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Sungjin Ra; Choljin Jang; Jinmyong Hong

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SEMICLASSICAL LIMIT OF A SIMPLIFIED QUANTUM ENERGY-TRANSPORT MODEL FOR BIPOLAR SEMICONDUCTORS

Sungjin Ra, Choljin Jang, Jinmyong Hong, Pyongyang

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Abstract. We are concerned with a simplified quantum energy-transport model for bipolar semiconductors, which consists of nonlinear parabolic fourth-order equations for the electron and hole density; degenerate elliptic heat equations for the electron and hole temperature; and Poisson equation for the electric potential. For the periodic boundary value problem in the torus $\mathbb{T}^d$, the global existence of weak solutions is proved, based on a time-discretization, an entropy-type estimate, and a fixed-point argument. Furthermore, the semiclassical limit is obtained by using a priori estimates independent of the scaled Planck constant.

Keywords: quantum energy-transport model; time-discretization; periodic boundary value problem; bipolar semiconductor

MSC 2020: 35K20, 82D37

1. Introduction

Semiconductor devices are scaled down into the nanoscale regime to achieve high circuit performance in the future integrated system. The performance of nanoscale semiconductor devices primarily relies on carrier transport properties in the short channels. Therefore, mathematical modelling for semiconductors has attracted tremendous attention. In much of the previous research, the device temperature is assumed to be constant. However, for the design of power devices it is often desirable to simulate interaction of electrothermal phenomena. In many applications, changes in the temperature and its distribution in the interior of the device can significantly influence the electrical behaviour of the device. Of particular interest is the effect of a thermal runaway, a mechanism in which the electrical energy dissipated causes a rise in temperature over an area of the device, resulting in increased power dissipation, which leads to irrecoverable device failure (burn-out).

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Quantum energy-transport (QET) models have been developed to understand such physical phenomena in scaled semiconductor devices. Grubin, Kreskovsky [7] seem to be the first who have proposed a quantum energy-balance system. A full QET model has been derived from collisional Wigner-Boltzmann equations using the entropy minimization principle (see [4]). They are alternatives to the dissipative Schrödinger equations whose numerical solution is generally very time-consuming. Later, a nonlocal quantum energy-transport model has been derived by Degond-Gallego-Méhats [3]. Another quantum energy-transport model has been studied by Chen, Liu [2]. Their model consists of a quantum drift-diffusion-type equation for the particle density, coupled to an energy equation.

A physically more precise model, called a simplified QET model (see (1.2)), has been formally derived by Jüngel, Milišić [13] in the large-time and small-velocity limit from the quantum hydrodynamic (QHD) model, and they have proven the existence of global-in-time weak solutions. Also, the semiclassical limit for the simplified QET model is obtained by Chen, Chen, Jüngel [1]. Motivated by [13], in this paper, we will analyze the following simplified version of a bipolar QET model (see Section 2 for the derivation of model equations):

$$
\begin{cases}
(\varrho_i)_t + \operatorname{div}\Big(\hbar^2\varrho_i\nabla\Big(\dfrac{\Delta\sqrt{\varrho_i}}{\sqrt{\varrho_i}}\Big) - \nabla(\varrho_i\theta_i) + (-1)^{i-1}\varrho_i\nabla V\Big) = 0, \\
-\operatorname{div}(\varrho_i\nabla\theta_i) = \dfrac{\varrho_i}{\tau_e}(\bar\theta(x) - \theta_i), \quad i = 1, 2, \\
\lambda^2\Delta V = \varrho_1 - \varrho_2 - D(x), \quad x \in \mathbb{T}^d,
\end{cases}
\tag{1.1}
$$

where $\mathbb{T}^d$ ($d = 1, 2, 3$) is the multi-dimensional torus, the unknown variables $\varrho_i(x,t)$, $\theta_i(x,t)$ are the electron ($i = 1$) and the hole ($i = 2$) density or temperature, respectively, and $V(x,t)$ is the electric field, while $\hbar$ is the scaled Planck constant, $\tau_e > 0$ is the energy relaxation-time, $\bar\theta(x)$ is the temperature of the semiconductor lattice in energy units, $\lambda > 0$ is the Debye length and $D(x)$ is the doping profile.

If the carrier in semiconductors is electron (or hole) only, then the corresponding unipolar model reads as

$$
\begin{cases}
\varrho_t + \operatorname{div}\Big(\hbar^2\varrho\nabla\Big(\dfrac{\Delta\sqrt{\varrho}}{\sqrt{\varrho}}\Big) - \nabla(\varrho\theta) + \varrho\nabla V\Big) = 0, \\
-\operatorname{div}(\varrho\nabla\theta) = \dfrac{\varrho}{\tau_e}(\bar\theta(x) - \theta), \\
\lambda^2\Delta V = \varrho - D(x), \quad x \in \mathbb{T}^d,
\end{cases}
\tag{1.2}
$$

which is the simplified QET model derived by [13].

Also, if we take $\hbar = 0$ in (1.1) and (1.2), then we get the following bipolar or unipolar classical energy transport models, respectively:

$$\begin{cases} (\varrho_i)_t = \operatorname{div}(\nabla(\varrho_i\theta_i) - (-1)^{i-1}\varrho_i\nabla V), \\ -\operatorname{div}(\varrho_i\nabla\theta_i) = \dfrac{\varrho_i}{\tau_e}(\bar{\theta}(x) - \theta_i), \quad i = 1, 2, \\ \lambda^2\Delta V = \varrho_1 - \varrho_2 - D(x), \quad (x,t) \in \mathbb{T}^d \times (0,\infty), \end{cases} \tag{1.3}$$

$$\begin{cases} \varrho_t = \operatorname{div}(\nabla(\varrho\theta) - \varrho\nabla V), \\ -\operatorname{div}(\varrho\nabla\theta) = \dfrac{\varrho}{\tau_e}(\bar{\theta}(x) - \theta), \\ \lambda^2\Delta V = \varrho - D(x), \quad (x,t) \in \mathbb{T}^d \times (0,\infty). \end{cases} \tag{1.4}$$

There are several results for the unipolar classical model (1.4). For the 1-dimensional case, the existence of classical solutions is proved in [5] and [6]. Recently, the global existence and uniqueness, exponential decay of the strong solutions to the Dirichlet boundary value problem of (1.4) with general conductivity are proved in [15]. For multi-dimensional case, the global existence of weak solutions to the initial boundary value problem (1.4) with Dirichlet-Neumann mixed boundary conditions is proved in [14]. This result is generalized to the case of $k(\varrho,\theta) = \varrho\theta$ in [20] and [17], where $k(\varrho,\theta)$ is heat conductivity. Moreover, the uniquness and energy relaxation-time limit is proved in [17].

We consider the periodic boundary value problem of the quantum energy-transport model (1.1) for bipolar semiconductors in the torus $\mathbb{T}^d$ with the initial data

$$\varrho_i(x,0) = \varrho_{i0}(x), \quad x \in \mathbb{T}^d. \tag{1.5}$$

We first state the result on the global existence for the problem (1.1), (1.5).

Theorem 1.1 (Global Existence). *Let $D(x)$, $\bar{\theta}(x) \in L^\infty(\mathbb{T}^d)$ and $0<m\leqslant\bar{\theta}(x)\leqslant M$ for all $x \in \mathbb{T}^d$. Assume that*

$$\begin{aligned} &\varrho_{i0} \in L^1(\mathbb{T}^d), \quad \varrho_{i0}(x) \geqslant 0 \quad \forall\, x \in \mathbb{T}^d, \\ &\int_{\mathbb{T}^d} \varrho_{i0}(x)\ln\varrho_{i0}(x)\,\mathrm{d}x < \infty, \quad i = 1, 2, \\ &\int_{\mathbb{T}^d} (\varrho_{10}(x) - \varrho_{20}(x) - D(x))\,\mathrm{d}x = 0. \end{aligned} \tag{1.6}$$

Then there exists a weak solution $(\varrho_i, \theta_i, V)(x,t)$ *to the problem* (1.1), (1.5) *such that the regularity properties*

$$
\begin{gathered}
\varrho_i(x,t) \geqslant 0, \quad 0 < \frac{m}{2} \leqslant \theta_i(x,t) \leqslant M+1, \\
\sqrt{\varrho_i} \in L^2(0,T;H^2(\mathbb{T}^d)) \cap L^\infty(0,T;L^2(\mathbb{T}^d)), \\
\varrho_i \in W^{1,11/10}(0,T;H^{-2}(\mathbb{T}^d)), \quad \sqrt{\varrho_i}\theta_i \in L^2(0,T;H^1(\mathbb{T}^d)), \\
\varrho_i\theta_i \in L^{4/3}(0,T;W^{1,4/3}(\mathbb{T}^d)), \quad V \in L^2(0,T;H^2(\mathbb{T}^d))
\end{gathered} \tag{1.7}
$$

hold for all $T > 0$. *Moreover, the equations*

$$
\begin{aligned}
&(\varrho_i)_t + \hbar^2\nabla^2 : (\sqrt{\varrho_i}\nabla^2\sqrt{\varrho_i} - \nabla\sqrt{\varrho_i} \otimes \nabla\sqrt{\varrho_i}) = \operatorname{div}(\nabla(\varrho_i\theta_i) - (-1)^{i-1}\varrho_i\nabla V), \\
&\quad - \operatorname{div}(\sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i) - \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i}) = \frac{\varrho_i}{\tau_e}(\bar{\theta}(x) - \theta_i), \\
&\lambda^2\Delta V = \varrho_1 - \varrho_2 - D(x)
\end{aligned} \tag{1.8}
$$

are satisfied in the sense of $L^1_{\mathrm{loc}}(0,T;H^{-2}(\mathbb{T}^d))$. *The initial condition* (1.5) *holds in the sense of* $H^{-2}(\mathbb{T}^d)$.

R e m a r k 1.1. Let $(\varrho_i, \theta_i, V)(x,t)$ be the classical solutions to the problem (1.1), and (1.5). Then, we have

$$
\begin{gathered}
\operatorname{div}\Big(\varrho_i\nabla\Big(\frac{\Delta\sqrt{\varrho_i}}{\sqrt{\varrho_i}}\Big)\Big) = \frac{1}{2}\nabla^2 : \varrho_i\nabla^2\ln\varrho_i = \nabla^2 : (\sqrt{\varrho_i}\nabla^2\sqrt{\varrho_i} - \nabla\sqrt{\varrho_i} \otimes \nabla\sqrt{\varrho_i}), \\
\varrho_i\nabla\theta_i = \sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i) - \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i}, \quad i = 1,2
\end{gathered}
$$

(see [13]).

Next, we consider the semiclassical limit for the problem (1.1), (1.5).

Theorem 1.2 (Semiclassical limit)**.** *Under the assumptions of Theorem* 1.1, *let* $\{(\varrho_i^\hbar, \theta_i^\hbar, V^\hbar)\}$ *be a weak solution to the problem* (1.1), (1.5), *guaranteed by Theorem* 1.1. *Then there exists a subsequence (not relabeled) such that, for any* $T > 0$ *and* $p \in [1, \frac{3}{2})$, *when* $\hbar \to 0$, *it holds that*

$$
\begin{aligned}
\varrho_i^\hbar &\to \varrho_i\colon \text{ strongly in } L^2(0,T;L^p(\mathbb{T}^d)), \\
\sqrt{\varrho_i^\hbar} &\to \sqrt{\varrho_i}\colon \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^d)), \\
(\varrho_i^\hbar)_t &\to (\varrho_i)_t\colon \text{ weakly in } L^{8/7}(0,T;H^{-3}(\mathbb{T}^d)), \\
\theta_i^\hbar &\to \theta_i\colon \text{ *-weakly in } L^\infty(0,T;L^\infty(\mathbb{T}^d)), \\
V^\hbar &\to V\colon \text{ weakly in } L^2(0,T;H^2(\mathbb{T}^d))
\end{aligned} \tag{1.9}
$$

and

$$
(1.10)\qquad \begin{cases} \sqrt{\varrho_i} \in L^2(0,T;H^1(\mathbb{T}^d)), \\ \varrho_i \in L^2(0,T;L^p(\mathbb{T}^d)) \cap W^{1,8/7}(0,T;H^{-3}(\mathbb{T}^d)), \\ \theta_i \in L^\infty(0,T;L^\infty(\mathbb{T}^d)), \quad \sqrt{\varrho_i}\theta_i \in L^2(0,T;H^1(\mathbb{T}^d)), \\ \varrho_i\theta_i \in L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d)), \quad V \in L^2(0,T;H^2(\mathbb{T}^d)). \end{cases}
$$

The limit $\{(\varrho_i,\theta_i,V)\}$ *solves the periodic boundary value problem for the bipolar classical energy transport model* (1.3) *in the following weak sense:*

$$
(1.11)\quad \int_0^T \langle(\varrho_i)_t,\varphi\rangle_{H^{-1},H^1}\,\mathrm{d}t = -\int_0^T\int_{\mathbb{T}^d} (\nabla(\varrho_i\theta_i) - (-1)^{i-1}\varrho_i\nabla V)\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t,
$$

$$
\int_0^T\int_{\mathbb{T}^d} (\sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i) - \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i})\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^d} \frac{\varrho_i}{\tau_e}(\bar\theta(x)-\theta_i)\varphi\,\mathrm{d}x\,\mathrm{d}t,
$$

$$
\lambda^2\int_0^T\int_{\mathbb{T}^d} \nabla V\cdot\nabla\varphi\,\mathrm{d}x\,\mathrm{d}t = \int_0^T\int_{\mathbb{T}^d} (\varrho_1-\varrho_2-D(x))\varphi\,\mathrm{d}x\,\mathrm{d}t \quad \forall\,\varphi\in L^\infty(0,T;H^1(\mathbb{T}^d)).
$$

The initial condition (1.5) *holds in the sense of* $H^{-1}(\mathbb{T}^d)$.

R e m a r k 1.2. For the unipolar QET model (1.2), the similar results in Theorems 1.1 and 1.2 are obtained by [13], Theorem 1, and [1], Theorem 1.2, respectively.

Highlight of this paper. The main purpose of this paper is to prove the existence of weak solutions and to justify the semiclassical limit for the problem (1.1), (1.5). The main steps in proving Theorems 1.1 and 1.2 are to derive the entropy-type estimates, independent of the time step τ, the square root of the densities $\overline{w}_i$ $(i=1,2)$ at time $t-\tau$ and the scaled Planck constant $\hbar$ for the time-discrete system (3.1). Here, we briefly review the main difference, compared with the previous works [13] and [1], where the authors obtained similar results for the unipolar model. The estimates for the temperature θ_i $(i=1,2)$ and electric field V can be derived using the maximum principle for the equation $(3.1)_2$ and the solution estimate for the Poisson equation $(3.1)_3$, respectively, by the same lines as in [13]. However, the estimates for the square root of densities w_i $(i=1,2)$ at time t cannot be derived by the same lines as in [13]. Indeed, for the unipolar model (1.2), they obtained the lower estimate on $\int_{\mathbb{T}^d} w^2\Delta V\,\mathrm{d}x$, which is an essential part, as follows:

$$
\int_{\mathbb{T}^d} w^2\Delta V\,\mathrm{d}x = \lambda^{-2}\int_{\mathbb{T}^d} w^2(\overline{w}^2 - D(x))\,\mathrm{d}x \geqslant -\lambda^{-2}\int_{\mathbb{T}^d} w^2 D(x)\,\mathrm{d}x
$$

relying on the Poisson equation $\lambda^2\Delta V = \overline{w}^2 - D(x)$ (see [13], (26)). But, for the bipolar model (1.1), we cannot borrow the idea, because it holds that

$$
\int_{\mathbb{T}^d} (-1)^{i-1} w_i^2\Delta V\,\mathrm{d}x = \lambda^{-2}(-1)^{i-1}\int_{\mathbb{T}^d} w_1^2(\overline{w}_1^2 - \overline{w}_2^2 - D(x))\,\mathrm{d}x
$$

due to the Poisson equation $\lambda^2\Delta V = \overline{w}_1^2 - \overline{w}_2^2 - D(x)$. To circumvent the difficulty, we modify the time-discrete system (22) in [13], that is, we choose the Poisson equation $\lambda^2\Delta V = w_1^2 - w_2^2 - D(x)$ in (3.1) instead of $\lambda^2\Delta V = \overline{w}_1^2 - \overline{w}_2^2 - D(x)$. Then, by using

$$\begin{aligned}\sum_{i=1}^{2}\int_{\mathbb{T}^d}(-1)^{i-1}w_i^2\Delta V\,\mathrm{d}x &= \lambda^{-2}\int_{\mathbb{T}^d}(w_1^2-w_2^2)(w_1^2-w_2^2-D(x))\,\mathrm{d}x\\ &\geqslant -\lambda^{-2}\int_{\mathbb{T}^d}(w_1^2-w_2^2)D(x)\,\mathrm{d}x,\end{aligned}$$

we can arrive at the lower estimate on $\sum_{i=1}^{2}\int_{\mathbb{T}^d}(-1)^{i-1}w_i^2\Delta V\,\mathrm{d}x$, and as a result, we can derive the estimates for the square root of the densities w_i $(i=1,2)$ (see (3.17) and (3.18)).

The paper is organized as follows. In Section 2, we give the derivation of the model equations. Section 3 is devoted to proving the existence of weak solutions. In Section 4, the semiclassical limit is obtained by using the estimates independent of $\hbar$.

2. Derivation of the model equations

The full quantum hydrodynamic (FQHD) model for bipolar semiconductors consists of the following balance equations for density, velocity, and energy density of the electron and hole, which have been derived from the Wigner equation:

$$\begin{cases}\varrho_{it}+\operatorname{div}(\varrho_i\mathbf{u}_i)=0,\\ (\varrho_i\mathbf{u}_i)_t+\operatorname{div}(\varrho_i\mathbf{u}_i\otimes\mathbf{u}_i)=\operatorname{div}\mathbb{S}^i+(-1)^{i-1}\varrho_i\mathbf{V}-\dfrac{\varrho_i\mathbf{u}_i}{\tau_m},\\ (\varrho_iE_i)_t+\operatorname{div}(\varrho_i\mathbf{u}_iE_i-\mathbb{S}^i\mathbf{u}_i)+\operatorname{div}\mathbf{q}_i\\ \qquad=(-1)^{i-1}\varrho_i\mathbf{u}_i\cdot\mathbf{V}-\frac{1}{\tau_e}\Big(\frac{1}{2}\varrho_i\mathbf{u}_i^2+\frac{3}{2}\varrho_i(\theta_i-\bar{\theta}(x))\Big),\quad i=1,2,\\ \lambda^2\operatorname{div}\mathbf{V}=\varrho_1-\varrho_2-D(x),\quad \operatorname{rot}\mathbf{V}=0,\end{cases}\tag{2.1}$$

where $x\in\mathbb{R}^3$, $t>0$, the unknown variables $\varrho_i(x,t)$, $\mathbf{u}_i(x,t)$, $\theta_i(x,t)$ are the electron $(i=1)$ and the hole $(i=2)$ density, velocity, temperature, respectively, and $\mathbf{V}(x,t)$ is the electric field, while $\mathbb{S}^i=(S^i_{kl})_{k,l=1,2,3}$, E_i and $\mathbf{q}_i$ are the stress tensor, the total energy density and the heat flux of the electron and the hole, respectively. Also, $\tau_m>0$ is the momentum relaxation time, $\tau_e>0$ is the energy relaxation time, $\bar{\theta}(x)$ is the temperature of the semiconductor lattice in energy units, $\lambda>0$ is the Debye length and $D(x)$ is the doping profile. Moreover, the stress tensor $\mathbb{S}^i$, the total energy

density E_i and the heat flux $\mathbf{q}_i$ are given by

$$
\begin{aligned}
S_{kl}^i &= -p_i\delta_{kl} + \frac{\hbar^2}{2}\varrho_i\partial_{x_k}\partial_{x_l}\ln\varrho_i, \quad k,l=1,2,3,\\
E_i &= e_i + \frac{1}{2}\mathbf{u}_i^2 - \frac{\hbar^2}{4}\Delta\ln\varrho_i,\\
\mathbf{q}_i &= -\kappa_i\nabla\theta_i - \frac{3\hbar^2}{4}\varrho_i\Delta\mathbf{u}_i,
\end{aligned}
\tag{2.2}
$$

respectively, where δ_{kl} is the Kronecker symbol, $\hbar > 0$ is the (scaled) Planck constant, $\kappa_i = k\ (\varrho_i, \theta_i)$ is the heat conductivity, while the pressure $p_i = p(\varrho_i, \theta_i)$ and internal energy $e_i = e(\varrho_i, \theta_i)$ are expressed by

$$
p_i = \varrho_i\theta_i \quad \text{and} \quad e_i = \frac{3}{2}\theta_i, \quad i = 1,2.
\tag{2.3}
$$

In the expression $(2.2)_1$ of the stress tensor $\mathbb{S}^i$, we add a force including the Bohm potential $\hbar^2\Delta\sqrt{\varrho_i}/\sqrt{\varrho_i}$ to the pressure p_i, because it holds

$$
\frac{\hbar^2}{2}\mathrm{div}(\varrho_i(\nabla\otimes\nabla)\ln\varrho_i) = \hbar^2\varrho_i\nabla\Big(\frac{\Delta\sqrt{\varrho_i}}{\sqrt{\varrho_i}}\Big).
\tag{2.4}
$$

The quantum correction to the total energy E_i in $(2.2)_2$ was first derived by Wigner [19]. The heat conduction term $\mathbf{q}_i$ consists of a classical Fourier law $-\kappa\nabla\theta_i$ plus a new quantum contribution $-(3\hbar^2/4)\varrho_i\Delta\mathbf{u}_i$ which can be interpreted as a dispersive heat flux [12]. For details on the more general quantum models for semiconductor devices, we refer to [10], [16], [21].

By substituting (2.2)–(2.4) into (2.1), a direct computation yields the multi-dimensional bipolar FQHD model for semiconductors as

$$
\begin{cases}
\varrho_{it} + \mathrm{div}(\varrho_i\mathbf{u}_i) = 0,\\
(\varrho_i\mathbf{u}_i)_t + \mathrm{div}(\varrho_i\mathbf{u}_i\otimes\mathbf{u}_i) + \nabla(\varrho_i\theta_i)\\
\qquad = \hbar^2\varrho_i\nabla\Big(\dfrac{\Delta\sqrt{\varrho_i}}{\sqrt{\varrho_i}}\Big) + (-1)^{i-1}\varrho_i\mathbf{V} - \dfrac{\varrho_i\mathbf{u}_i}{\tau_m},\\
\varrho_i\theta_{it} + \varrho_i\mathbf{u}_i\cdot\nabla\theta_i + \dfrac{2\varrho_i\theta_i}{3}\mathrm{div}\,\mathbf{u}_i - \dfrac{\hbar^2}{3}\mathrm{div}(\varrho_i\Delta\mathbf{u}_i)\\
\qquad = \dfrac{2}{3}\mathrm{div}(k(\varrho_i,\theta_i)\nabla\theta_i) + \dfrac{2\tau_e-\tau_m}{3\tau_m\tau_e}\varrho_i\mathbf{u}_i^2 - \dfrac{\varrho_i(\theta_i-\bar\theta(x))}{\tau_e}, \quad i=1,2,\\
\lambda^2\mathrm{div}\,\mathbf{V} = \varrho_1 - \varrho_2 - D(x), \quad \mathrm{rot}\,\mathbf{V} = 0,
\end{cases}
\tag{2.5}
$$

where we used

$$
-\varrho_i(\Delta\ln\varrho_i)_t - \varrho_i\mathbf{u}_i\cdot\nabla(\Delta\ln\varrho_i) - 2\varrho_i((\nabla\otimes\nabla)\ln\varrho_i) : \nabla\mathbf{u}_i = \mathrm{div}(\varrho_i\Delta\mathbf{u}_i)
$$

due to $\varrho_{it} = -\mathrm{div}(\varrho_i \mathbf{u}_i)$. Comparing with the classical bipolar full hydrodynamic (FHD) model (for example, see [8], [9]), the new feature of the bipolar FQHD model is the term $\hbar^2\varrho_i\nabla(\Delta\sqrt{\varrho_i}/\sqrt{\varrho_i})$ including the Bohm potential $\hbar^2\Delta\sqrt{\varrho_i}/\sqrt{\varrho_i}$ in the momentum equation $(2.5)_2$ and the dispersive velocity term $\frac{1}{2}\hbar^2\mathrm{div}(\varrho_i\Delta\mathbf{u}_i)$ in the energy equation $(2.5)_3$. Both of them are called quantum correction terms (or dispersive terms) and belong to the third-order derivative terms of system (2.5).

Now, we turn to the analysis of relaxation-time limits. To this end, let us introduce the diffusion scaling as

$$x \to x, \quad t \to \frac{t}{\tau_m}, \quad (\varrho_i^{\tau_m}, \mathbf{u}_i^{\tau_m}, \theta_i^{\tau_m}, \mathbf{V}^{\tau_m})(x,t) = \Big(\varrho_i, \frac{\mathbf{u}_i}{\tau_m}, \theta_i, \mathbf{V}\Big)\Big(x, \frac{t}{\tau_m}\Big).$$

Then (2.5) can be rewritten as

$$\begin{aligned}
(2.6)\quad & \varrho_{it}^{\tau_m} + \mathrm{div}(\varrho_i^{\tau_m}\mathbf{u}_i^{\tau_m}) = 0,\\
& \tau_m^2\varrho_i^{\tau_m}\mathbf{u}_{it}^{\tau_m} + \tau_m^2\varrho_i^{\tau_m}\mathbf{u}_i^{\tau_m}\cdot\nabla\mathbf{u}_i^{\tau_m} + \nabla(\varrho_i^{\tau_m}\theta_i^{\tau_m})\\
&\qquad = \hbar^2\varrho_i^{\tau_m}\nabla\Big(\frac{\Delta\sqrt{\varrho_i^{\tau_m}}}{\sqrt{\varrho_i^{\tau_m}}}\Big) + (-1)^{i-1}\varrho_i^{\tau_m}\mathbf{V}^{\tau_m} - \varrho_i^{\tau_m}\mathbf{u}_i^{\tau_m},\\
& \tau_m\varrho_i^{\tau_m}\theta_{it}^{\tau_m} + \tau_m\varrho_i^{\tau_m}\mathbf{u}_i^{\tau_m}\cdot\nabla\theta_i^{\tau_m} + \tau_m\frac{2\varrho_i^{\tau_m}\theta_i^{\tau_m}}{3}\mathrm{div}\,\mathbf{u}_i^{\tau_m} - \frac{\tau_m\hbar^2}{3}\mathrm{div}(\varrho_i^{\tau_m}\Delta\mathbf{u}_i^{\tau_m})\\
&\qquad = \frac{2}{3}\mathrm{div}(k(\varrho_i^{\tau_m},\theta_i^{\tau_m})\nabla\theta_i^{\tau_m}) + \frac{(2\tau_e-\tau_m)\tau_m}{3\tau_e}\varrho_i^{\tau_m}(\mathbf{u}_i^{\tau_m})^2\\
&\qquad\qquad - \frac{\varrho_i^{\tau_m}(\theta_i^{\tau_m}-\bar\theta(x))}{\tau_e}, \quad i=1,2,\\
& \lambda^2\mathrm{div}\,\mathbf{V}^{\tau_m} = \varrho_1^{\tau_m} - \varrho_2^{\tau_m} - D(x), \quad \mathrm{rot}\,\mathbf{V}^{\tau_m} = 0, \quad x\in\mathbb{R}^3,\ t>0.
\end{aligned}$$

Formally, if we take only the moment relaxation-time limit $\tau_m \to 0$ for the fixed energy relaxation-time $\tau_e > 0$, then the simplified bipolar QET model

$$\begin{aligned}
(2.7)\quad & \varrho_{it} + \mathrm{div}\Big[(-1)^{i-1}\varrho_i\mathbf{V} - \nabla(\varrho_i\theta_i) + \hbar^2\varrho_i\nabla\Big(\frac{\Delta\sqrt{\varrho_i}}{\sqrt{\varrho_i}}\Big)\Big] = 0,\\
& \frac{2}{3}\mathrm{div}(k(\varrho_i,\theta_i)\nabla\theta_i) = \frac{\varrho_i}{\tau_e}(\theta_i - \bar\theta(x)), \quad i=1,2,\\
& \lambda^2\mathrm{div}\,\mathbf{V} = \varrho_1 - \varrho_2 - D(x), \quad \mathrm{rot}\,\mathbf{V} = 0
\end{aligned}$$

Choosing $k(\varrho_i,\theta_i) = \varrho_i$ in (2.7), we deduce (1.1). Physically, we expect that the heat conductivity k_i depends on the thermal energy $\frac{3}{2}\varrho_i\theta_i$. Our choice $k(\varrho_i,\theta_i) = \varrho_i$ has a purely technical reason.

3. Global existence of weak solutions

In this section, we will prove Theorem 1.1.

3.1. Existence of a time-discrete solution. We replace (1.1) by a semidiscrete system. To this end, let $\tau > 0$ be a time step and $\overline{w}_i$ $(i = 1, 2)$ be a given function. We wish to find a solution to the problem

$$
\begin{aligned}
&\frac{w_i^2 - \overline{w}_i^2}{\tau} + \hbar^2 \nabla^2 : (w_i \nabla^2 w_i - \nabla w_i \otimes \nabla w_i) = \operatorname{div}(\nabla(w_i^2 \theta_i) - (-1)^{i-1} w_i^2 \nabla V), \\
&- \operatorname{div}(w_i^2 \nabla \theta_i) = \frac{w_i^2}{\tau_e}(\bar{\theta}(x) - \theta_i), \quad i = 1, 2, \\
&\lambda^2 \Delta V = w_1^2 - w_2^2 - D(x), \quad x \in \mathbb{T}^d.
\end{aligned} \tag{3.1}
$$

Here, w_i represents the square root of the electron $(i = 1)$ and hole $(i = 2)$ density at some time t and $\overline{w}_i$ is the corresponding quantity at time $t - \tau$.

Lemma 3.1. *Let the assumptions of Theorem* 1.1 *hold and assume that*

$$
\overline{w}_i \in L^\infty(\mathbb{T}^d), \quad i = 1, 2, \quad \int_{\mathbb{T}^d} (\overline{w}_1^2(x) - \overline{w}_2^2(x) - D(x))\,\mathrm{d}x = 0. \tag{3.2}
$$

Then there exists a weak solution $(w_i, \theta_i, V) \in H^2(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) \times H^2(\mathbb{T}^d)$ *to the problem* (3.1) *such that*

$$
\begin{aligned}
&w_i(x) \geqslant 0, \quad 0 < \frac{m}{2} \leqslant \theta_i(x) \leqslant M + 1 \quad \forall\, x \in \mathbb{T}^d, \\
&w_i \theta_i \in H^1(\mathbb{T}^d), \quad w_i^2 \nabla \theta_i \in L^2(\mathbb{T}^d), \quad i = 1, 2, \\
&\int_{\mathbb{T}^d} (w_1^2(x) - w_2^2(x) - D(x))\,\mathrm{d}x = 0.
\end{aligned} \tag{3.3}
$$

P r o o f. The proof is performed in several steps.

Step 1: Definition of a regularized problem. The solution to (3.1) is obtained as the limit of solutions to a regularized problem. In particular, we add a strongly elliptic operator in $y_i = 2 \ln w_i$ $(i = 1, 2)$:

$$
\begin{aligned}
&\frac{w_i^2 - \overline{w}_i^2}{\tau} + \frac{\hbar^2}{2} \nabla^2 : (w_i^2 \nabla^2 y_i) + \delta(\Delta^2 y_i + y_i) = \operatorname{div}(\nabla(w_i^2 \theta_i) - (-1)^{i-1} w_i^2 \nabla V), \\
&- \operatorname{div}((w_i^2 + \delta) \nabla \theta_i) = \frac{w_i^2 + \delta}{\tau_e}(\bar{\theta}(x) - \theta_i), \quad i = 1, 2, \\
&\lambda^2 \Delta V = w_1^2 - w_2^2 - D(x), \quad x \in \mathbb{T}^d,
\end{aligned} \tag{3.4}
$$

where $\delta > 0$ is a regularization parameter. The fourth-order operator $\delta(\Delta^2 y_i + y_i)$ guarantees the coercivity of the left-hand side of $(3.4)_1$ with respect to y_i.

Step 2: Solution of a regularized problem. We solve (3.4) by employing the Leray-Schauder fixed-point theorem. For given

$$w_i \in W^{1,4}(\mathbb{T}^d)\ (i=1,2), \quad \int_{\mathbb{T}^d} (w_1^2 - w_2^2 - D(x))\,\mathrm{d}x = 0, \tag{3.5}$$

it is well-known that there exists the unique solution

$$V \in H^2(\mathbb{T}^d), \quad \int_{\mathbb{T}^d} V(x)\,\mathrm{d}x = 0$$

to the Poisson equation

$$\lambda^2 \Delta V = w_1^2 - w_2^2 - D(x), \quad x \in \mathbb{T}^d. \tag{3.6}$$

Next, for given $\sigma \in [0,1]$ and $w_i \in W^{1,4}(\mathbb{T}^d)$ $(i=1,2)$, we consider the linear equation

$$-\mathrm{div}((w_i^2+\delta)\nabla\theta_i) = \frac{\sigma w_i^2 + \delta}{\tau_e}(\bar{\theta}(x) - \theta_i), \quad x \in \mathbb{T}^d. \tag{3.7}$$

As the coefficient of the zeroth-order term $(\sigma w_i^2 + \delta)\theta_i/\tau_e$ is uniformly positive, the problem (3.7) has the unique solution $\theta_i \in H^1(\mathbb{T}^d)$.

Finally, for given $\sigma \in [0,1]$, $w_i \in W^{1,4}(\mathbb{T}^d)$, $\theta_i \in H^1(\mathbb{T}^d)$ $(i=1,2)$ and $V \in H^2(\mathbb{T}^d)$, we consider the linear equation

$$\begin{aligned} \frac{\hbar^2}{2}\nabla^2 : (w_i^2 \nabla^2 y_i) + \delta(\Delta^2 y_i + y_i) \\ = -\frac{\sigma}{\tau}(w_i^2 - \overline{w_i^2}) + \sigma \mathrm{div}(\nabla(w_i^2\theta_i) - (-1)^{i-1} w_i^2 \nabla V), \quad x \in \mathbb{T}^d. \end{aligned} \tag{3.8}$$

Define the bilinear form $b(y_i, z)$ and the linear form $f_i(z)$ by

$$\begin{aligned} b(y_i,z) &= \frac{\hbar^2}{2}\int_{\mathbb{T}^d} w_i^2 \nabla^2 y_i : \nabla^2 z\,\mathrm{d}x + \delta \int_{\mathbb{T}^d} (\Delta y_i \Delta z + y_i z)\,\mathrm{d}x, \\ f_i(z) &= -\frac{\sigma}{\tau}\int_{\mathbb{T}^d} (w_i^2 - \overline{w_i^2}) z\,\mathrm{d}x - \sigma \int_{\mathbb{T}^d} (\nabla(w_i^2\theta_i) - (-1)^{i-1} w_i^2 \nabla V)\cdot \nabla z\,\mathrm{d}x \end{aligned} \tag{3.9}$$

for $y_i, z \in H^2(\mathbb{T}^d)$, respectively. Then, (3.8) is equivalent to the variational problem

$$b(y_i, z) = f_i(z) \quad \forall\, z \in H^2(\mathbb{T}^d). \tag{3.10}$$

Because the bilinear form $b(y_i, z)$ is continuous and coercive for $\delta > 0$, and the linear form $f_i(z)$ is continuous, the Lax-Milgram lemma provides the existence of a unique solution $y_i \in H^2(\mathbb{T}^d)$ to the problem (3.10).

This defines the fixed-point operator S as

$$S : W^{1,4}(\mathbb{T}^d) \times W^{1,4}(\mathbb{T}^d) \times [0,1] \to W^{1,4}(\mathbb{T}^d) \times W^{1,4}(\mathbb{T}^d),$$
$$S(w_1, w_2, \sigma) = (v_1, v_2) := (\mathrm{e}^{y_1/2}, \mathrm{e}^{y_2/2}).$$

We shall now verify the hypotheses of the Leray-Schauder theorem which provides a solution (w_1, w_2) to $S(w_1, w_2, 1) = (w_1, w_2)$. The operator S is constant at $\sigma = 0$, that is, $S(w_1, w_2, 0) = (1, 1)$. We prove the operator S is continuous. Let V', V'' be the solution to (3.6) with given w_i', w_i'' $(i = 1, 2)$, respectively, θ_i', θ_i'' be the solution to (3.7), and y_i', y_i'' be the solution to (3.8). Put $y_i = y_i' - y_i''$. By (3.10) and (3.9), we have

$$\begin{aligned}
&\delta \int_{\mathbb{T}^d} (\Delta y_i \Delta z + y_i z)\,\mathrm{d}x + \frac{\hbar^2}{2} \int_{\mathbb{T}^d} ((w_i')^2 \nabla^2 y_i' - (w_i'')^2 \nabla^2 y_i'') : \nabla^2 z\,\mathrm{d}x \\
&\quad = -\frac{\sigma}{\tau} \int_{\mathbb{T}^d} ((w_i')^2 - (w_i'')^2) z\,\mathrm{d}x - \sigma \int_{\mathbb{T}^d} \nabla(((w_i')^2 \theta_i') - ((w_i'')^2 \theta_i'')) \cdot \nabla z\,\mathrm{d}x \\
&\qquad + (-1)^{i-1} \sigma \int_{\mathbb{T}^d} ((w_i')^2 \nabla V' - (w_i'')^2 \nabla V'') \cdot \nabla z\,\mathrm{d}x.
\end{aligned}$$

Therefore, setting $z = y_i$, we get the estimate

(3.11)

$$\begin{aligned}
&\delta(\|\Delta y_i\|^2 + \|y_i\|^2) + \frac{\hbar^2}{2} \int_{\mathbb{T}^d} (w_i')^2 |\nabla^2 y_i|^2\,\mathrm{d}x \\
&\quad = -\frac{\hbar^2}{2} \int_{\mathbb{T}^d} ((w_i')^2 - (w_i'')^2) \nabla^2 y_i'' : \nabla^2 y_i\,\mathrm{d}x - \frac{\sigma}{\tau} \int_{\mathbb{T}^d} ((w_i')^2 - (w_i'')^2) y_i\,\mathrm{d}x \\
&\qquad - \sigma \int_{\mathbb{T}^d} \nabla(((w_i')^2 \theta_i') - ((w_i'')^2 \theta_i'')) \cdot \nabla y_i\,\mathrm{d}x \\
&\qquad + (-1)^{i-1} \sigma \int_{\mathbb{T}^d} ((w_i')^2 \nabla V' - (w_i'')^2 \nabla V'') \cdot \nabla y_i\,\mathrm{d}x \\
&\quad \leqslant C \|w_i' - w_i''\|_{W^{1,4}(\mathbb{T}^d)} \|y_i\|_{H^2(\mathbb{T}^d)} (1 + \|\theta_i' - \theta_i''\|_{L^2(\mathbb{T}^d)} + \|V' - V''\|_{L^2(\mathbb{T}^d)}).
\end{aligned}$$

By similar arguments, we obtain from (3.6) and (3.7) that

$$\|\theta_i' - \theta_i''\|_{L^2(\mathbb{T}^d)} + \|V' - V''\|_{L^2(\mathbb{T}^d)} \leqslant C(\|w_1' - w_1''\|_{W^{1,4}(\mathbb{T}^d)} + \|w_2' - w_2''\|_{W^{1,4}(\mathbb{T}^d)}). \tag{3.12}$$

By using (3.11) and (3.12), it is proved that the operator S is continuous.

By standard results for elliptic equations, S is compact, since the embedding $H^2(\mathbb{T}^d) \subset W^{1,4}(\mathbb{T}^d)$is compact. It remains to show a uniform bound for all fixed points of $S(\cdot, \cdot, \sigma)$. This is achieved by suitable entropy estimates.

Step 3: Estimates for the regularized problem. Let w_i be a fixed point of $S(\cdot, \cdot, \sigma)$ for some $\sigma \in [0, 1]$. Then there exists $y_i \in H^2(\mathbb{T}^d)$ such that $w_i = \mathrm{e}^{y_i/2}$.

We first derive some bounds for θ_i. With the test function $(\theta_i(x) - \widehat{M})^+ = \max\{0, \theta_i(x) - \widehat{M}\} \in H^1(\mathbb{T}^d)$ in (3.7), where $\widehat{M} = \sup_{\mathbb{T}^d} \bar{\theta}(x) + 1$, we infer that

$$\begin{aligned}\int_{\mathbb{T}^d} (w_i^2 + \delta)|\nabla(\theta_i(x) - \widehat{M})^+|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}((\theta_i(x) - \widehat{M})^+)^2 \,\mathrm{d}x \\ = \int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}(\bar{\theta}(x) - \widehat{M})(\theta_i(x) - \widehat{M})^+ \,\mathrm{d}x \leqslant 0.\end{aligned}$$

Hence $\theta_i(x) \leqslant \widehat{M}$. The test function $(\theta_i(x) - \widehat{m})^- = \min\{0, \theta_i(x) - \widehat{m}\} \in H^1(\mathbb{T}^d)$ with $\widehat{m} = \frac{1}{2} \inf_{\mathbb{T}^d} \bar{\theta}(x) > 0$ leads to

$$\begin{aligned}\int_{\mathbb{T}^d} (w_i^2 + \delta)|\nabla(\theta_i(x) - \widehat{m})^-|^2 \,\mathrm{d}x + \int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}((\theta_i(x) - \widehat{m})^-)^2 \,\mathrm{d}x \\ = \int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}(\bar{\theta}(x) - \widehat{m})(\theta_i(x) - \widehat{m})^- \,\mathrm{d}x \leqslant 0,\end{aligned}$$

from which we conclude that $\widehat{m} \leqslant \theta_i(x)$. That is

$$0 < \frac{m}{2} \leqslant \theta_i(x) \leqslant M + 1 \quad \forall\, x \in \mathbb{T}^d. \tag{3.13}$$

Next we derive some bounds for $w_i \nabla \theta_i$ and $w_i \theta_i$. Employing the test function θ_i in (3.7), we have

$$\begin{aligned}\int_{\mathbb{T}^d} (w_i^2 + \delta)|\nabla \theta_i|^2 \,\mathrm{d}x &= \int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}(\bar{\theta}(x) - \theta_i)\theta_i \,\mathrm{d}x \\ &\leqslant -\int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e}\theta_i^2 \,\mathrm{d}x + M(M+1)\int_{\mathbb{T}^d} \frac{\sigma w_i^2 + \delta}{\tau_e} \,\mathrm{d}x.\end{aligned}$$

We derive an approximate L^2 bound for w_i by using the test function $z = 1$ in (3.10):

$$\sigma \int_{\mathbb{T}^d} w_i^2 \,\mathrm{d}x = \sigma \int_{\mathbb{T}^d} \overline{w}_i^2 \,\mathrm{d}x - \delta\tau \int_{\mathbb{T}^d} y_i \,\mathrm{d}x. \tag{3.14}$$

Thus,

$$\begin{aligned}\int_{\mathbb{T}^d} w_i^2 |\nabla \theta_i|^2 \,\mathrm{d}x + \frac{\sigma}{\tau_e} \int_{\mathbb{T}^d} w_i^2 \theta_i^2 \,\mathrm{d}x \\ \leqslant \frac{M(M+1)}{\tau_e}\bigg(\sigma \int_{\mathbb{T}^d} \overline{w}_i^2 \,\mathrm{d}x - \delta\tau \int_{\mathbb{T}^d} y_i \,\mathrm{d}x + \delta\bigg).\end{aligned} \tag{3.15}$$

Next we derive some bounds for w_i. Taking the test function $z_i = 2\ln w_i$ in (3.10), we find that

$$\begin{aligned}
&2\hbar^2 \int_{\mathbb{T}^d} (w_i^2+\delta)|\nabla^2 \ln w_i|^2 \,\mathrm{d}x \\
&\quad + \delta \int_{\mathbb{T}^d} (|\Delta y_i|^2 + |y_i|^2)\,\mathrm{d}x + \frac{\sigma}{\tau}\int_{\mathbb{T}^d} (w_i^2 - \overline{w}_i^2)\ln(w_i^2)\,\mathrm{d}x \\
&= -2\sigma \int_{\mathbb{T}^d} (\nabla(w_i^2\theta_i) - (-1)^{i-1} w_i^2 \nabla V)\cdot \nabla \ln w_i \,\mathrm{d}x.
\end{aligned}$$

Setting $\phi(s) = s(\ln s - 1) + 1, s \geqslant 0$, and noticing that

$$\phi(s) - \phi(t) \leqslant \phi'(s)(s-t), \quad \phi'(s) = \ln s$$

due to the convexity of the function, we obtain

$$\begin{aligned}
&2\hbar^2 \int_{\mathbb{T}^d} (w_i^2+\delta)|\nabla^2 \ln w_i|^2 \,\mathrm{d}x + \delta \int_{\mathbb{T}^d} (|\Delta y_i|^2 + |y_i|^2)\,\mathrm{d}x \\
&\quad + \frac{\sigma}{\tau}\int_{\mathbb{T}^d} (\phi(w_i^2) - \phi(\overline{w}_i^2))\,\mathrm{d}x \\
&\leqslant -2\sigma \int_{\mathbb{T}^d} \Big(2\theta_i |\nabla w_i|^2 + w_i \nabla\theta_i \cdot \nabla w_i + \frac{(-1)^{i-1} w_i^2}{2}\Delta V\Big)\,\mathrm{d}x.
\end{aligned} \tag{3.16}$$

By using the Young inequality, (3.13), (3.6), (3.14)–(3.15), we obtain from (3.16) that

$$\begin{aligned}
&2\hbar^2 \int_{\mathbb{T}^d} (w_i^2+\delta)|\nabla^2 \ln w_i|^2 \,\mathrm{d}x + \delta \int_{\mathbb{T}^d} (|\Delta y_i|^2 + |y_i|^2)\,\mathrm{d}x \\
&\quad + \frac{\sigma}{\tau}\int_{\mathbb{T}^d} (\phi(w_i^2) - \phi(\overline{w}_i^2))\,\mathrm{d}x \\
&\leqslant \sigma \int_{\mathbb{T}^d} \Big(-m|\nabla w_i|^2 + \frac{w_i^2}{m}|\nabla\theta_i|^2 - \frac{(-1)^{i-1}w_i^2}{2\lambda^2}(w_1^2 - w_2^2 - D(x))\Big)\,\mathrm{d}x \\
&\leqslant -\sigma m \int_{\mathbb{T}^d} |\nabla w_i|^2\,\mathrm{d}x + \frac{\sigma}{m}\int_{\mathbb{T}^d} w_i^2 |\nabla\theta_i|^2\,\mathrm{d}x \\
&\quad - \frac{\sigma(-1)^{i-1}}{2\lambda^2}\int_{\mathbb{T}^d} w_i^2(w_1^2 - w_2^2 - D(x))\,\mathrm{d}x \\
&\leqslant -\sigma m \int_{\mathbb{T}^d} |\nabla w_i|^2\,\mathrm{d}x + \frac{M(M+1)\sigma}{\tau_e m}\Big(\sigma\int_{\mathbb{T}^d} \overline{w}_i^2\,\mathrm{d}x - \delta\tau\int_{\mathbb{T}^d} y_i\,\mathrm{d}x + \delta\Big) \\
&\quad - \frac{\sigma(-1)^{i-1}}{2\lambda^2}\int_{\mathbb{T}^d} w_i^2(w_1^2 - w_2^2 - D(x))\,\mathrm{d}x,
\end{aligned}$$

which implies

$$
\begin{aligned}
(3.17)\quad & 2\hbar^2\sum_{i=1}^{2}\int_{\mathbb{T}^d}(w_i^2+\delta)|\nabla^2\ln w_i|^2\,\mathrm{d}x+\delta\sum_{i=1}^{2}\int_{\mathbb{T}^d}(|\Delta y_i|^2+|y_i|^2)\,\mathrm{d}x\\
&\quad+\frac{\sigma}{\tau}\sum_{i=1}^{2}\int_{\mathbb{T}^d}(\phi(w_i^2)-\phi(\overline{w}_i^2))\,\mathrm{d}x+\sigma m\sum_{i=1}^{2}\int_{\mathbb{T}^d}|\nabla w_i|^2\,\mathrm{d}x\\
&\quad+\frac{\sigma}{2\lambda^2}\int_{\mathbb{T}^d}(w_1^2-w_2^2)^2\,\mathrm{d}x\leqslant K_1\sum_{i=1}^{2}\biggl(\int_{\mathbb{T}^d}\overline{w}_i^2\,\mathrm{d}x+\delta\int_{\mathbb{T}^d}|y_i|\,\mathrm{d}x\biggr)+K_2,
\end{aligned}
$$

where we used

$$
\frac{\sigma(-1)^{i-1}}{2\lambda^2}\int_{\mathbb{T}^d}w_i^2D(x)\,\mathrm{d}x\leqslant C\biggl(\int_{\mathbb{T}^d}\overline{w}_i^2\,\mathrm{d}x+\delta\int_{\mathbb{T}^d}|y_i|\,\mathrm{d}x\biggr)
$$

due to (3.14), and K_i $(i=1,2)$, C are positive constants independent of σ, τ, δ. Therefore, we have

$$
\begin{aligned}
(3.18)\quad & 2\hbar^2\sum_{i=1}^{2}\int_{\mathbb{T}^d}(w_i^2+\delta)|\nabla^2\ln w_i|^2\,\mathrm{d}x+\delta\sum_{i=1}^{2}\int_{\mathbb{T}^d}|\Delta y_i|^2\,\mathrm{d}x+\frac{\delta}{2}\sum_{i=1}^{2}\int_{\mathbb{T}^d}|y_i|^2\,\mathrm{d}x\\
&\quad+\frac{\sigma}{\tau}\sum_{i=1}^{2}\int_{\mathbb{T}^d}(\phi(w_i^2)-\phi(\overline{w}_i^2))\,\mathrm{d}x+\sigma m\sum_{i=1}^{2}\int_{\mathbb{T}^d}|\nabla w_i|^2\,\mathrm{d}x\leqslant K_3,
\end{aligned}
$$

where $K_3>0$ is a positive constant independent of σ, τ, δ.

By (3.18), y_i and Δy_i are uniformly bounded in $L^2(\mathbb{T}^d)$ for any fixed $\delta>0$. This implies that y_i and also $w_i=\mathrm{e}^{y_i/2}$ are uniformly bounded in $H^2(\mathbb{T}^d)$. Then the Leray-Schauder theorem provides a solution (w_1,w_2) to $S(w_1,w_2,1)=(w_1,w_2)$, which we denote by $w_{i\delta}$. Let $(w_{i\delta},\theta_{i\delta},V_\delta)$ be the solutions to (3.4).

We prove $w_{i\delta}$ is strictly positive. By using (3.18) and the embedding $H^2(\mathbb{T}^d)\subset L^\infty(\mathbb{T}^d)$, we get $\|y_i\|_{L^\infty(\mathbb{T}^d)}\leqslant K_4$, which implies

$$
w_{i\delta}=\mathrm{e}^{y_i/2}\geqslant\mathrm{e}^{-K_4/2}>0. \tag{3.19}
$$

Finally, we prove H^1-uniform bounds for $w_{i\delta}\theta_{i\delta}$. The estimate (3.18) shows that $\sqrt{\delta}\|y_i\|_{L^2(\mathbb{T}^d)}$ is uniformly bounded. Hence, by (3.15) with $\sigma=1$,

$$
\int_{\mathbb{T}^d}w_{i\delta}^2|\nabla\theta_{i\delta}|^2\,\mathrm{d}x+\frac{1}{\tau_e}\int_{\mathbb{T}^d}w_{i\delta}^2\theta_{i\delta}^2\,\mathrm{d}x\leqslant K_5. \tag{3.20}
$$

Noticing that

$$
\int_{\mathbb{T}^d}|\nabla(w_{i\delta}\theta_{i\delta})|^2\,\mathrm{d}x\leqslant 2\int_{\mathbb{T}^d}w_{i\delta}^2|\nabla\theta_{i\delta}|^2\,\mathrm{d}x+2(M+1)^2\int_{\mathbb{T}^d}|\nabla w_{i\delta}|^2\,\mathrm{d}x,
$$

and using (3.20) yield that

$$\|w_{i\delta}\theta_{i\delta}\|_{H^1(\mathbb{T}^d)} \leqslant K_6, \tag{3.21}$$

where $K_6 > 0$ is a positive constant independent of σ, τ, δ.

Step 4: The limit $\delta \to 0$. By (3.18), the sequences $\phi(w_{i\delta}^2)$ and $|\nabla w_{i\delta}|^2$ are bounded in $L^1(\mathbb{T}^d)$. Since $s \leqslant \phi(s) + \mathrm{e} - 1$ for all $s \geqslant 0$, the sequence $w_{i\delta}^2$ is bounded in $L^1(\mathbb{T}^d)$, too.

Next, we employ Lemma 2.2 of [11] which gives the inequality

$$\int_{\mathbb{T}^d} w_{i\delta}^2 |\nabla^2 \ln w_{i\delta}|^2 \,\mathrm{d}x \geqslant K_7 \int_{\mathbb{T}^d} |\Delta w_{i\delta}|^2 \,\mathrm{d}x \tag{3.22}$$

for some constant $K_7 > 0$ (depending on the space dimension only). This inequality, together with the estimate (3.18), shows that $w_{i\delta}$ is bounded in $H^2(\mathbb{T}^d)$. Thus, for a subsequence which is not relabeled, as $\delta \to 0$,

$$\begin{aligned} &w_{i\delta} \to w_i \colon \text{ weakly in } H^2(\mathbb{T}^d), \\ &w_{i\delta} \to w_i \colon \text{ strongly in } W^{1,4}(\mathbb{T}^d) \end{aligned} \tag{3.23}$$

for some $w_i \in H^2(\mathbb{T}^d)$. Also, we get

$$w_i \geqslant 0 \quad \forall\, x \in \mathbb{T}^d \tag{3.24}$$

by using (3.19), and

$$\begin{aligned} w_{i\delta}^2 \nabla^2 \ln w_{i\delta} &= w_{i\delta} \nabla^2 w_{i\delta} - \nabla w_{i\delta} \otimes \nabla w_{i\delta} \\ &\to w_i \nabla^2 w_i - \nabla w_i \otimes \nabla w_i = w_i^2 \nabla^2 \ln w_i \colon \text{ weakly in } L^2(\mathbb{T}^d) \end{aligned} \tag{3.25}$$

by using (3.23).

By (3.18), it follows that $\sqrt{\delta} y_{i\delta}$ is the bound in $H^2(\mathbb{T}^d)$ and then we have

$$\delta(\Delta^2 y_{i\delta} + y_{i\delta}) \to 0 \colon \text{ weakly in } H^{-2}(\mathbb{T}^d). \tag{3.26}$$

The sequence V_δ is bounded in $H^2(\mathbb{T}^d)$ since (3.6) and $w_{i\delta}^2$ is bounded in $L^2(\mathbb{T}^d)$. Moreover, $\theta_{i\delta}$ is bounded in $L^\infty(\mathbb{T}^d)$ by (3.13) and $w_{i\delta}\theta_{i\delta}$ is bounded in $H^1(\mathbb{T}^d)$ by (3.21). As a consequence, up to subsequences,

$$\begin{aligned} V_\delta &\to V \colon \text{ weakly in } H^2(\mathbb{T}^d), \\ V_\delta &\to V \colon \text{ strongly in } W^{1,4}(\mathbb{T}^d), \\ \theta_{i\delta} &\to \theta_i \colon \text{ *-weakly in } L^\infty(\mathbb{T}^d), \\ w_{i\delta}\theta_{i\delta} &\to w_i\theta_i \colon \text{ weakly in } H^1(\mathbb{T}^d) \end{aligned} \tag{3.27}$$

for some $(\theta_i, V) \in L^\infty(\mathbb{T}^d) \times H^2(\mathbb{T}^d)$.

We will prove that $(w_i,\theta_i,V)\in H^2(\mathbb{T}^d)\times L^\infty(\mathbb{T}^d)\times H^2(\mathbb{T}^d)$ is a solution to (3.1). By using $(3.27)_1$ and (3.6), we know that V satisfy $(3.1)_3$. Also, it is proved that θ_i satisfy $(3.1)_2$ by using (3.27), (3.23) and

$$
\begin{aligned}
(3.28)\qquad w_{i\delta}^2\nabla\theta_{i\delta} &= w_{i\delta}\nabla(w_{i\delta}\theta_{i\delta}) - w_{i\delta}\theta_{i\delta}\nabla w_{i\delta}\\
&\to w_i\nabla(w_i\theta_i) - w_i\theta_i\nabla w_i = w_i^2\nabla\theta_i\colon \text{ weakly in } L^2(\mathbb{T}^d).
\end{aligned}
$$

By (3.27) and (3.23), we have

$$
\begin{aligned}
&\int_{\mathbb{T}^d}(\nabla(w_{i\delta}^2\theta_{i\delta}) - (-1)^{i-1}w_{i\delta}^2\nabla V_\delta)\cdot\nabla z\,\mathrm{d}x\\
&\qquad = \int_{\mathbb{T}^d}(w_{i\delta}\nabla(w_{i\delta}\theta_{i\delta}) + w_{i\delta}\theta_{i\delta}\nabla w_{i\delta} - (-1)^{i-1}w_i^2\nabla V)\cdot\nabla z\,\mathrm{d}x\\
&\to \int_{\mathbb{T}^d}(w_i\nabla(w_i\theta_i) + w_i\theta_i\nabla w_i - (-1)^{i-1}w_i^2\nabla V)\cdot\nabla z\,\mathrm{d}x\\
&\qquad = \int_{\mathbb{T}^d}(\nabla(w_i^2\theta_i) - (-1)^{i-1}w_i^2\nabla V)\cdot\nabla z\,\mathrm{d}x,
\end{aligned}
$$

where $z\in H^1(\mathbb{T}^d)$, which implies, together with (3.25), that w_i satisfy $(3.1)_1$. Integrating $(3.4)_1$ and using (3.2) yields that

$$
\int_{\mathbb{T}^d} w_i^2\,\mathrm{d}x = \int_{\mathbb{T}^d}\overline{w}_i^2\,\mathrm{d}x,\quad i=1,2,
$$

which implies (3.3), together with (3.24), (3.13), $(3.27)_3$ and (3.28). The proof of lemma is completed. □

P r o o f of Theorem 1.1. Let an arbitrary final time $T>0$ be fixed. Define the step functions $(w_i^\tau,\theta_i^\tau,V^\tau)\colon [0,T)\to L^2(\mathbb{T}^d)$ recursively as follows. Let $w_i^0=\sqrt{\varrho_{i0}}$ and for given $k\in\mathbb{N}$, let $(w_i^k,\theta_i^k,V^k)\in H^2(\mathbb{T}^d)\times L^\infty(\mathbb{T}^d)\times H^2(\mathbb{T}^d)$ be a solution to (3.1) with $\overline{w}_i=w_i^{k-1}$ (according to Lemma 3.1). Now we put

$$
w_i^\tau(t)=w_i^k,\quad \theta_i^\tau(t)=\theta_i^k,\quad V^\tau(t)=V^k,\quad (k-1)\tau<t\leqslant k\tau.
$$

These step functions satisfy

$$
\begin{aligned}
(3.29)\qquad &\frac{(w_i^\tau)^2-(\sigma_\tau w_i^\tau)^2}{\tau} + \hbar^2\nabla^2 : (w_i^\tau\nabla^2 w_i^\tau - \nabla w_i^\tau\otimes\nabla w_i^\tau)\\
&\qquad = \operatorname{div}(\nabla((w_i^\tau)^2\theta_i^\tau) - (-1)^{i-1}(w_i^\tau)^2\nabla V^\tau),\\
&-\operatorname{div}((w_i^\tau)^2\nabla\theta_i^\tau) = \frac{(w_i^\tau)^2}{\tau_e}(\bar\theta(x)-\theta_i^\tau),\quad i=1,2,\\
&\lambda^2\Delta V^\tau = (w_1^\tau)^2 - (w_2^\tau)^2 - D(x)
\end{aligned}
$$

in $\mathbb{T}^d$, where σ_τ denotes the shift operator $(\sigma_\tau w_i^\tau)(t)=w_i^\tau(t-\tau)$, $\tau\leqslant t<T$.

In order to pass to the continuum limit $\tau \to 0$ in the above system, we need the following a priori estimates.

By a similar argument as in (3.18), we obtain from (3.29) that

$$
\begin{aligned}
(3.30)\qquad & 2\hbar^2 \sum_{i=1}^{2} \int_{\mathbb{T}^d} (w_i^\tau)^2 |\nabla^2 \ln w_i^\tau|^2 \,\mathrm{d}x \\
&\quad + \frac{1}{\tau} \sum_{i=1}^{2} \int_{\mathbb{T}^d} (\phi((w_i^\tau)^2) - \phi((\sigma_\tau w_i^\tau)^2)) \,\mathrm{d}x + m \sum_{i=1}^{2} \int_{\mathbb{T}^d} |\nabla w_i^\tau|^2 \,\mathrm{d}x \\
&\leqslant c + C \sum_{i=1}^{2} \int_{\mathbb{T}^d} (\sigma_\tau w_i^\tau)^2 \,\mathrm{d}x \leqslant c + C \sum_{i=1}^{2} \int_{\mathbb{T}^d} (w_i^\tau)^2 \,\mathrm{d}x \\
&\leqslant c + C \sum_{i=1}^{2} \int_{\mathbb{T}^d} \phi((w_i^\tau)^2) \,\mathrm{d}x,
\end{aligned}
$$

where we used

$$
\int_{\mathbb{T}^d} (\sigma_\tau w_i^\tau)^2 \,\mathrm{d}x = \int_{\mathbb{T}^d} (w_i^\tau)^2 \,\mathrm{d}x
$$

and the inequality $s \leqslant \phi(s) + \mathrm{e} - 1$ for all $s \geqslant 0$. It is proved by (3.30) that $\phi((w_i^\tau)^2), |\nabla w_i^\tau|^2, (w_i^\tau)^2$ are uniformly bounded in $L^1(\mathbb{T}^d)$ for τ.

On the other hand, by (3.22) and (3.30), it is proved that Δw_i^τ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$ and w_i^τ is bounded in $L^2(0,T;H^2(\mathbb{T}^d))$. Thus,

$$
(3.31)\qquad \|w_i^\tau\|_{L^\infty(0,T;L^2(\mathbb{T}^d))} + \|w_i^\tau\|_{L^2(0,T;H^2(\mathbb{T}^d))} \leqslant K_8,
$$

where $K_8 > 0$ is independent of τ.

Also, by a similar argument as in (3.13) and (3.21), we have

$$
(3.32)\qquad \|\theta_i^\tau\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + \|w_i^\tau \theta_i^\tau\|_{L^2(0,T;H^1(\mathbb{T}^d))} \leqslant K_9,
$$

where $K_9 > 0$ is independent of τ.

Finally, by a similar argument as in [13], Lemma 4, we get

$$
\begin{aligned}
(3.33)\quad & \|(w_i^\tau)^2\|_{L^{11/10}(0,T;H^2(\mathbb{T}^d))} + \frac{1}{\tau} \|(w_i^\tau)^2 - (\sigma_\tau w_i^\tau)^2\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))} \leqslant K_{10}, \\
& \|w_i^\tau\|_{L^{8/d+2}(0,T;L^{8/d+2}(\mathbb{T}^d))} + \|V^\tau\|_{L^{4/(d+1)}(0,T;H^2(\mathbb{T}^d))} \leqslant K_{11},
\end{aligned}
$$

where K_{10} and $K_{11}$are indepemdent of τ.

The a priori estimates of the previous subsection are sufficient to pass to the limit $\tau \to 0$. By (3.31), $(3.33)_1$ and the compactness result for the imbedding $H^2(\mathbb{T}^d) \subset$

$W^{1,4}(\mathbb{T}^d)$ (see [18], Corollary 5), there exists a subsequence of $\{w_i^\tau\}$ (denoted as $\{w_i^\tau\}$ still) such that when $\tau \to 0$, it holds that

$$
\begin{aligned}
(3.34)\qquad & w_i^\tau \to \sqrt{\varrho_i}\colon \text{ *-weakly in } L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^2(\mathbb{T}^d)),\\
& (w_i^\tau)^2 \to \varrho_i\colon \text{ weakly in } L^{11/10}(0,T;H^2(\mathbb{T}^d)),\\
& \frac{1}{\tau}((w_i^\tau)^2 - (\sigma_\tau w_i^\tau)^2) \to (\varrho_i)_t\colon \text{ weakly in } L^{11/10}(0,T;H^{-2}(\mathbb{T}^d)),\\
& w_i^\tau \to \sqrt{\varrho_i}\colon \text{ strongly in } L^2(0,T;W^{1,4}(\mathbb{T}^d)),\\
& (w_i^\tau)^2 \to \varrho_i\colon \text{ strongly in } L^{11/10}(0,T;W^{1,4}(\mathbb{T}^d))
\end{aligned}
$$

satisfying

$$
(3.35)\quad \sqrt{\varrho_i} \in L^2(0,T;H^2(\mathbb{T}^d)) \cap L^\infty(0,T;L^2(\mathbb{T}^d)), \quad \varrho_i \in W^{1,11/10}(0,T;H^{-2}(\mathbb{T}^d)).
$$

By (3.34), we have

$$
\begin{aligned}
(3.36)\qquad & w_i^\tau \nabla^2 w_i^\tau \to \sqrt{\varrho_i}\nabla^2\sqrt{\varrho_i}\colon \text{ weakly in } L^1(0,T;L^2(\mathbb{T}^d)),\\
& \nabla w_i^\tau \otimes \nabla w_i^\tau \to \nabla\sqrt{\varrho_i} \otimes \nabla\sqrt{\varrho_i}\colon \text{ weakly in } L^1(0,T;L^2(\mathbb{T}^d)).
\end{aligned}
$$

Also, by $(3.33)_2$ and (3.32), we get

$$
\begin{aligned}
(3.37)\qquad & w_i^\tau \to \sqrt{\varrho_i}\colon \text{ weakly in } L^{8/d+2}(0,T;L^{8/d+2}(\mathbb{T}^d)),\\
& V^\tau \to V\colon \text{ weakly in } L^{4/(d+1)}(0,T;H^2(\mathbb{T}^d)),\\
& \theta_i^\tau \to \theta_i\colon \text{ *-weakly in } L^\infty(0,T;L^\infty(\mathbb{T}^d)),\\
& w_i^\tau\theta_i^\tau \to \sqrt{\varrho_i}\theta_i\colon \text{ *-weakly in } L^\infty(0,T;L^2(\mathbb{T}^d)) \cap L^2(0,T;H^1(\mathbb{T}^d)),\\
& \nabla((w_i^\tau)^2\theta_i^\tau) = w_i^\tau\nabla(w_i^\tau\theta_i^\tau) + w_i^\tau\theta_i^\tau\nabla w_i^\tau\\
& \qquad \to \sqrt{\varrho_i}\nabla\sqrt{\varrho_i}\theta_i + \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i} = \nabla(\varrho_i\theta_i)\colon\\
& \qquad\quad \text{weakly in } L^{4/3}(0,T;L^{4/3}(\mathbb{T}^d)),\\
& (w_i^\tau)^2\nabla V^\tau \to \varrho_i\nabla V\colon \text{ weakly in } L^{4/3}(0,T;L^2(\mathbb{T}^d))
\end{aligned}
$$

satisfying

$$
\begin{aligned}
(3.38)\qquad & V \in L^2(0,T;H^2(\mathbb{T}^d)), \quad \theta_i \in L^\infty(0,T;L^\infty(\mathbb{T}^d)),\\
& \sqrt{\varrho_i}\theta_i \in L^2(0,T;H^1(\mathbb{T}^d)), \quad \varrho_i\theta_i \in L^{4/3}(0,T;W^{1,4/3}(\mathbb{T}^d)).
\end{aligned}
$$

Finally, by $(3.33)_1$ and $(3.34)_5$, we obtain

$$
\begin{aligned}
(3.39)\qquad & \|(\sigma_\tau w_i^\tau)^2 - \varrho_i\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))}\\
& \leqslant \|(\sigma_\tau w_i^\tau)^2 - (w_i^\tau)^2\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))}\\
& \quad + \|(w_i^\tau)^2 - \varrho_i\|_{L^{11/10}(0,T;H^{-2}(\mathbb{T}^d))}\\
& \leqslant K_{10}\tau + C\|(w_i^\tau)^2 - \varrho_i\|_{L^{11/10}(0,T;L^2(\mathbb{T}^d))} \to 0.
\end{aligned}
$$

We prove that these limit functions $\{(\varrho_i, \theta_i, V)\}$ satisfy (1.8). By (3.36), we get

$$\int_0^T \int_{\mathbb{T}^d} (w_i^\tau \nabla^2 w_i^\tau - \nabla w_i^\tau \otimes \nabla w_i^\tau) : \nabla^2 \varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} (\sqrt{\varrho_i}\nabla^2\sqrt{\varrho_i} - \nabla\sqrt{\varrho_i} \otimes \nabla\sqrt{\varrho_i}) : \nabla^2\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{3.40}$$

for all $\varphi \in L^\infty(0,T;H^3(\mathbb{T}^d))$. Also, by $(3.37)_{5,6,7}$,

$$\int_0^T \int_{\mathbb{T}^d} (\nabla((w_i^\tau)^2\theta_i^\tau) - (-1)^{i-1}(w_i^\tau)^2\nabla V^\tau) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} (\nabla(\varrho_i\theta_i) - (-1)^{i-1}\varrho_i\nabla V) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \tag{3.41}$$

for all $\varphi \in L^\infty(0,T;H^3(\mathbb{T}^d))$. By using (3.40), (3.41) and $(3.34)_3$, we obtain $(1.8)_1$ from $(3.29)_1$. Using $(3.34)_4$, $(3.37)_4$ and $(3.34)_5$ yields

$$\int_0^T \int_{\mathbb{T}^d} w_i^\tau \nabla(w_i^\tau\theta_i^\tau) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t,$$

$$\int_0^T \int_{\mathbb{T}^d} w_i^\tau\theta_i^\tau \nabla w_i^\tau \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}^d} w_i^\tau(\nabla(w_i^\tau\theta_i^\tau) \cdot \nabla\varphi + w_i^\tau\theta_i^\tau\Delta\varphi) \,\mathrm{d}x\,\mathrm{d}t$$

$$\to \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}(\nabla(\sqrt{\varrho_i}\theta_i) \cdot \nabla\varphi + \sqrt{\varrho_i}\theta_i\Delta\varphi) \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i} \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^T \int_{\mathbb{T}^d} \frac{(w_i^\tau)^2}{\tau_e}(\bar\theta(x) - \theta_i^\tau) \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \frac{\varrho_i}{\tau_e}(\bar\theta(x) - \theta_i) \,\mathrm{d}x\,\mathrm{d}t,$$

respectively. Therefore, we obtain $(1.8)_2$ from $(3.29)_2$.

Finally, by $(3.37)_2$, (3.39) and $(3.37)_3$, we have

$$\int_0^T \int_{\mathbb{T}^d} \nabla V^\tau \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \nabla V \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^T \int_{\mathbb{T}^d} ((w_1^\tau)^2 - (w_2^\tau)^2 - D(x))\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} (\varrho_1 - \varrho_2 - D(x))\varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Therefore, we obtain $(1.8)_3$ from $(3.29)_3$. Moreover, by using $\varrho_i \geqslant 0$, (3.13), (3.35) and (3.38) we get (1.7). The proof of Theorem 1.1 is completed. □

4. Semiclassical limit

In this section, we will prove Theorem 1.2.

4.1. Estimates independent of $\hbar$.

Lemma 4.1. *Let the assumptions of Theorem* 1.1 *hold. Then there exists a weak solution* $\{(\varrho_i^\hbar, \theta_i^\hbar, V^\hbar)\}$ *to* (1.1), (1.5) *with* $\hbar > 0$ *satisfying the following bounds for all* $T > 0$*:*

$$
\begin{aligned}
(4.1)\qquad &\|\varrho_i^\hbar \ln \varrho_i^\hbar\|_{L^\infty(0,T;L^1(\mathbb{T}^d))} + \Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^2(0,T;H^1(\mathbb{T}^d))} \\
&\qquad + \|\varrho_i^\hbar\|_{L^2(0,T;L^2(\mathbb{T}^d))} + \hbar\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^2(0,T;H^2(\mathbb{T}^d))} \leqslant C_1, \\
&\|\theta_i^\hbar\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + \Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|_{L^2(0,T;H^1(\mathbb{T}^d))} \leqslant C_2,
\end{aligned}
$$

where C_j $(j = 1, 2)$ *are independent of* $\hbar$.

Proof. Let an arbitrary final time $\tau > 0$, $\delta > 0$ be fixed. Define the step functions $(\varrho_{i\tau\delta}^\hbar, \theta_{i\tau\delta}^\hbar, V_{\tau\delta}^\hbar) : [0, T) \to L^2(\mathbb{T}^d)$ recursively as follows. For given $k \in \mathbb{N}$, let $(w_i^k, \theta_i^k, V^k) \in H^2(\mathbb{T}^d) \times L^\infty(\mathbb{T}^d) \times H^2(\mathbb{T}^d)$ be a solution to (3.4) with $\overline{w}_i = w_i^{k-1}$ (according to Lemma 3.1). Now we put

$$
\sqrt{\varrho_{i\tau\delta}^\hbar}(t) = w_i^k, \quad \theta_{i\tau\delta}^\tau(t) = \theta_i^k, \quad V_{\tau\delta}^\tau(t) = V^k, \quad (k-1)\tau < t \leqslant k\tau.
$$

These step functions satisfy

$$
\begin{aligned}
(4.2)\qquad &\frac{\varrho_{i\tau\delta}^\hbar - \sigma_\tau \varrho_{i\tau\delta}^\hbar}{\tau} + \frac{\hbar^2}{2}\nabla^2 : (\varrho_{i\tau\delta}^\hbar \nabla^2 \ln \varrho_{i\tau\delta}^\hbar) + \delta(\Delta^2 \ln \varrho_{i\tau\delta}^\hbar + \varrho_{i\tau\delta} i^\hbar) \\
&\qquad = \operatorname{div}(\nabla(\varrho_{i\tau\delta}^\hbar \theta_{i\tau\delta}^\hbar) - (-1)^{i-1}\varrho_{i\tau\delta}^\hbar \nabla V_{\tau\delta}^\hbar), \\
&-\operatorname{div}(\varrho_{i\tau\delta}^\hbar \nabla \theta_{i\tau\delta}^\hbar) = \frac{\varrho_{i\tau\delta}^\hbar}{\tau_e}(\bar{\theta}(x) - \theta_{i\tau\delta}^\hbar), \quad i = 1, 2, \\
&\lambda^2 \Delta V_{\tau\delta}^\hbar = \varrho_{1\tau\delta}^\hbar - \varrho_{2\tau\delta}^\hbar - D(x)
\end{aligned}
$$

in $\mathbb{T}^d$, where $w_i^0 = \sqrt{\varrho_{i0}}$.

By a similar argument as in (3.30), we obtain from (4.2) that

$$
\begin{aligned}
(4.3)\quad &2\hbar^2 \sum_{i=1}^2 \int_{\mathbb{T}^d} \varrho_{i\tau\delta}^\hbar |\nabla^2 \ln \varrho_{i\tau\delta}^\hbar|^2 \,\mathrm{d}x + \delta \sum_{i=1}^2 \int_{\mathbb{T}^d} |\Delta \ln \varrho_{i\tau\delta}^\hbar|^2 \,\mathrm{d}x \\
&\qquad + \frac{\delta}{2} \sum_{i=1}^2 \int_{\mathbb{T}^d} |\ln \varrho_{i\tau\delta}^\hbar|^2 \,\mathrm{d}x + \frac{1}{\tau} \sum_{i=1}^2 \int_{\mathbb{T}^d} (\phi(\varrho_{i\tau\delta}^\hbar) - \phi(\sigma_\tau \varrho_{i\tau\delta}^\hbar)) \,\mathrm{d}x \\
&\qquad + m \sum_{i=1}^2 \int_{\mathbb{T}^d} \Big|\nabla \sqrt{\varrho_{i\tau\delta}^\hbar}\Big|^2 \,\mathrm{d}x \leqslant C(T) + c(T) \sum_{i=1}^2 \int_{\mathbb{T}^d} \phi(\varrho_{i\tau\delta}^\hbar) \,\mathrm{d}x,
\end{aligned}
$$

where $C(T)$, $c(T) > 0$ are independent of $\hbar$, τ, δ. Using (4.3), (3.22) and the inequality $s \leqslant \phi(s) + \mathrm{e} - 1$, we get

$$
\begin{aligned}
(4.4)\quad & \hbar^2 \int_0^T \int_{\mathbb{T}^d} |\Delta \varrho_{i\tau\delta}^{\hbar}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \varrho_{i\tau\delta}^{\hbar} \,\mathrm{d}x\,\mathrm{d}t \\
& \leqslant C(T)\bigg(2\hbar^2 \int_0^T \int_{\mathbb{T}^d} \varrho_{i\tau\delta}^{\hbar} |\nabla^2 \ln \varrho_{i\tau\delta}^{\hbar}|^2 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^d} \phi(\varrho_{i\tau\delta}^{\hbar}) \,\mathrm{d}x\,\mathrm{d}t + 1\bigg) \\
& \leqslant C(T).
\end{aligned}
$$

By using Gagliardo-Nirenberg inequality with $\alpha = d/(d+4)$ and (4.4), we have

$$
\begin{aligned}
(4.5)\quad & \Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\Big\|^{8/d+2}_{L^{8/d+2}(0,T;L^{8/d+2}(\mathbb{T}^d))} \\
& \leqslant C \int_0^T \Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\Big\|^{(8+2d)\alpha/d}_{H^2(\mathbb{T}^d)} \Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\Big\|^{(8+2d)(1-\alpha)/d}_{L^2(\mathbb{T}^d)} \,\mathrm{d}t \\
& \leqslant C\Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\Big\|^{(8+2d)(1-\alpha)/d}_{L^\infty(0,T;L^2(\mathbb{T}^d))} \int_0^T \Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\Big\|^2_{H^2(\mathbb{T}^d)} \,\mathrm{d}t \leqslant C(T)\hbar^{-2},
\end{aligned}
$$

which implies that $\varrho_{i\tau\delta}^{\hbar}$ is uniformly bounded in $L^{4/d+1}(0,T;L^{4/d+1}(\mathbb{T}^d))$ for τ, δ.

By the same lines, $\sigma_\tau \varrho_{i\tau\delta}^{\hbar}$ is uniformly bounded in $L^{4/d+1}(0,T;\ L^{4/d+1}(\mathbb{T}^d))$ for τ, δ. Also, by the same lines as in (3.13), (3.20) and (3.21), we have

$$
(4.6)\qquad 0 < \frac{m}{2} \leqslant \theta_{i\tau\delta}^{\hbar}(x,t) \leqslant M + 1 \quad \forall\, (x,t) \in \mathbb{T}^d \times (0,T),
$$

$$
(4.7)\qquad \int_0^T \int_{\mathbb{T}^d} \varrho_{i\tau\delta}^{\hbar} |\nabla \theta_{i\tau\delta}^{\hbar}|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C(T),
$$

and

$$
(4.8)\qquad \int_0^T \int_{\mathbb{T}^d} \Big|\nabla\Big(\sqrt{\varrho_{i\tau\delta}^{\hbar}}\theta_{i\tau\delta}^{\hbar}\Big)\Big|^2 \,\mathrm{d}x\,\mathrm{d}t \leqslant C(T),
$$

respectively.

We now will prove Lemma 4.1. Take the limit $(\tau,\delta) \to (0,0)$ in the estimates (4.3)–(4.8) independent of τ and δ. Then, by the same lines as in Theorem 1.1, there exists a subsequence of $(\varrho_{i\tau\delta}^{\hbar}, \theta_{i\tau\delta}^{\hbar}, V_{\tau\delta}^{\hbar})$ such that when $\tau \to 0$, its limit functions $(\varrho_i^{\hbar}, \theta_i^{\hbar}, V^{\hbar})$ are solutions to (1.1), (1.5). Moreover, by (4.6)–(4.8), we have

$$
\begin{aligned}
(4.9)\quad & \|\theta_i^{\hbar}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + \Big\|\sqrt{\varrho_i^{\hbar}}\theta_i^{\hbar}\Big\|_{L^2(0,T;H^1(\mathbb{T}^d))} \\
& \leqslant \liminf_{(\tau,\delta)\to(0,0)} \Big(\|\theta_{i\tau\delta}^{\hbar}\|_{L^\infty(0,T;L^\infty(\mathbb{T}^d))} + \Big\|\sqrt{\varrho_{i\tau\delta}^{\hbar}}\theta_{i\tau\delta}^{\hbar}\Big\|_{L^2(0,T;H^1(\mathbb{T}^d))}\Big) \leqslant C(T).
\end{aligned}
$$

Since $\sigma_\tau \varrho_{i\tau\delta}^\hbar$ is uniformly bounded in $L^2(0,T;L^2(\mathbb{T}^d))$ for τ,δ by (4.5), we obtain from (4.3) and (4.4), respectively, that

(4.10)
$$\begin{aligned}
&\frac{1}{\tau_e}\int_{\mathbb{T}^d}\phi(\varrho_i^\hbar)\,\mathrm{d}x + m\int_{\mathbb{T}^d}\Big|\nabla\sqrt{\varrho_i^\hbar}\Big|^2\,\mathrm{d}x + \lambda\int_{\mathbb{T}^d}(\varrho_i^\hbar)^2\,\mathrm{d}x\\
&\quad\leqslant \liminf_{(\tau,\delta)\to(0,0)}\bigg(\frac{1}{\tau_e}\int_{\mathbb{T}^d}\phi(\varrho_{i\tau\delta}^\hbar)\,\mathrm{d}x + m\int_{\mathbb{T}^d}\Big|\nabla\sqrt{\varrho_{i\tau\delta}^\hbar}\Big|^2\,\mathrm{d}x + \lambda\int_{\mathbb{T}^d}\varrho_{i\tau\delta}^\hbar\sigma_\tau\varrho_{i\tau\delta}^\hbar\,\mathrm{d}x\bigg)\\
&\quad\leqslant C(T)
\end{aligned}$$

and

(4.11)
$$\begin{aligned}
&\hbar^2\int_0^T\int_{\mathbb{T}^d}|\Delta\varrho_i^\hbar|^2\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}\varrho_i^\hbar\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leqslant \liminf_{(\tau,\delta)\to(0,0)}\bigg(\hbar^2\int_0^T\int_{\mathbb{T}^d}\Delta\varrho_{i\tau\delta}^\hbar\,\mathrm{d}x\,\mathrm{d}t + \int_0^T\int_{\mathbb{T}^d}\varrho_{i\tau\delta}^\hbar\,\mathrm{d}x\,\mathrm{d}t\bigg)\leqslant C(T),
\end{aligned}$$

respectively. The estimate (4.1) follows from (4.9)–(4.11). The proof of Lemma 4.1 is completed. □

Lemma 4.2. *It follows that*

$$\begin{aligned}
&\|\varrho_i^\hbar\theta_i^\hbar\|_{L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d))} + \|V^\hbar\|_{L^2(0,T;H^2(\mathbb{T}^d))}\leqslant C_3,\\
&\|\varrho_i^\hbar\|_{L^2(0,T;W^{1,1}(\mathbb{T}^d))} + \|(\varrho_i^\hbar)_t\|_{L^{8/7}(0,T;H^{-3}(\mathbb{T}^d))}\leqslant C_4
\end{aligned}\tag{4.12}$$

for all $T>0$, where C_j $(j=3,4)$ are independent of $\hbar$.

P r o o f. By using Gagliardo-Nirenberg inequality with $\alpha=\frac{1}{4}d$ and (4.1), we have

(4.13)
$$\begin{aligned}
&\Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|^{8/d}_{L^{8/d}(0,T;L^4(\mathbb{T}^d))}\\
&\quad\leqslant C\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|^{8\alpha/d}_{H^1(\mathbb{T}^d)}\Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|^{8(1-\alpha)/d}_{L^2(\mathbb{T}^d)}\,\mathrm{d}t\\
&\quad\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|^{8(1-\alpha)/d}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\|\theta_i^\hbar\|^{8(1-\alpha)/d}_{L^\infty(0,T;L^\infty(\mathbb{T}^d))}\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|^2_{H^1(\mathbb{T}^d)}\,\mathrm{d}t\leqslant C(T).
\end{aligned}$$

Using (4.13) and $d\leqslant 3$ yields

$$\begin{aligned}
&\|\nabla(\varrho_i^\hbar\theta_i^\hbar)\|_{L^{8/7}(0,T;L^{4/3}(\mathbb{T}^d))}\\
&\quad\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/3}(0,T;L^4(\mathbb{T}^d))}\Big\|\nabla\Big(\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big)\Big\|_{L^2(0,T;L^2(\mathbb{T}^d))}\\
&\qquad + C\Big\|\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big\|_{L^{8/3}(0,T;L^4(\mathbb{T}^d))}\Big\|\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^2(0,T;L^2(\mathbb{T}^d))}\\
&\quad\leqslant C(T).
\end{aligned}\tag{4.14}$$

By (4.13) and (4.14), we get

$$\|\varrho_i^\hbar\theta_i^\hbar\|_{L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d))}\leqslant C(T). \tag{4.15}$$

Since $\varrho_i^\hbar$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$, we obtain from the estimate of the solutions to the Poisson equation $(1.1)_3$ that

$$\|V^\hbar\|_{L^2(0,T;H^2(\mathbb{T}^d))}\leqslant C(T). \tag{4.16}$$

Since $\sqrt{\varrho_i^\hbar}$ is bounded in $L^\infty(0,T;L^2(\mathbb{T}^d))$ and because $\nabla\sqrt{\varrho_i^\hbar}$ is bounded in $L^2(0,T;L^2(\mathbb{T}^d))$, $\nabla\varrho_i^\hbar=2\sqrt{\varrho_i^\hbar}\nabla\sqrt{\varrho_i^\hbar}$ is bounded in $L^2(0,T;L^1(\mathbb{T}^d))$. Thus,

$$\|\varrho_i^\hbar\|_{L^2(0,T;W^{1,1}(\mathbb{T}^d))}\leqslant C(T). \tag{4.17}$$

We prove that $(\varrho_i^\hbar)_t$ is bounded. By the Gagliardo-Nirenberg inequality with $\alpha=\frac{1}{12}d$,

$$\begin{aligned}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^8(0,T;L^{12/5}(\mathbb{T}^d))}&\leqslant C\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{H^1(\mathbb{T}^d)}^{8\alpha}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^2(\mathbb{T}^d)}^{8(1-\alpha)}\,\mathrm{d}t\\&\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}^{8(1-\alpha)}\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{H^1(\mathbb{T}^d)}^{2d/3}\,\mathrm{d}t\\&\leqslant C(T),\end{aligned}$$

where we used $2d/3\leqslant 2$. Thus,

$$\begin{aligned}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/3}(0,T;L^{3/2}(\mathbb{T}^d))}&\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^4(0,T;L^4(\mathbb{T}^d))}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^8(0,T;L^{12/5}(\mathbb{T}^d))}\\&\leqslant C(T).\end{aligned} \tag{4.18}$$

On the other hand, by the Gagliardo-Nirenberg inequality with $\alpha=(d+12)/24$,

$$\begin{aligned}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))}^{16/5}&\leqslant C\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{H^2(\mathbb{T}^d)}^{16\alpha/5}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^2(\mathbb{T}^d)}^{16(1-\alpha)/5}\,\mathrm{d}t\\&\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^\infty(0,T;L^2(\mathbb{T}^d))}^{16(1-\alpha)/5}\int_0^T\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{H^1(\mathbb{T}^d)}^{2(d+12)/15}\,\mathrm{d}t\\&\leqslant C\|\varrho_i^\hbar\|_{L^\infty(0,T;L^1(\mathbb{T}^d))}^{8(1-\alpha)/5}\Big\|\sqrt{\varrho_i^\hbar}\Big\|_{L^2(0,T;H^2(\mathbb{T}^d))}^{2(d+12)/15}\\&\leqslant C(T)\hbar^{-2(d+12)/15}\end{aligned}$$

and by the Gagliardo-Nirenberg inequality with $\alpha = d/12$,

$$\begin{aligned}\Big\|\sqrt{\varrho_i^\hbar}\Big\|^8_{L^8(0,T;L^3(\mathbb{T}^d))} &\leqslant C\int_0^T \Big\|\sqrt{\varrho_i^\hbar}\Big\|^{8\alpha}_{H^2(\mathbb{T}^d)}\Big\|\sqrt{\varrho_i^\hbar}\Big\|^{8(1-\alpha)}_{L^2(\mathbb{T}^d)}\,\mathrm{d}t\\ &\leqslant C\Big\|\sqrt{\varrho_i^\hbar}\Big\|^{8(1-\alpha)}_{L^\infty(0,T;L^2(\mathbb{T}^d))}\int_0^T \Big\|\sqrt{\varrho_i^\hbar}\Big\|^{2d/3}_{H^1(\mathbb{T}^d)}\,\mathrm{d}t\\ &\leqslant C(T)\hbar^{-2d/3}.\end{aligned}$$

Thus, we get

$$\hbar^{2(d+12)/15}\Big\|\sqrt{\varrho_i^\hbar}\Big\|^{16/5}_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))} + \hbar^{2d/3}\Big\|\sqrt{\varrho_i^\hbar}\Big\|^8_{L^8(0,T;L^3(\mathbb{T}^d))} \leqslant C(T),$$

which implies, together with (4.1),

$$\begin{aligned}(4.19)\qquad &\hbar^2\Big\|\sqrt{\varrho_i^\hbar}\nabla^2\sqrt{\varrho_i^\hbar} - \nabla\sqrt{\varrho_i^\hbar}\otimes\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/5}(0,T;L^{6/5}(\mathbb{T}^d))}\\ &\quad\leqslant C\hbar^{(12-d)/12}\Big\|\hbar^{d/12}\sqrt{\varrho_i^\hbar}\Big\|_{L^8(0,T;L^3(\mathbb{T}^d))}\Big\|\hbar\sqrt{\varrho_i^\hbar}\Big\|_{L^2(0,T;H^2(\mathbb{T}^d))}\\ &\qquad + C\hbar^{(12-d)/12}\Big\|\hbar^{(12+d)/24}\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^{16/5}(0,T;L^{12/5}(\mathbb{T}^d))}\\ &\quad\leqslant C(T)\hbar^{(12-d)/12}.\end{aligned}$$

By (4.15) and (4.16), we have

$$\begin{aligned}(4.20)\qquad &\|\nabla(\varrho_i^\hbar\theta_i^\hbar) - (-1)^{i-1}\varrho_i^\hbar\nabla V^\hbar\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))}\\ &\quad\leqslant C\|\nabla(\varrho_i^\hbar\theta_i^\hbar)\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))}\\ &\qquad + C\|\varrho_i^\hbar\|_{L^{8/3}(0,T;L^{3/2}(\mathbb{T}^d))}\|\nabla V^\hbar\|_{L^2(0,T;L^6(\mathbb{T}^d))}\\ &\quad\leqslant C(T).\end{aligned}$$

By (4.19), (4.20) and the embbeding $L^{6/5}(\mathbb{T}^d) = (L^6(\mathbb{T}^d))^* \subset H^{-1}(\mathbb{T}^d)$, we obtain from $(1.8)_1$ that

$$\begin{aligned}(4.21)\qquad &\|(\varrho_i^\hbar)_t\|_{L^{8/7}(0,T;H^{-3}(\mathbb{T}^d))}\\ &\quad\leqslant C\hbar^2\Big\|\sqrt{\varrho_i^\hbar}\nabla^2\sqrt{\varrho_i^\hbar} - \nabla\sqrt{\varrho_i^\hbar}\otimes\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/7}(0,T;H^{-1}(\mathbb{T}^d))}\\ &\qquad + C\|\nabla(\varrho_i^\hbar\theta_i^\hbar) - (-1)^{i-1}\varrho_i^\hbar\nabla V^\hbar\|_{L^{8/7}(0,T;H^{-1}(\mathbb{T}^d))}\\ &\quad\leqslant C\hbar^2\Big\|\sqrt{\varrho_i^\hbar}\nabla^2\sqrt{\varrho_i^\hbar} - \nabla\sqrt{\varrho_i^\hbar}\otimes\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/5}(0,T;L^{6/5}(\mathbb{T}^d))}\\ &\qquad + C\|\nabla(\varrho_i^\hbar\theta_i^\hbar) - (-1)^{i-1}\varrho_i^\hbar\nabla V^\hbar\|_{L^{8/7}(0,T;L^{6/5}(\mathbb{T}^d))}\\ &\quad\leqslant C(T).\end{aligned}$$

The estimate (4.12) follows from (4.15)–(4.17) and (4.21). The proof of Lemma 4.2 is completed. □

P r o o f of Theorem 1.2. By using (4.12), (4.1) and the compactness result of the imbedding $W^{1,1}(\mathbb{T}^d) \subset L^p(\mathbb{T}^d)$, $p < \frac{3}{2}$ (see [18], Corollary 5), there exist subsequences of $\{(\varrho_i^\hbar, \theta_i^\hbar, V^\hbar)\}$ (denoted as $\{(\varrho_i^\hbar, \theta_i^\hbar, V^\hbar)\}$ still) such that when $\hbar \to 0$, it holds that

$$
\begin{aligned}
(4.22)\qquad \varrho_i^\hbar &\to \varrho_i\colon \text{ strongly in } L^2(0,T;L^p(\mathbb{T}^d)) \quad \forall\, p;\ 1 \leqslant p < \frac{3}{2},\\
\sqrt{\varrho_i^\hbar} &\to \sqrt{\varrho_i}\colon \text{ weakly in } L^2(0,T;H^1(\mathbb{T}^d)),\\
(\varrho_i^\hbar)_t &\to (\varrho_i)_t\colon \text{ weakly in } L^{8/7}(0,T;H^{-3}(\mathbb{T}^d)),\\
\theta_i^\hbar &\to \theta_i\colon \text{ *-weakly in } L^\infty(0,T;L^\infty(\mathbb{T}^d)),\\
V^\hbar &\to V\colon \text{ weakly in } L^2(0,T;H^2(\mathbb{T}^d))
\end{aligned}
$$

satisfying

$$
\begin{aligned}
(4.23)\quad &\sqrt{\varrho_i} \in L^2(0,T;H^1(\mathbb{T}^d)), \quad \varrho_i \in L^2(0,T;L^p(\mathbb{T}^d)) \cap W^{1,8/7}(0,T;H^{-3}(\mathbb{T}^d)),\\
&\theta_i \in L^\infty(0,T;L^\infty(\mathbb{T}^d)), \quad V \in L^2(0,T;H^2(\mathbb{T}^d)).
\end{aligned}
$$

Moreover, by the compactness result of the imbedding $H^2(\mathbb{T}^d) \subset W^{1,6}(\mathbb{T}^d)$, we get

$$
\begin{aligned}
(4.24)\qquad \varrho_i^\hbar \nabla V^\hbar &\to \varrho_i \nabla V\colon \text{ weakly in } L^1(0,T;L^{6p/(6+p)}(\mathbb{T}^d)),\\
\varrho_i^\hbar \theta_i^\hbar &\to \varrho_i \theta_i\colon \text{ weakly in } L^2(0,T;L^p(\mathbb{T}^d)),\\
\nabla(\varrho_i^\hbar \theta_i^\hbar) &\to \nabla(\varrho_i \theta_i)\colon \text{ weakly in } L^{8/7}(0,T;L^{4/3}(\mathbb{T}^d)),\\
\sqrt{\varrho_i^\hbar}\theta_i^\hbar &\to \sqrt{\varrho_i}\theta_i\colon \text{ weakly in } L^4(0,T;L^p(\mathbb{T}^d)),\\
\nabla\Big(\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big) &\to \nabla(\sqrt{\varrho_i}\theta_i)\colon \text{ weakly in } L^2(0,T;L^2(\mathbb{T}^d)).
\end{aligned}
$$

Thus,

$$
(4.25)\qquad \varrho_i\theta_i \in L^{8/7}(0,T;W^{1,4/3}(\mathbb{T}^d)), \quad \sqrt{\varrho_i}\theta_i \in L^2(0,T;H^1(\mathbb{T}^d)).
$$

The estimate (1.9) follows from (4.22), the estimate (1.10) follows from (4.23) and (4.25).

We prove the limit functions $\{(\varrho_i, \theta_i, V)\}$ satisfy (1.11). By $(4.24)_{1,3}$,

$$
\begin{aligned}
(4.26)\qquad &\int_0^T \int_{\mathbb{T}^d} (\nabla(\varrho_i^\hbar \theta_i^\hbar) - (-1)^{i-1} \varrho_i^\hbar \nabla V^\hbar) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t\\
&\qquad \to \int_0^T \int_{\mathbb{T}^d} (\nabla(\varrho_i\theta_i) - (-1)^{i-1} \varrho_i \nabla V) \cdot \nabla\varphi \,\mathrm{d}x\,\mathrm{d}t
\end{aligned}
$$

for all $\varphi \in L^\infty(0,T;H^3(\mathbb{T}^d))$. Also, by (4.19), we get

(4.27)
$$\begin{aligned}
&\hbar^2 \int_0^T \int_{\mathbb{T}^d} \Big(\sqrt{\varrho_i^\hbar}\nabla^2\sqrt{\varrho_i^\hbar} - \nabla\sqrt{\varrho_i^\hbar}\otimes\nabla\sqrt{\varrho_i^\hbar}\Big) : \nabla^2\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad\leqslant \hbar^2 \Big\| \sqrt{\varrho_i^\hbar}\nabla^2\sqrt{\varrho_i^\hbar} - \nabla\sqrt{\varrho_i^\hbar}\otimes\nabla\sqrt{\varrho_i^\hbar}\Big\|_{L^{8/5}(0,T;L^{6/5}(\mathbb{T}^d))} \|\varphi\|_{L^{8/3}(0,T;W^{2,6}(\mathbb{T}^d))} \\
&\quad\leqslant C(T)\hbar^{(12-d)/12}\|\varphi\|_{L^{8/3}(0,T;W^{2,6}(\mathbb{T}^d))} \to 0.
\end{aligned}$$

Therefore, we obtain $(1.11)_1$ from (4.26) and(4.27). Noticing that

$$\sqrt{\varrho_i^\hbar}\nabla\Big(\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big) \to \sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i) \text{: weakly in } L^{4/3}(0,T;L^{2p/(1+p)}(\mathbb{T}^d))$$

due to $(4.22)_1$ and $(4.24)_5$, and using $(4.24)_2$ yields that

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i^\hbar}\nabla\Big(\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\to \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}\nabla(\sqrt{\varrho_i}\theta_i)\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i^\hbar}\theta_i^\hbar\nabla\sqrt{\varrho_i^\hbar}\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i^\hbar}\Big(\nabla\Big(\sqrt{\varrho_i^\hbar}\theta_i^\hbar\Big)\cdot\nabla\varphi + \sqrt{\varrho_i^\hbar}\theta_i^\hbar\Delta\varphi\Big)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad\quad\to \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}(\nabla(\sqrt{\varrho_i}\theta_i)\cdot\nabla\varphi + \sqrt{\varrho_i}\theta_i\Delta\varphi)\,\mathrm{d}x\,\mathrm{d}t \\
&\qquad= \int_0^T \int_{\mathbb{T}^d} \sqrt{\varrho_i}\theta_i\nabla\sqrt{\varrho_i}\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t, \\
&\int_0^T \int_{\mathbb{T}^d} \frac{\varrho_i^\hbar}{\tau_e}(\bar\theta(x)-\theta_i^\hbar)\,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \frac{\varrho_i}{\tau_e}(\bar\theta(x)-\theta_i)\,\mathrm{d}x\,\mathrm{d}t.
\end{aligned}$$

Therefore, we obtain $(1.11)_2$ By using $(4.24)_1$ and $(4.22)_1$, we have

$$\int_0^T \int_{\mathbb{T}^d} \nabla V^\hbar\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} \nabla V\cdot\nabla\varphi \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\int_0^T \int_{\mathbb{T}^d} (\varrho_1^\hbar - \varrho_2^\hbar - D(x))\varphi \,\mathrm{d}x\,\mathrm{d}t \to \int_0^T \int_{\mathbb{T}^d} (\varrho_1 - \varrho_2 - D(x))\varphi \,\mathrm{d}x\,\mathrm{d}t,$$

respectively. Therefore, we obtain $(1.11)_3$. The proof of Theorem 1.2 is completed. □

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Authors' address: *Sungjin Ra* (corresponding author), *Choljin Jang*, *Jinmyong Hong*, University of Sciences, Unjong District 355, 950003, Pyongyang, Democratic People's Republic of Korea, e-mail: `sjra0702@star-co.net.kp`, `cjjang@star.co-net.kp`, `jmhong@star.co-net.kp`.