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# SOME EXTENSIONS OF CHU'S FORMULAS AND FURTHER COMBINATORIAL IDENTITIES

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*Abstract.* We present some extensions of Chu's formulas and several striking generalizations of some well-known combinatorial identities. As applications, some new identities on binomial sums, harmonic numbers, and the generalized harmonic numbers are also derived.

*Keywords:* partial fraction decomposition; polynomial; combinatorial identity; harmonic number; generalized harmonic number; complete Bell polynomial

*MSC 2020:* 05A10, 05A19, 11B65

## 1. INTRODUCTION

First, let us recall that the generalized harmonic numbers denoted by  $H_n^{(r)}$  are defined to be partial sums of the Riemann zeta series:

$$(1.1) \quad H_0^{(r)} = 0 \quad \text{and} \quad H_n^{(r)} = \sum_{k=1}^n \frac{1}{k^r} \quad \text{for } n, r = 1, 2, \dots$$

When  $r = 1$ , these numbers reduce to the classical harmonic numbers, shortened as  $H_n = H_n^{(1)}$ .

Secondly, we recall that the complete Bell polynomials can be explicitly expressed as in [9]

$$\mathbf{B}_n(x_1, x_2, \dots, x_n) = \sum_{m_1+2m_2+\dots+nm_n=n} \frac{n!}{m_1! m_2! \dots m_n!} \left(\frac{x_1}{1!}\right)^{m_1} \left(\frac{x_2}{2!}\right)^{m_2} \dots \left(\frac{x_n}{n!}\right)^{m_n}.$$

Combinatorial identities is a classical topic in combinatorics that have always been of great importance since Euler's era. In [21], Karatsuba indicated that combinatorial

identities are used in several combinatorial problems, number theory, probability, the construction of computational algorithms, and mathematical physics. For some specific references of these applications see, for example, [1], [2], [7], [11], [14], [18], [19], [20], [24], [29], [30], [31], [33], [36].

There are various formulas and identities involving binomial coefficients. One of these combinatorial identities is

$$\frac{n!}{x(x+1)\dots(x+n)} = \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{x+j},$$

which appeared, for example, in [12], page 3, [14], equation (1.41) and [16], page 188. In recent years, there has been considerable interest in providing simple probabilistic proofs for this identity (see, for example, [26], [27], [32], [34]).

The second identity is the formula

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} j^n = n!.$$

In the literature, this identity is usually called the Boole formula because it appears in Boole's classical book (see [6]). Actually, it goes back to Euler, so Gould (see [15]) renamed it to Euler's formula.

In recent decades, Euler's formula has received a regain of interest, therefore several papers have been devoted to provide new proofs. The interested reader can consult [1], [3], [4], [5], [13], [17], [22], [28].

Involving complex numbers, Katsuura (in [22]) generalized Euler's formula as

$$\sum_{j=0}^n (-1)^j \binom{n}{j} (b+aj)^l = \begin{cases} 0 & \text{if } 0 \leq l < n, \\ (-1)^n a^n n! & \text{if } l = n, \end{cases}$$

where  $a$  and  $b$  are two complex numbers.

Extending Katsuura's formula, Pohoata (see [28]) considered the following identity in terms of polynomials with real coefficients:

$$\sum_{j=0}^n (-1)^{n-j} \binom{n}{j} P(\alpha + \beta j) = \beta^n n! a_n,$$

where  $P(x)$  is a polynomial of degree  $n$  with leading coefficient  $a_n$ .

In [8], among other results, Chu established for any two natural numbers  $\lambda$  and  $\theta$  with  $0 \leq \theta < \lambda(n+1)$  the partial fraction decompositions of the two rational functions

$$\frac{1}{x^\lambda(x+1)^\lambda \dots (x+n)^\lambda} \quad \text{and} \quad \frac{x^\theta}{x^\lambda(x+1)^\lambda \dots (x+n)^\lambda},$$

then he obtained several striking harmonic number identities and recovered a conjectured identity due to Weideman (see [35]):

$$(1.2) \quad \sum_{k=0}^n (-1)^k \binom{n}{k}^3 (3(H_k - H_{n-k})^2 + (H_k^{(2)} + H_{n-k}^{(2)})) = 0.$$

In [10], Driver and his collaborators confirmed this formula via computer algebra and symbolic calculus. It is important to note that Weideman (see [35]) declared that this formula is one of the hardest challenges among algebraic identities.

In [37], Zhu and Luo rewrote these two identities of Chu (see [8]) in another form as

$$(1.3) \quad \frac{1}{x^\lambda(x+1)^\lambda \dots (x+n)^\lambda} = \sum_{k=0}^n \frac{(-1)^{k\lambda}}{(n!)^\lambda} \binom{n}{k}^\lambda \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(x_1, x_2, \dots, x_j)}{j! (x+k)^{\lambda-j}},$$

and for  $\lambda \leq M < \lambda(n+1)$ ,

$$(1.4) \quad \frac{x^M}{x^\lambda(x+1)^\lambda \dots (x+n)^\lambda} = \sum_{k=0}^n \frac{(-1)^{k\lambda+M}}{(n!)^\lambda} \binom{n}{k}^\lambda k^M \sum_{j=0}^{\lambda-1} \frac{\mathbf{B}_j(x_1, x_2, \dots, x_j)}{j! (x+k)^{\lambda-j}},$$

and gave a novel proof of these two main results of Chu (see [8]) using an appropriate contour integral and Cauchy's residue theorem.

Motivated by these results, our purpose is to establish the following general combinatorial identities which are a common generalization of these important works introduced before.

Let  $m$  and  $n$  be two positive integers. Let  $P(x) = x^m(x+1)^m(x+2)^m \dots (x+n)^m$  and  $Q(x) \in \mathbb{C}[x]$  be two polynomials such that  $\deg(Q) < m(n+1)$ . Then the following algebraic identity holds true:

$$\frac{(n!)^m Q(x)}{P(x)} = \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^k \mathbf{B}_k(x_1, x_2, \dots, x_k) Q^{(i)}(-j)}{i! k! (x+j)^{m-i-k}},$$

where

$$x_l = m(l-1)! (H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

Here and further,  $Q^{(i)}(x)$  denotes the  $i$ th derivative of  $Q(x)$ .

In addition, if  $Q(x) \in \mathbb{C}[x]$  is a polynomial of degree  $l$  with leading coefficient  $a_l$ , then we have the following identity:

$$\begin{aligned} \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \frac{(-1)^{m-1-i} \mathbf{B}_{m-1-i}(x_1, x_2, \dots, x_{m-1-i}) Q^{(i)}(-j)}{i! (m-1-i)!} \\ = \begin{cases} 0 & \text{if } 0 \leq l < m(n+1) - 1, \\ (n!)^m a_l & \text{if } l = m(n+1) - 1. \end{cases} \end{aligned}$$

Consequently, we obtain

$$\begin{aligned} & \sum_{j=0}^n (-1)^j \binom{n}{j}^3 \left( \frac{Q^{(2)}(-j)}{2} - 3Q'(-j)(H_{n-j} - H_j) \right. \\ & \quad \left. + \frac{3Q(-j)}{2}(3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)})) \right) \\ & = \begin{cases} 0 & \text{if } 0 \leq l < 3n+2, \\ (n!)^3 a_l & \text{if } l = 3n+2. \end{cases} \end{aligned}$$

Setting  $Q(x) = 1$ , the last expression reduces to the conjectured identity of Weideman (see [35]):

$$\sum_{j=0}^n (-1)^j \binom{n}{j}^3 (3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)})) = 0.$$

## 2. PRELIMINARIES AND THE PROOF OF THE MAIN IDENTITIES

We first formulate the following important result.

**Theorem 2.1.** *Let  $\alpha_1, \alpha_2, \dots, \alpha_s$  be distinct elements in the field of complex numbers  $\mathbb{C}$ . For a positive integer  $m$ , let  $P(x) = (x - \alpha_1)^m (x - \alpha_2)^m \dots (x - \alpha_s)^m$ . For any polynomial  $Q(x) \in \mathbb{C}[x]$  with  $\deg(Q) < \deg(P)$ , we have*

$$(2.1) \quad \frac{Q(x)}{P(x)} = \sum_{j=1}^s \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^k g_j(\alpha_j) \mathbf{B}_k(x_1, \dots, x_k) Q^{(i)}(\alpha_j)}{i! k! (x - \alpha_j)^{m-i-k}},$$

where

$$x_l = m(l-1)! \sum_{i=1, i \neq j}^s \frac{1}{(\alpha_j - \alpha_i)^l} \quad \text{and} \quad g_j(x) = \prod_{i=1, i \neq j}^s (x - \alpha_i)^{-m_i}.$$

**Proof.** From [25], equation (4) we have

$$Q(x) = \sum_{j=1}^s \sum_{i=0}^{m-1} \frac{1}{i!} Q^{(i)}(\alpha_j) L_{ji}(x) [P],$$

where

$$L_{ji}(x) [P] = P_j(x) (x - \alpha_j)^i \sum_{k=0}^{m-1-i} \frac{1}{k!} g_j^{(k)}(\alpha_j) (x - \alpha_j)^k$$

and

$$P_j(x) = \prod_{i=1, i \neq j}^s (x - \alpha_i)^m = \frac{P(x)}{(x - \alpha_j)^m}, \quad g_j(x) = (P_j(x))^{-1}.$$

As a consequence, we obtain the identity

$$L_{ji}(x)[P] = P(x) \sum_{k=0}^{m-1-i} \frac{g_j^{(k)}(\alpha_j)}{k! (x - \alpha_j)^{m-i-k}}.$$

Therefore, by combining these identities, we can write

$$\frac{Q(x)}{P(x)} = \sum_{j=1}^s \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{g_j^{(k)}(\alpha_j) Q^{(i)}(\alpha_j)}{i! k! (x - \alpha_j)^{m-i-k}}.$$

On the other hand, we have

$$g_j(x) = \varphi(x) \circ f_j(x),$$

where  $\varphi(x) = \exp(mx)$  and  $f_j(x) = \ln\left(\prod_{i=1, i \neq j}^s (x - \alpha_i)^{-1}\right)$ . It is clear that  $\varphi^{(k)}(x) = m^k \exp(mx)$  and  $f_j^{(k)}(x) = (-1)^k (k-1)! \mathcal{H}_{k, \alpha_s[j]}(x)$ , where  $\mathcal{H}_{l, \alpha_s[j]}(x) = \sum_{i=1, i \neq j}^s 1/(x - \alpha_i)^l$ . Now by applying the Faà di Bruno formula [23], equation (1.13), we get

$$g_j^{(k)}(x) = (-1)^k g_j(x) \sum_{m_1+2m_2+\dots+km_k=k} \frac{k!}{m_1! m_2! \dots m_k!} \prod_{l=1}^k \left( \frac{m(l-1)! \mathcal{H}_{l, \alpha_s[j]}(x)}{l!} \right)^{m_l}.$$

It follows that

$$g_j^{(k)}(\alpha_j) = (-1)^k g_j(\alpha_j) \mathbf{B}_k(x_1, \dots, x_k).$$

Therefore, the rest follows easily.  $\square$

Theorem 2.1 has the following corollary.

**Corollary 2.1.** *Let  $m$  and  $n$  be two positive integers. Let  $P(x) = x^m \times (x-1)^m \dots (x-n)^m$  and  $Q(x) \in \mathbb{C}[x]$  be two polynomials such that  $\deg(Q) < \deg(P)$ . We have*

$$\frac{(n!)^m Q(x)}{P(x)} = \sum_{j=0}^n \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \dots, x_k) Q^{(i)}(j)}{i! k! (x-j)^{m-i-k}},$$

where

$$x_l = m(l-1)! (H_j^{(l)} + (-1)^l H_{n-j}^{(l)}).$$

In particular, we get

$$\frac{(n!)^m x^l}{P(x)} = \sum_{j=0}^n \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{l}{i} j^{l-i} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \dots, x_k)}{k! (x-j)^{m-i-k}},$$

where  $1 \leq l < m(n+1)$ , and

$$\frac{(n!)^m}{P(x)} = \sum_{j=0}^n \binom{n}{j}^m \sum_{k=0}^{m-1} \frac{(-1)^{m(n-j)+k} \mathbf{B}_k(x_1, x_2, \dots, x_k)}{k! (x-j)^{m-k}}.$$

According to the expression of Theorem 2.1, we can easily obtain the following result.

**Corollary 2.2.** *Let  $m$  and  $n$  be two positive integers. Let  $P(x) = x^m(x+1)^m \times (x+2)^m \dots (x+n)^m$  and  $Q(x) \in \mathbb{C}[x]$  be two polynomials such that  $\deg(Q) < m(n+1)$ . The following algebraic identity holds true:*

$$(2.2) \quad \frac{(n!)^m Q(x)}{P(x)} = \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \frac{(-1)^k \mathbf{B}_k(x_1, x_2, \dots, x_k) Q^{(i)}(-j)}{i! k! (x+j)^{m-i-k}},$$

where

$$x_l = m(l-1)! (H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

By multiplying both sides of (2.2) by  $x$  and letting  $x$  to  $\infty$ , we obtain the following result.

**Theorem 2.2.** *Let  $m$  and  $n$  be two positive integers. Let  $Q(x) \in \mathbb{C}[x]$  be a polynomial of degree  $l$  with leading coefficient  $a_l$ . Then we have the following identity:*

$$\begin{aligned} \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \frac{(-1)^{m-1-i} \mathbf{B}_{m-1-i}(x_1, x_2, \dots, x_{m-1-i}) Q^{(i)}(-j)}{i! (m-1-i)!} \\ = \begin{cases} 0 & \text{if } 0 \leq l < m(n+1) - 1, \\ (n!)^m a_l & \text{if } l = m(n+1) - 1, \end{cases} \end{aligned}$$

where

$$x_l = m(l-1)! (H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

Setting  $m = 1, 2, 3$  in Theorem 2.2, we gain the following identities.

**Corollary 2.3.** *Let  $n$  be a positive integer and  $Q(x) \in \mathbb{C}[x]$  be a polynomial of degree  $l$  with leading coefficient  $a_l$ . Then we have*

(a) for  $m = 1$

$$(2.3) \quad \sum_{j=0}^n (-1)^j \binom{n}{j} Q(-j) = \begin{cases} 0 & \text{if } 0 \leq l < n, \\ n! a_l & \text{if } l = n, \end{cases}$$

(b) for  $m = 2$

$$(2.4) \quad \sum_{j=0}^n \binom{n}{j}^2 (Q'(-j) - 2(H_{n-j} - H_j)Q(-j)) = \begin{cases} 0 & \text{if } 0 \leq l < 2n + 1, \\ (n!)^2 a_l & \text{if } l = 2n + 1, \end{cases}$$

(c) for  $m = 3$

$$(2.5) \quad \sum_{j=0}^n (-1)^j \binom{n}{j}^3 \left( \frac{Q^{(2)}(-j)}{2} - 3Q'(-j)(H_{n-j} - H_j) \right. \\ \left. + \frac{3Q(-j)}{2} (3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)})) \right) \\ = \begin{cases} 0 & \text{if } 0 \leq l < 3n + 2, \\ (n!)^3 a_l & \text{if } l = 3n + 2. \end{cases}$$

**Remark 2.1.** In the following, we derive Euler's formula, Katsuura's formula, and Pohoata's formula.

- ▷ When  $Q(x) = x^n$ , identity (2.3) gives Euler's formula.
- ▷ When  $Q(x) = (b - ax)^l$ , identity (2.3) reduces to Katsuura's formula.
- ▷ Setting  $Q(x) = P(\alpha - \beta x)$  in identity (2.3), we obtain Pohoata's formula.

The following example is an illustration of (2.4).

**Example 2.1.** Choose  $Q(x) = x$  and  $Q(x) = x^{2n+1}$  in (2.4), we derive

$$\sum_{j=0}^n \binom{n}{j}^2 (1 + 2j(H_{n-j} - H_j)) = 0$$

and

$$\sum_{j=0}^n \binom{n}{j}^2 ((2n+1)j^{2n} + 2j^{2n+1}(H_{n-j} - H_j)) = (n!)^2,$$

respectively.

**Example 2.2.** By choosing special polynomials in Corollary 2.3, we obtain interesting identities.

▷ Setting  $Q(x) = 1$ , the expression of (2.5) becomes

$$\sum_{j=0}^n (-1)^j \binom{n}{j}^3 (3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)})) = 0.$$

The last identity is declared as one of the hardest challenges among identities. It is conjectured by [35], equation (20) and proved in [10] by means of symbolic calculus and computer algebra package Sigma.

▷ When  $Q(x) = x$ , the formula of (2.5) reduces to the identity

$$\sum_{j=0}^n (-1)^j \binom{n}{j}^3 (2(H_{n-j} - H_j) + j(3(H_{n-j} - H_j)^2 + (H_{n-j}^{(2)} + H_j^{(2)}))) = 0.$$

When we set  $Q(x) = 1$  and  $Q(x) = x^\theta$ , where  $\theta$  is a positive integer, in the formula (2.2), we can easily reformulate the two algebraic identities appeared in the work of Chu (see [8]), anticipated at the beginning of this paper, as follows.

**Corollary 2.4.** *Let  $m, n$  and  $\theta$  be three positive integers with  $0 \leq \theta < m(n+1)$ . Then*

$$(2.6) \quad \frac{(n!)^m x^\theta}{x^m(x+1)^m \dots (x+n)^m} = \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{i=0}^{m-1} \sum_{k=0}^{m-1-i} \binom{\theta}{i} j^{\theta-i} \frac{(-1)^{k+\theta-i} \mathbf{B}_k(x_1, x_2, \dots, x_k)}{k! (x+j)^{m-i-k}},$$

where

$$x_l = m(l-1)! (H_{n-j}^{(l)} + (-1)^l H_j^{(l)}).$$

In particular, we have

$$(2.7) \quad \frac{(n!)^m}{x^m(x+1)^m \dots (x+n)^m} = \sum_{j=0}^n (-1)^{jm} \binom{n}{j}^m \sum_{k=0}^{m-1} \frac{(-1)^k \mathbf{B}_k(x_1, x_2, \dots, x_k)}{k! (x+j)^{m-k}}.$$

According to (2.6), we can provide a list of identities, for example the following two examples.

**Example 2.3.** Let  $n$  and  $\theta$  be two positive integers with  $\theta < n + 1$ . Then

$$\frac{n! x^\theta}{x(x+1) \dots (x+n)} = \sum_{j=0}^n \binom{n}{j} (-1)^{j+\theta} \frac{j^\theta}{(x+j)}.$$

The last identity appeared in the work of Chu (see [8], Example 1).

**Example 2.4.** Let  $n$  and  $\theta$  be two positive integers with  $\theta < 2(n + 1)$ . Then

$$\frac{(n!)^2 x^\theta}{x^2(x+1)^2 \dots (x+n)^2} = \sum_{j=0}^n \binom{n}{j}^2 \left( \frac{(-1)^\theta j^\theta}{(x+j)^2} - (-1)^\theta \frac{j^\theta (H_{n-j} - H_j) + \theta j^{\theta-1}}{(x+j)} \right)$$

and the corresponding harmonic number identity is

$$\sum_{j=0}^n j^{\theta-1} \binom{n}{j}^2 (\theta - 2j(H_j - H_{n-j})) = \begin{cases} 0 & \text{if } 0 \leq \theta < 2n + 1, \\ (n!)^2 & \text{if } \theta = 2n + 1. \end{cases}$$

We note that the last formula has been conjectured by Weideman in [35], equation (11) and proved in [10], Theorem 1 and recovered by Chu in [8], Example 2.

According to (2.7), we can obtain several expansion expressions involving the generalized harmonic numbers, for example for  $m = 2, 3$ , one obtains the following two identities.

**Remark 2.2.**

$$\begin{aligned} \frac{(n!)^2}{x^2(x+1)^2 \dots (x+n)^2} &= \sum_{j=0}^n \binom{n}{j}^2 \left( \frac{1}{(x+j)^2} - \frac{2}{(x+j)} (H_{n-j} - H_j) \right), \\ \sum_{j=0}^n (-1)^j \binom{n}{j}^3 &\left( \frac{1}{(x+j)^3} - \frac{3(H_{n-j} - H_j)}{(x+j)^2} + \frac{9(H_{n-j} - H_j)^2 + 3(H_{n-j}^{(2)} + H_j^{(2)})}{(x+j)} \right) \\ &= \frac{(n!)^3}{x^3(x+1)^3 \dots (x+n)^3}. \end{aligned}$$

**Corollary 2.5.** Let  $n$  be a positive integer. Let  $Q(x) \in \mathbb{C}[x]$  be a polynomial such that  $\deg(Q) < n + 1$ . We have

$$(2.8) \quad \frac{n! Q(x)}{x(x+1) \dots (x+n)} = \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j Q(-j)}{(x+j)}.$$

When  $Q(x) = x^l$ ,  $l = 0, 1, \dots, n$ , formula (2.8) reduces to

$$\frac{n! x^l}{x(x+1) \dots (x+n)} = \sum_{j=0}^n \binom{n}{j} \frac{(-1)^{j+l} j^l}{(x+j)}.$$

As a consequence, we recover the well-known identity (see, for example, [12], page 3, [14], equation (1.41), and [16], page 188):

$$\frac{n!}{x(x+1) \dots (x+n)} = \sum_{j=0}^n \binom{n}{j} \frac{(-1)^j}{x+j}.$$

### 3. CONCLUSION

In this concluding section, we encourage the interested reader to develop the results of this paper and examine other important algebraic identities.

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