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THE MINIMAL CLOSED MONOIDS FOR THE GALOIS CONNECTION End-Con

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Abstract. The minimal nontrivial endomorphism monoids $M = \text{EndCon}(A, F)$ of congruence lattices of algebras (A, F) defined on a finite set A are described. They correspond (via the Galois connection End-Con) to the maximal nontrivial congruence lattices $\text{Con}(A, F)$ investigated and characterized by the authors in previous papers. Analogous results are provided for endomorphism monoids of quasiorder lattices $\text{Quord}(A, F)$.

Keywords: endomorphism monoid; congruence lattice; quasiorder lattice; finite algebra

MSC 2020: 08A35, 08A30, 08A60

1. PRELIMINARIES

In this short note we describe the minimal nontrivial endomorphism monoids $M = \text{EndCon}(A, F)$ of congruence lattices of algebras (A, F) defined on a finite set A .

Congruence relations (i.e., compatible equivalence relations) are one of the basic tools for the investigation of universal algebras (A, F) . A nice property of equivalence relations (or, more general, of quasiorders, i.e., reflexive and transitive relations) is that their compatibility with the operations F of an algebra depends only on their compatibility with unary polynomial functions $f \in A^A$. Thus, one can focus on unary algebras (A, F) with $F \subseteq A^A$ or even on monounary algebras (A, f) via $\text{Con}(A, F) = \bigcap_{f \in F} \text{Con}(A, f)$ (for the investigation of monounary algebras we refer to the monograph [5], but they are also discussed in numerous recent publications, of which we mention only [1], [2], [9] because these are close to our own research).

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For fixed A , the congruence lattices $\text{Con}(A, F)$ themselves form a lattice (with respect to inclusion), which can be characterized as

$$\mathcal{E}_A := \{\text{Con}(A, F) : F \subseteq A^A\},$$

and was investigated, e.g., in [7] (see also [9]). Each congruence lattice is a complete sublattice of $\text{Eq}(A)$ (the lattice of all equivalence relations), in particular, it contains the trivial congruences $\Delta_A = \{(x, x) : x \in A\}$ and $\nabla_A = A \times A$. Due to the Galois connection End-Con (see below), the endomorphism monoids $M = \text{End Con}(A, F)$ of such congruence lattices also form a lattice

$$\mathcal{M}_A := \{\text{End Con}(A, F) : F \subseteq A^A\},$$

which is dual to $\mathcal{E}_A$.

The coatoms of $\mathcal{E}_A$ (i.e., the maximal elements below the top element $\text{Eq}(A) \in \mathcal{E}_A$) were determined in [7], Theorem 4.3 as congruence lattices of the form $\text{Con}(A, f)$ for special functions f of type I, II and III (and their structure was studied in detail in [8]). It is natural to ask for the other (the monoid) side of the Galois connection, that is, to consider the atoms of $\mathcal{M}_A$, in other words, the minimal elements of $\mathcal{M}_A$ above the least element $T \in \mathcal{M}_A$. This is done in the present short note.

We explicitly describe the atoms of $\mathcal{M}_A$, i.e., $\text{End Con}(A, f)$ for these special functions f (Theorem 2.1 (A)). As shown in [6], the same functions of type I, II and III also give the coatoms in the lattice of quasiorder lattices $\text{Quord}(A, F)$, therefore we also shall characterize the corresponding atoms $\text{End Quord}(A, f)$ (Theorem 2.1 (B)).

To fix the notions and notation, recall that a binary relation $\theta \subseteq A \times A$ is *compatible* with (or *invariant* for) a function $f \in A^A$; we also say f *preserves* ϱ , denoted by $f \triangleright \varrho$, if

$$\forall x, y \in A : (x, y) \in \theta \Rightarrow (fx, fy) \in \theta.$$

Equivalently, this expresses the fact that f is an *endomorphism* of θ ($f \in \text{End } \theta$) and (provided that θ is an equivalence relation ($\theta \in \text{Eq}(A)$)) that θ is a *congruence* of the algebra (A, f) ($\theta \in \text{Con}(A, f)$).

The relation $\triangleright$ induces a Galois connection, namely End-Con , between unary mappings and equivalence relations, defined by

$$\text{End } Q := \{f \in A^A : \forall \varrho \in Q : f \triangleright \varrho\} \quad \text{for } Q \subseteq \text{Eq}(A)$$

and

$$\text{Con}(A, F) := \text{Con } F := \{\theta \in \text{Eq}(A) : \forall f \in F : f \triangleright \theta\} \quad \text{for } F \subseteq A^A.$$

The Galois closures are just the congruence lattices $\text{Con}(A, F) \in \mathcal{E}_A$ and the monoids $\text{End } Q \in \mathcal{M}_A$. Thus, we are looking for the minimal nontrivial Galois closed endomorphism monoids. For a two-element base set A there exist only two (namely

the trivial) equivalence relations. Thus, A^A is the only Galois closed endomorphism monoid. We exclude this trivial case and, from now on, make the assumption $3 \leq |A| < \infty$.

The least monoid $T \in \mathcal{M}_A$ consists of all unary functions that preserve all equivalence relations on A , that is, we have $T = \text{End Eq}(A)$. Therefore, the monoid T and the functions in it are called *trivial*. Since $3 \leq |A|$, it is known (e.g., [11], [10]) that $T := \{\text{id}_A\} \cup C_A$, where id_A is the identity mapping and C_A denotes the set of all unary constant functions on A .

We now define the above mentioned functions of type I, II and III, which play the central role for describing the minimal endomorphism monoids of congruence lattices.

Definition 1.1. A unary function $f \in A^A$ is called a function of type I, II or III, respectively, if it is *nontrivial* and satisfies the following conditions:

- (I) $f^2 = f$,
- (II) f^2 is a constant, say u , and $|\{x \in A : fx = u\}| \geq 3$,
- (III) $f^p = \text{id}_A$ for a prime p such that the permutation f has at least two cycles of length p .

Some examples of special functions f of type I and II can be seen in Figure 1.

Every function $f \in A^A$ can be presented as a (directed) graph with vertex set A and edge set $f^\bullet := \{(x, y) \in A^2 : fx = y\}$. Two elements $x, y \in A$ are called *connected* if there exist $m, n \in \mathbb{N}$ such that $f^m x = f^n y$ (see, e.g., [5], page 11). This is an equivalence relation and its equivalence classes are called (*connected*) *components* of f . A *nontrivial component* of a function f of type I is an at least two-element component of the graph of f , i.e., any at least two-element set of the form $f^{-1}(z)$ for a fixed point z of f .

2. THE ENDOMORPHISM MONOIDS OF THE COATOMS OF $\mathcal{E}_A$

As already mentioned, the coatoms of $\mathcal{E}_A$ are well-known as $\text{Con}(A, f)$ for functions of type I, II, III (see [7], Theorem 4.3). In addition, we mention that the same functions also determine the coatoms $\text{Quord}(A, f)$ in the lattice of quasiorder lattices of algebras on the base set A (see [6], Theorem 3.1).

We now consider the other side of the Galois connection End-Con and determine $\text{End Con}(A, f)$ for all coatoms $\text{Con}(A, f)$, i.e., the minimal nontrivial endomorphism monoids in $\mathcal{M}_A$. In fact, this means to determine $\text{End Con}(A, f)$ for all functions of type I, II and III. Before we give the result in Theorem 2.1, we need two more definitions.

For a function f of type I with exactly one nontrivial component (whose fixed point is denoted by z) let $\hat{f}$ be defined as follows:

$$(*) \quad \hat{f}x := \begin{cases} z & \text{if } fx = x, \\ x & \text{otherwise.} \end{cases}$$

For a function f with a 2-element image $\text{Im}(f) = \{z, u\}$, let f' be defined as follows:

$$(**) \quad f'x := \begin{cases} u & \text{if } fx = z, \\ z & \text{if } fx = u. \end{cases}$$

These functions $\hat{f}$ and f' (for f of type I or II) are shown in Figure 1, where $\{a, \dots, a'\}$ and $\{b, \dots, b'\}$ schematically represent arbitrary nonempty subsets of A (i.e., also one-element sets are allowed) while $\{b', \dots, b''\}$ is an arbitrary (possibly empty) subset. Note that $\hat{f} = f$ and $f'' = f$. It is straightforward to check that $\text{Con}(A, \hat{f}) = \text{Con}(A, f)$ and $\text{Con}(A, f') = \text{Con}(A, f)$.

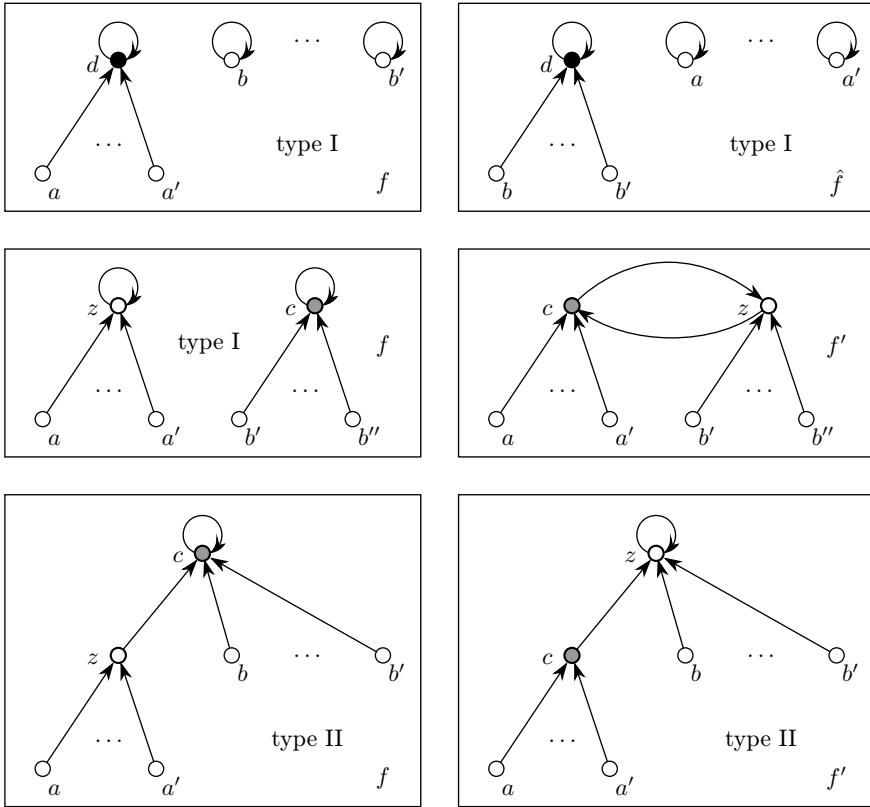


Figure 1. The functions $\hat{f}$ and f' .

Theorem 2.1. Let $3 \leq |A| < \infty$.

- (A) The following table describes the Galois closure $\text{End Con}(A, f)$ for all functions f of type I, II or III. The number s indicates the number of nontrivial functions in the closure.

	type of f	$ \text{Im}(f) $	number of nontrivial components K of f	other conditions	Galois closure $\text{End Con}(A, f)$	s
(1)	I	≥ 3	≥ 2		$\{f\} \cup T$	1
(2)	I	≥ 3	1	$ K \geq 3$	$\{f, \hat{f}\} \cup T$	2
(3)	I	≥ 3	1	$ K = 2$	$\{f, \hat{f}, (\hat{f})'\} \cup T$	3
(4)	I	2	2		$\{f, f'\} \cup T$	2
(5)	I	2	1	$ A > 3$	$\{f, f', \hat{f}\} \cup T$	3
(6)	I	2	1	$ A = 3$	$\{f, f', \hat{f}, (\hat{f})'\} \cup T$	4
(7)	II	≥ 3			$\{f\} \cup T$	1
(8)	II	2			$\{f, f'\} \cup T$	2
(9)	III			cycle length p	$\{f, f^2, \dots, f^{p-1}\} \cup T$	$p - 1$

- (B) The Galois closures $\text{End Quord}(A, f)$ for the functions of type I and II are always $\{f\} \cup T$ and for functions of type III we have $\text{End Quord}(A, f) = \text{End Con}(A, f) = \{f, f^2, \dots, f^{p-1}\} \cup T$.

Proof. (A): Observe that $g \in \text{End Con}(A, f)$ (for nontrivial g) is equivalent to $\text{Con}(A, g) = \text{Con}(A, f)$ (because $g \in \text{End Con}(A, f)$ implies $\text{Con}(A, f) \subseteq \text{Con}(A, g)$, which yields equality since $\text{Con}(A, f)$ is a coatom in $\mathcal{E}_A$ for the functions f of type I, II and III).

Let f be as described in one of the cases (1)–(8) and let $M := \text{End Con}(A, f)$. The description of the functions $g \in M \setminus T$, i.e., those with $\text{Con}(A, g) = \text{Con}(A, f) \subsetneq \text{Eq}(A)$, follows from results in [3] summarized in [3], Proposition 4.11. Moreover, we know $1 \leq |M \setminus T| \leq 4$ by [3], Proposition 4.12. The functions which appear in each line of the last but one column of the table in (A), are all different and have the same congruences as f (as mentioned above); hence, they belong to $M \setminus T$. Therefore it is enough to know the cardinality s of $M \setminus T$ (with the notation from [3] this is $|\mathcal{R}(f)|$). If it coincides with the number s in the last column, then we are done.

At first, we note that the cases (3) and (5) are equivalent: in fact, if f is a function of form (3), then $g := \hat{f}$ is of form (5) and we have $\{g, g', \hat{g}\} = \{\hat{f}, (\hat{f})', f\}$ (since $\hat{\hat{f}} = f$). Thus (3) follows from (5) (and vice versa).

Let f_i denote a function of the form (i) ($i \in \{1, 2, 4, 5, 6, 7, 8\}$). Now we shall discuss each case in more detail.

- ⊃ f_1 satisfies the conditions in [3], Lemma 3.17, therefore $s = |M \setminus T| = 1$ according to [3], Lemma 4.1.
- ⊃ f_2 satisfies the conditions in [3], Lemma 3.15, therefore $s = 2$ according to [3], Lemma 4.7.
- ⊃ f_4 satisfies the conditions in [3], Lemma 3.16 (a), therefore $s = 2$ according to [3], Lemma 4.6.
- ⊃ f_5 satisfies the conditions in [3], Lemma 3.14 (a), therefore $s = 3$ according to [3], Lemma 4.8.
- ⊃ f_6 satisfies the conditions in [3], Lemma 3.12 (b), therefore $s = 4$ according to [3], Lemma 4.9, cf. also [3], Figure 3.12.
- ⊃ f_7 satisfies the conditions in [3], Lemma 3.7, therefore $s = 1$ according to [3], Lemma 4.1.
- ⊃ f_8 satisfies the conditions in [3], Lemma 3.6, therefore $s = 2$ according to [3], Lemma 4.3.

The remaining case (9) for functions f of type III directly follows from [7], Proposition 2.8, where for permutations f of prime power order p^m it was shown that $\{g \in A^A \setminus T : \text{Con}(A, f) \subseteq \text{Con}(A, g)\} = \{f, f^2, \dots, f^{p^m-1}\}$. Here we have to take $m = 1$. We remark that the result also could be derived from results in [3], [4].

(B): Since congruences are special invariant quasiorders, for every $f \in A^A$ we have $\text{End Quord}(A, f) \subseteq \text{End Con}(A, f)$. Thus, one only has to check which functions from $\text{End Con}(A, f)$ (as described in (A)) “survive” in the possibly smaller monoid $\text{End Quord}(A, f)$. As in Part (A), for $g \in A^A$ we have $g \in \text{End Quord}(A, f)$ if and only if $\text{Quord}(A, g) = \text{Quord}(A, f)$, cf. [6].

It is straightforward to see that the quasiorder lattices $\text{Quord}(A, f)$, $\text{Quord}(A, \hat{f})$, $\text{Quord}(A, f')$ and $\text{Quord}(A, (\hat{f})')$ differ. For this one can choose suitable $(x, y) \in A^2$ such that the principal quasiorders $\alpha_f(x, y) \in \text{Quord}(A, f)$ generated by (x, y) differ for the functions f under consideration. For example, we have $(a, z) \in \alpha_{\hat{f}}(a, b) \setminus \alpha_f(a, b)$, $(u, z) \in \alpha_{f'}(a, u) \setminus \alpha_f(a, u)$ and $(a, z) \in \alpha_{(\hat{f})'}(b, a) \setminus \alpha_f(b, a)$ (for notation see Figure 1).

Concerning functions f of type III, the equality $\text{End Quord}(A, f) = \text{End Con}(A, f)$ was proved in [7], Lemma 2.7. □

Some of the results of Theorem 2.1 are already contained implicitly in [7], Proposition 4.8.

Dealing with the functions of type I, II and III, we noticed a small but interesting lattice theoretic application concerning the lattice $\mathcal{E}_A$ of congruence lattices, which one can derive directly from results in [12]. We are closing our paper with this application.

Proposition 2.2. *For a finite set A , every distributive sublattice of $\text{Eq}(A)$ containing Δ_A and ∇_A is the meet of coatoms of type I or II in $\mathcal{E}_A$.*

Proof. In [12], it is shown that any distributive sublattice of $\text{Eq}(A)$ containing Δ_A and ∇_A (denoted $\mathbf{L}'$ in [12]) can be represented as a congruence lattice $\text{Con}(A, F)$. For this representation the authors use algebras (A, F) with functions $f \in F$ which arise in the following situation: Let $\theta, \theta_1, \theta_2 \in \text{Eq}(A)$ such that $\theta_1 \not\subseteq \theta$ and $\theta \not\subseteq \theta_2$. This implies $|A| \geq 3$ and we can fix elements $a, b, c, d \in A$ with $(a, b) \in \theta \setminus \theta_2$ and $(c, d) \in \theta_1 \setminus \theta$. Then f is defined as follows:

$$fx := \begin{cases} c & \text{if } (a, x) \in \theta_2, \\ d & \text{otherwise.} \end{cases}$$

Claim. *There exist functions g, h of type I or II such that $\text{Con}(A, f) = \text{Con}(A, g) \cap \text{Con}(A, h)$ (the cases $g = h$ or $f = g = h$ are not excluded).*

We observe that only the following four cases are possible and prove the claim for each case. Note that $a \neq b, c \neq d, fa = c, fb = d$ and $\text{Im}(f) = \{c, d\}$ by definition of f , in particular, f is nontrivial.

Case 1: $(a, d) \notin \theta_2$ and $(a, c) \in \theta_2$. Then $fd = d$ and $fc = c$, thus both c and d are fixed points and every element is mapped to one of them, i.e., f is of type I (and we can take $g := h := f$).

Case 2: $(a, d) \notin \theta_2$ and $(a, c) \notin \theta_2$. Then we have $fa = c, fc = fd = d$ and $fx \in \{c, d\}$. If there exists $u \notin \{c, d\}$ with $fu = d$, then f is of type II and we are done ($g := h := f$). Otherwise, $fx = c$ for every $x \in A \setminus \{c, d\}$. Consider the maps

$$gx := \begin{cases} d & \text{if } x = d, \\ c & \text{otherwise,} \end{cases} \quad \text{and} \quad hx := \begin{cases} d & \text{if } x \in \{c, d\}, \\ x & \text{otherwise.} \end{cases}$$

Both g and h are functions of type I, see Figure 2. Moreover, $f = g \circ h$, therefore $\text{Con}(A, g) \cap \text{Con}(A, h) \subseteq \text{Con}(A, f)$. We show equality. In fact, let $\psi \in \text{Con}(A, f)$.

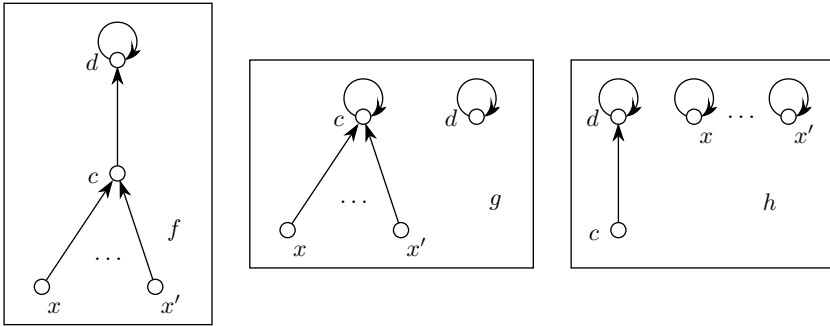


Figure 2. Functions f, g, h with $\text{Con}(A, f) = \text{Con}(A, g) \cap \text{Con}(A, h)$ (Case 2).

If $(c, d) \notin \psi$, then, as $f \triangleright \psi \not\triangleright (c, d)$, the ψ -equivalence classes of c and d do not contain elements from $A \setminus \{c, d\}$, respectively, and are hence singletons. Thus, we have $\psi = \psi_1 \cup \Delta_A$, where $\psi_1 = \psi \cap (A \setminus \{c, d\})^2$. One then immediately checks that both g and h preserve ψ .

If, on the other hand, $(c, d) \in \psi$, then clearly $g \triangleright \psi$, but we can also show $h \triangleright \psi$. To see this, take $(x, y) \in \psi$ with $x \in A \setminus \{c, d\}$, then $(hx, hy) = (x, y) \in \psi$ if $y \in A \setminus \{c, d\}$, $(hx, hy) = (x, d) \in \psi$ if $y = d$, and $(hx, hy) = (x, d)$ if $y = c$, but $(x, d) \in \psi$ by transitivity, since $(x, c), (c, d) \in \psi$; from this one concludes that $(hx, hy) \in \psi$ for every $(x, y) \in \psi$, i.e., $h \triangleright \psi$, equivalently $\psi \in \text{Con}(A, h)$. Consequently, $\psi \in \text{Con}(A, g) \cap \text{Con}(A, h)$.

Case 3: $(a, d) \in \theta_2$ and $(a, c) \in \theta_2$. Then $fd = fc = c$ and $fx \in \{c, d\}$. If there exists $u \notin \{c, d\}$ with $fu = c$, then f is of type II and we are done. Otherwise, $fx = d$ for all $x \in A \setminus \{c, d\}$ and we can proceed analogously as in Case 2 using the functions of type I

$$gx := \begin{cases} c & \text{if } x = c, \\ d & \text{otherwise,} \end{cases} \quad \text{and} \quad hx := \begin{cases} c & \text{if } x \in \{c, d\}, \\ x & \text{otherwise,} \end{cases}$$

giving $f = g \circ h$ and $\text{Con}(A, f) = \text{Con}(A, g) \cap \text{Con}(A, h)$.

Case 4: $(a, d) \in \theta_2$ and $(a, c) \notin \theta_2$. Then $fd = c$ and $fc = d$ and $fx \in \{c, d\}$ for every $x \in A$. Further, $g := f'$ (as defined in (**) above, see also Figure 1) is a function of type I and we have $\text{Con}(A, f) = \text{Con}(A, g)$ (formally put $h := g$). Thus, the claim is proved.

$\text{Con}(A, F)$ is the intersection of all $\text{Con}(A, f)$ with $f \in F$. Because of the just proved claim, $\text{Con}(A, F)$ is also the intersection of $\text{Con}(A, g)$ with functions $g \in A^A$ of type I or II only, i.e., it is the intersection of coatoms (due to the already mentioned result [7], Theorem 4.3). $\square$

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