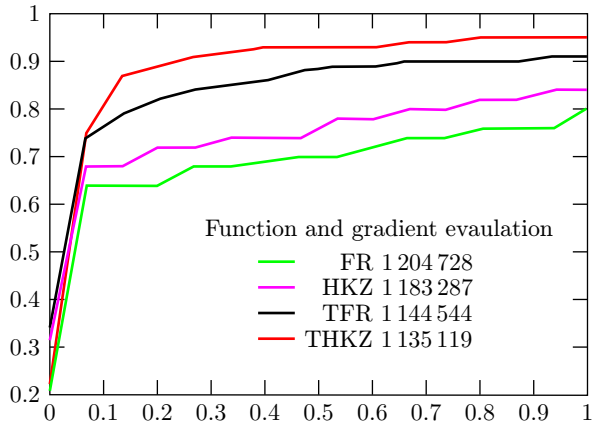


Figure 2. Performance profiles on iteration

5. Conclusions

To summarize, the paper introduces a modified version of the Fletcher-Revees conjugate gradient algorithm to address large-scale unconstrained problems. The adjustment involves incorporating a modified parameter, denoted as β_k, which alters the Fletcher and Reeves computational scheme. The modification ensures that at each iteration $\beta_k \neq 1$, thereby making $d_{k+1} \neq d_k$. By applying strong Wolfe line search conditions, the descent condition and global convergence can be established as consequences of Theorem 3.1.

Image of Baboon				
Noies	TTZK1	TZK1	TTFR	FR
bmp/0.30	64/3.27/23.28	96/7.07/23.19	74/3.25/23.27	96/7.50/23.03
bmp/0.50	68/4.89/21.85	48/5.46/21.77	119/7.30/21.72	69/8.59/20.60
bmp/0.70	69/6.56/20.42	62/9.11/20.36	163/12.06/20.27	5/2.80/13.38
bmp/0.90	71/8.22/18.74	68/12.45/18.66	174/15.93/18.60	58/12.59/17.30
Image of Barbara				
Noies	TTZK1	TZK1	TTFR	FR
png/0.30	141/22.91/28.37	165/50.95/28.08	154/24.52/28.29	168/50.26/27.48
png/0.50	102/30.03/26.36	51/27.46/26.26	152/40.33/25.86	142/75.89/25.24
png/0.70	91/37.64/24.39	100/73.35/24.27	233/78.46/23.89	105/78.58/22.61
png/0.90	84/48.27/22.24	92/82.90/22.00	201/91.52/21.89	160/144.09/20.85
Image of Lena				
Noies	TTZK1	TZK1	TTFR	FR
png/0.30	226/40.59/37.01	138/46.39/34.08	89/21.43/35.28	245/85.96/34.22
png/0.50	134/41.29/34.47	80/49.18/34.18	166/45.86/32.01	99/62.93/26.24
png/0.70	83/38.84/31.18	93/66.56/30.68	250/82.86/29.22	5/14.67/14.19
png/0.90	87/49.44/26.36	152/133.55/26.11	144/69.46/25.18	130/118.58/23.43

Table 3. Numerical results of the image restoration problems.

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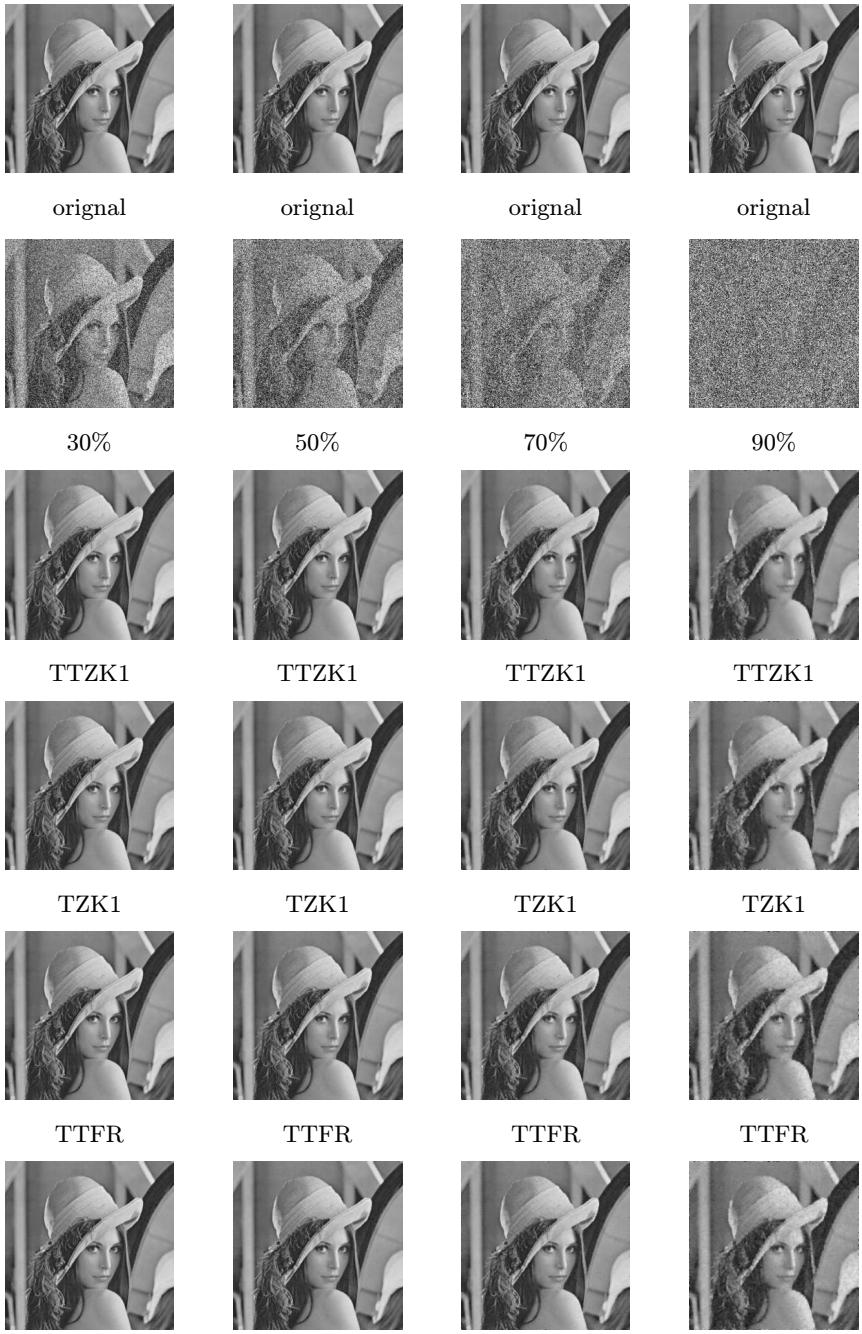


Figure 3. Lena image. (In the second row are desplayed original images with percentage of salt-and-pepper noise.)

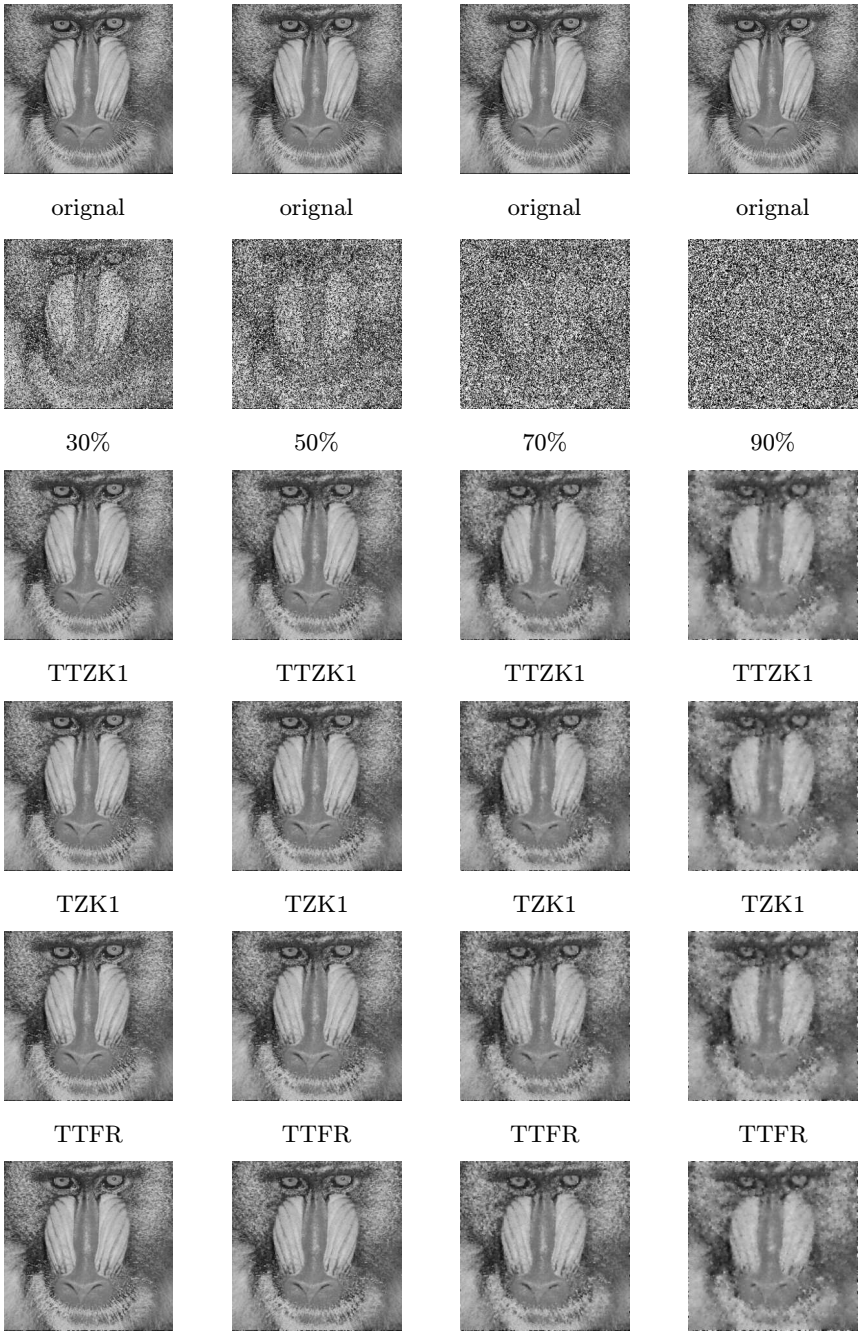


Figure 4. Monkey image. (In the second row are desplayed original images with percentage of salt-and-pepper noise.)

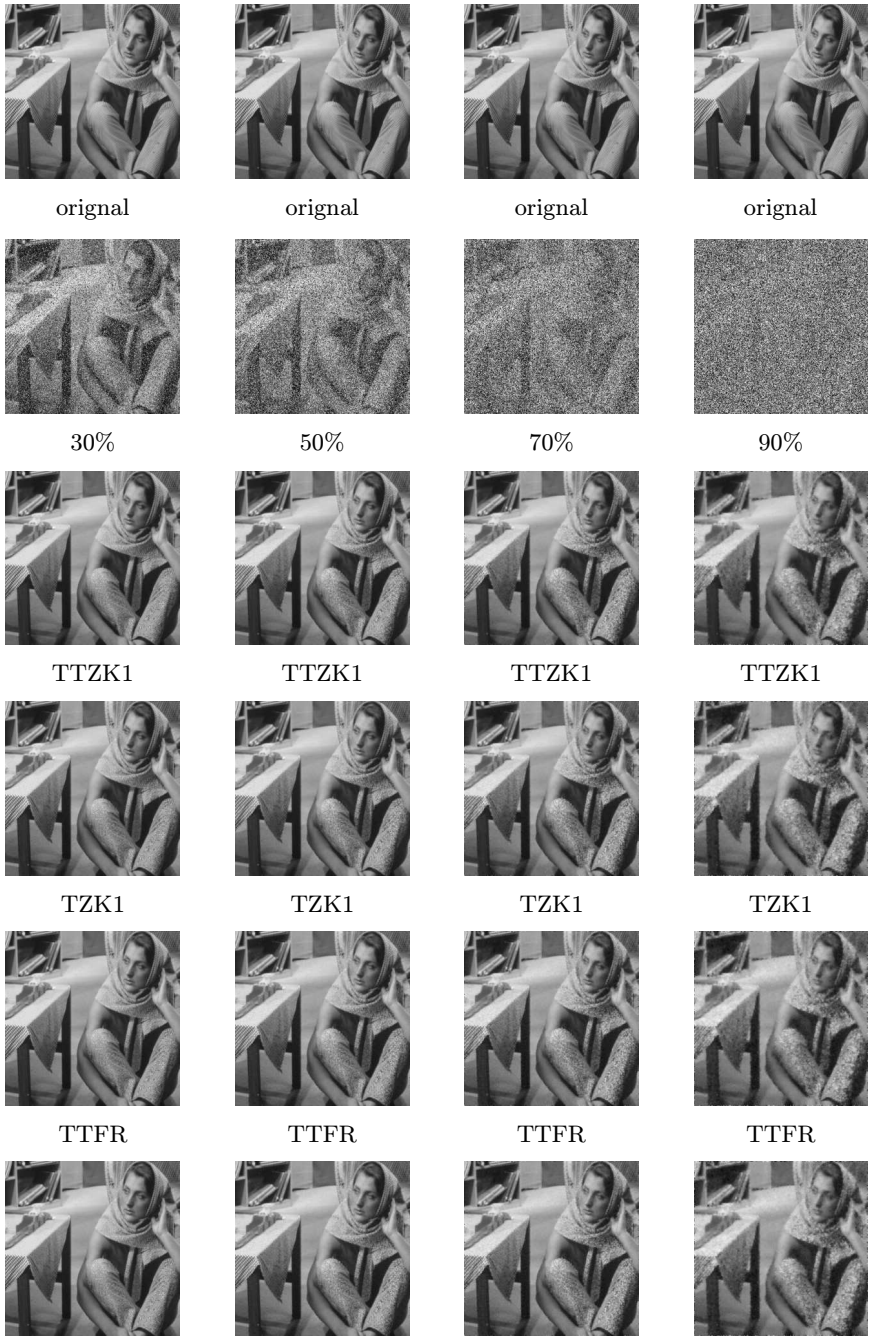


Figure 5. Woman image. (In the second row are desplayed original images with percentage of salt-and-pepper noise.)

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