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COTORSION PAIRS IN COMMA CATEGORIES

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Abstract. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projective and injective objects, and $T\colon \mathcal{A} \to \mathcal{B}$ a left exact additive functor. Then one has a comma category $(\mathcal{B} \downarrow T)$. It is shown that if $T\colon \mathcal{A} \to \mathcal{B}$ is $\mathcal{X}$-exact, then $({}^{\perp}\mathcal{X}, \mathcal{X})$ is a (hereditary) cotorsion pair in $\mathcal{A}$ and $({}^{\perp}\mathcal{Y}, \mathcal{Y}))$ is a (hereditary) cotorsion pair in $\mathcal{B}$ if and only if $\big(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}\big), \langle \mathbf{h}(\mathcal{X}, \mathcal{Y})\rangle)$ is a (hereditary) cotorsion pair in $(\mathcal{B} \downarrow T)$ and $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions. Furthermore, we characterize when special preenveloping classes in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce special preenveloping classes in $(\mathcal{B} \downarrow T)$.

Keywords: comma category; cocompatible functor; cotorsion pair

MSC 2020: 18A25, 16E30, 18G25

1. Introduction

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $T\colon \mathcal{A} \to \mathcal{B}$ be a left exact additive functor and $G\colon \mathcal{B} \to \mathcal{A}$ a right exact additive functor. It follows from Fossum, Griffith and Reiten (see [4]) and Marmaridis (see [10]) that there are two abelian comma categories $(\mathcal{B} \downarrow T)$ and $(G \downarrow \mathcal{A})$. Notice that module categories of upper triangular matrix rings are comma categories. For each recollement of triangulated categories, it follows from Chen and Le (see [2]) that there is an epivalence (provided that it is full and essentially surjective, and detects isomorphisms between objects; see Gabriel and Roĭter [5]) between the middle category and the comma category associated with a triangle functor from the category on the right to the category on the left. Hu and

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Zhu in [8] characterize when special precovering classes in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce special precovering classes in comma category $(G \downarrow \mathcal{A})$.

In this paper, we study the property of the extension closure of some classes of objects in $(\mathcal{B} \downarrow T)$, the exactness of the functor $\mathbf{h}$ and the detailed description of orthogonal classes of a given class $\mathbf{h}(\mathcal{X}, \mathcal{Y})$ in $(\mathcal{B} \downarrow T)$. Moreover, we characterize when complete hereditary cotorsion pairs in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce complete hereditary cotorsion pairs in $(\mathcal{B} \downarrow T)$. Furthermore, we characterize when special preenveloping classes in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce special preenveloping classes in $(\mathcal{B} \downarrow T)$.

2. Notation and terminology

Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ are abelian categories. Let $\mathcal{X}$ be a subclass of $\mathcal{A}$. Set

$$^{\perp}\mathcal{X} = \{M \in \mathcal{A}\colon \operatorname{Ext}^1_{\mathcal{A}}(M, X) = 0 \text{ for every } X \in \mathcal{X}\},$$
$$\mathcal{X}^{\perp} = \{M \in \mathcal{A}\colon \operatorname{Ext}^1_{\mathcal{A}}(X, M) = 0 \text{ for every } X \in \mathcal{X}\}.$$

Let $f\colon X \to M$ be a morphism. It is called *special $\mathcal{X}$-precover* of M if f is surjective, $X \in \mathcal{X}$ and $\ker f \in \mathcal{X}^{\perp}$. Dually, a morphism $g\colon M \to Y$ is called *special $\mathcal{Y}$-preenvelop* of M if g is injective, $Y \in \mathcal{Y}$ and $\operatorname{coker} g \in {}^{\perp}\mathcal{Y}$. The class $\mathcal{X}$ is called *special-precovering* (or *special-preevenloping*) if every object in $\mathcal{A}$ has *special $\mathcal{X}$-precover* (or *special $\mathcal{X}$-preenvelop*).

Let $\mathcal{X}$ and $\mathcal{Y}$ be subclasses of $\mathcal{A}$. Recall from [11] that a pair $(\mathcal{X}, \mathcal{Y})$ is called a *cotorsion pair* if $\mathcal{X} = {}^{\perp}\mathcal{Y}, \mathcal{Y} = \mathcal{X}^{\perp}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *hereditary* (see [7]) if $\operatorname{Ext}^i_{\mathcal{A}}(X, Y) = 0$ for all $i \geqslant 1$, $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, equivalently, if whenever $0 \to X_1 \to X_2 \to X_3 \to 0$ is exact with $X_2, X_3 \in \mathcal{X}$, then X_1 is also in $\mathcal{X}$, or equivalently, if whenever $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ is exact with $Y_1, Y_2 \in \mathcal{Y}$, then Y_3 is also in $\mathcal{Y}$. A cotorsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be *complete* (see [6]) if $\mathcal{X}$ is special precovering and $\mathcal{Y}$ is special preenveloping. Moreover, if $\mathcal{A}$ has enough projective objects, the condition that $(\mathcal{X}, \mathcal{Y})$ is complete is equivalent to that $\mathcal{Y}$ is special preenveloping. Similarly, if $\mathcal{A}$ has enough injective objects, the condition that $(\mathcal{X}, \mathcal{Y})$ is complete is equivalent to that $\mathcal{X}$ is special precovering. Let $\mathcal{X}$ be a subclass of $\mathcal{A}$. The functor $T\colon \mathcal{A} \to \mathcal{B}$ is called *$\mathcal{X}$-exact* if T preserves exactness of exact sequence $0 \to X \to A_2 \to A_1 \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$.

Let $\mathcal{A}$ be an abelian category and X a complex in $\mathcal{A}$. With homological grading, X as the form

$$\ldots \longrightarrow X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \longrightarrow \ldots.$$

We use the notations $\mathrm{Z}_i(X)$ for the kernel of differential ∂_i^X and $\mathrm{C}_i(X)$ for the cokernel of the differential ∂_{i+1}^X. A complex X is called *acyclic* if the homology complex $\mathrm{H}(X)$ is the zero complex. A complex $\mathbf{I}$ of injective objects is called *totally acyclic* if it is acyclic and $\mathrm{Hom}_{\mathcal{A}}(E, \mathbf{I})$ is acyclic for any injective object E in $\mathcal{A}$. An object G in $\mathcal{A}$ is called *Gorenstein injective* if one has $G \cong \mathrm{Z}_0(\mathbf{I})$ for a totally acyclic complex $\mathbf{I}$ of injective objects in $\mathcal{A}$. In what follows, we denote by $\mathcal{GI}_{\mathcal{A}}$ the subcategory of $\mathcal{A}$ consisting of Gorenstein injective objects.

Let $\mathcal{X}$ be a class of objects in an abelian category $\mathcal{A}$. A class $\mathcal{X}$ is said to be *closed under extensions* if whenever $0 \to X_1 \to X_2 \to X_3 \to 0$ is an exact sequence in $\mathcal{A}$ with $X_1, X_3 \in \mathcal{X}$, then $X_2 \in \mathcal{X}$.

Recall the following definition of comma categories from [4], [10].

Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, and $T\colon \mathcal{A} \to \mathcal{B}$ a left exact additive functor. The comma category, denoted by $(\mathcal{B} \downarrow T)$, is defined as follows. The objects of the category are triplets $\binom{A}{B}_{\varphi}$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$ and $\varphi \in \mathrm{Hom}_{\mathcal{B}}(B, TA)$, and the morphisms of the category are $\binom{\alpha}{\beta}$ in $\mathcal{A} \times \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\beta} & B' \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} \\ TA & \xrightarrow{T\alpha} & TA' \end{array}$$

is commutative.

3. Homological behavior of the functor $\mathbf{h}$

Firstly, recall the following definition from [4], [10].

Definition 3.1. Let $T\colon \mathcal{A} \to \mathcal{B}$ be a left exact additive functor. Then we have the following functor: $\mathbf{h}\colon \mathcal{A} \times \mathcal{B} \to (\mathcal{B} \downarrow T)$ via $\mathbf{h}(A, B) = \binom{A}{B \oplus TA}$ and $\mathbf{h}(\alpha, \beta) = \binom{\alpha}{\beta \oplus T\alpha}$, where (A, B) is an object in $\mathcal{A} \times \mathcal{B}$ and (α, β) is a morphism in $\mathcal{A} \times \mathcal{B}$.

Remark 3.2.

(1) $\mathbf{h}(A, B) = \mathbf{h}(A, 0) \oplus \mathbf{h}(0, B)$ with $A \in \mathcal{A}$, $B \in \mathcal{B}$. Moreover, $\mathbf{h}$ preseres injective objects if $\mathcal{A}$ and $\mathcal{B}$ have enough injective objects.

(2) Define $\mathbf{q}\colon (\mathcal{B} \downarrow T) \to \mathcal{A} \times \mathcal{B}$ via $\mathbf{q}\binom{A}{B} = (A, B)$ and $\mathbf{q}\binom{\alpha}{\beta} = (\alpha, \beta)$, where $\binom{A}{B}$ is an object in $(\mathcal{B} \downarrow T)$ and $\binom{\alpha}{\beta}$ is a morphism in $(\mathcal{B} \downarrow T)$. Then $\mathbf{q}$ is a left adjoint of $\mathbf{h}$. Hence, $\mathbf{h}$ is a left exact functor.

(3) $\mathbf{h}(\mathcal{X}, \mathcal{Y}) = \{\mathbf{h}(X, Y)\colon\ X \in \mathcal{X} \text{ and } Y \in \mathcal{Y}\}$.

Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $(\mathcal{B}\downarrow T)$ the comma category, $\mathcal{X}$ and $\mathcal{Y}$ subclasses of $\mathcal{A}$ and $\mathcal{B}$, respectively. For convenience, we set

$$\mathfrak{D}^{\mathcal{X}}_{\mathcal{Y}} = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \in (\mathcal{B}\downarrow T)\colon\ X\in\mathcal{X},\ \varphi \text{ is epimorphism and } \ker\varphi\in\mathcal{Y} \right\},$$

$$\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle = \left\{ \begin{pmatrix} X \\ Y \end{pmatrix}\colon \text{ there is an exact sequence } 0\to \begin{pmatrix} X' \\ Y' \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix} \to \begin{pmatrix} X'' \\ Y'' \end{pmatrix} \to 0, \right.$$
$$\left. \text{where } \begin{pmatrix} X' \\ Y' \end{pmatrix} \text{ and } \begin{pmatrix} X'' \\ Y'' \end{pmatrix} \text{ are objects in } \mathbf{h}(\mathcal{X},\mathcal{Y}) \right\}.$$

Proposition 3.3. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $\mathcal{X}$ be a subclass of $\mathcal{A}$ with $0\in\mathcal{X}$, $\mathcal{Y}$ be a subclass of $\mathcal{B}$ with $0\in\mathcal{Y}$. Consider the following conditions:*

(1) *$\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions in $\mathcal{A}$ and $\mathcal{B}$, respectively.*

(2) *$\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle = \mathfrak{D}^{\mathcal{X}}_{\mathcal{Y}}$ is the smallest subclass of $(\mathcal{B}\downarrow T)$ containing $\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ and closed under extensions.*

(3) *$\mathfrak{D}^{\mathcal{X}}_{\mathcal{Y}}$ is closed under extensions.*

Then one has (1) $\Rightarrow$ (2) $\Rightarrow$ (3). The converse holds if $T\colon\ \mathcal{A}\to\mathcal{B}$ is $\mathcal{X}$-exact.

P r o o f. (1) $\Rightarrow$ (2) Let $\binom{X}{Y}_{\varphi}\in\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$. There is an exact sequence by definition

$$0\to \begin{pmatrix} X' \\ Y'\oplus TX' \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \to \begin{pmatrix} X'' \\ Y''\oplus TX'' \end{pmatrix} \to 0,$$

which yields an exact sequence $0\to X'\to X\to X''\to 0$ in $\mathcal{A}$. Consequently, one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Y'\oplus TX' & \longrightarrow & Y & \longrightarrow & Y''\oplus TX'' & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow & & \\ 0 & \longrightarrow & TX' & \longrightarrow & TX & \longrightarrow & TX''. & & \end{array}$$

We can get that $TX\to TX''$ is an epimorphism. It follows from the Five lemma that φ is an epimorphism, and $X\in\mathcal{X}$ since $\mathcal{X}$ is closed under extensions. By the Snake lemma, there is an exact sequence $0\to Y'\to\ker\varphi\to Y''\to 0$, which implies $\ker\varphi\in\mathcal{Y}$ since $\mathcal{Y}$ is closed under extensions.

Conversely, assume that $\binom{X}{Y}_{\varphi}\in\mathfrak{D}^{\mathcal{X}}_{\mathcal{Y}}$. Then the exact sequence

$$0\to \begin{pmatrix} 0 \\ \ker\varphi \end{pmatrix} \to \begin{pmatrix} X \\ Y \end{pmatrix}_{\varphi} \to \begin{pmatrix} X \\ TX \end{pmatrix} \to 0$$

yields that $\binom{X}{Y}_{\varphi}\in\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$. By the proof above, one can get that $\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ is closed under extensions. So $\langle \mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ is the smallest subclass of $(\mathcal{B}\downarrow T)$ containing $\mathbf{h}(\mathcal{X},\mathcal{Y})$ and closed under extensions.

(2) $\Rightarrow$ (3) It is trivial.

(3) $\Rightarrow$ (1) Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with $A', A'' \in \mathcal{X}$. Then there is an exact sequence $0 \to TA' \to TA \to TA'' \to 0$ in $\mathcal{B}$ with T being $\mathcal{X}$-exact by hypothesis. Now there is an exact sequence $0 \to \binom{A'}{TA'} \to \binom{A}{TA} \to \binom{A''}{TA''} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{A'}{TA'}, \binom{A''}{TA''} \in \mathfrak{D}_{\mathcal{Y}}^{\mathcal{X}}$ by hypothesis. It follows that $\binom{A}{TA} \in \mathfrak{D}_{\mathcal{Y}}^{\mathcal{X}}$, which implies that $A \in \mathcal{X}$. Moreover, let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{B}$ with $B', B'' \in \mathcal{Y}$. Accordingly, we get an exact sequence $0 \to \binom{0}{B'} \to \binom{0}{B} \to \binom{0}{B''} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{0}{B'}, \binom{0}{B''} \in \mathfrak{D}_{\mathcal{Y}}^{\mathcal{X}}$. Thus, $\binom{0}{B} \in \mathfrak{D}_{\mathcal{Y}}^{\mathcal{X}}$ by hypothesis, and hence we get that $B \in \mathcal{Y}$. So $\mathcal{Y}$ is closed under extensions, as desired. □

Proposition 3.4. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be a left exact functor. The following are true:*

(1) *T is $\mathcal{X}$-exact if and only if the two exact sequences, $0 \to X \to A \to A'' \to 0$ in $\mathcal{A}$ and $0 \to B' \to B \to B'' \to 0$ in $\mathcal{B}$ with $X \in \mathcal{X}$, induce an exact sequence $0 \to \mathbf{h}(X, B') \to \mathbf{h}(A, B) \to \mathbf{h}(A'', B'') \to 0$ in $(\mathcal{B} \downarrow T)$.*

(2) *T is exact if and only if $\mathbf{h}\colon \mathcal{A} \times \mathcal{B} \to (\mathcal{B} \downarrow T)$ is exact.*

P r o o f. (1) Assume that $T\colon \mathcal{A} \to \mathcal{B}$ is $\mathcal{X}$-exact. Let $0 \to X \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with $X \in \mathcal{X}$ and $0 \to B' \to B \to B'' \to 0$ in $\mathcal{B}$ be an exact sequence. Accordingly, there is an exact sequence $0 \to TX \to TA \to TA'' \to 0$ in $\mathcal{B}$. So the sequence $0 \to \mathbf{h}(X, B') \to \mathbf{h}(A, B) \to \mathbf{h}(A'', B'') \to 0$ is exact in $(\mathcal{B} \downarrow T)$.

Conversely, let $0 \to X \to A \to A'' \to 0$ be an exact sequence in $\mathcal{A}$ with $X \in \mathcal{X}$. Note that $0 \to \mathbf{h}(X, 0) \to \mathbf{h}(A, 0) \to \mathbf{h}(A'', 0) \to 0$ is an exact sequence in $(\mathcal{B} \downarrow T)$ by hypothesis. So $0 \to TX \to TA \to TA'' \to 0$ is an exact sequence in $\mathcal{B}$, T is X-exact, and (2) is a direct consequence of (1). □

Proposition 3.5. *Let $T\colon \mathcal{A} \to \mathcal{B}$ be $\mathcal{X}$-exact. Then ${}^{\perp}\langle \mathbf{h}(\mathcal{X}, \mathcal{Y})\rangle = \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$ holds in the category $(\mathcal{B} \downarrow T)$.*

P r o o f. It follows from Remark 3.2 that

$$ {}^{\perp}\langle \mathbf{h}(\mathcal{X}, \mathcal{Y})\rangle = {}^{\perp}\mathbf{h}(\mathcal{X}, \mathcal{Y}) = {}^{\perp}\mathbf{h}(\mathcal{X}, 0) \cap {}^{\perp}\mathbf{h}(0, \mathcal{Y}). $$

Firstly, we claim that $\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}} \subseteq {}^{\perp}\mathbf{h}(\mathcal{X}, \mathcal{Y})$. Let $\binom{A}{B} \in \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$. It is sufficient to show the exact sequences

$$ \zeta\colon 0 \to \binom{X}{TX} \xrightarrow{\binom{m}{n}} \binom{M}{N} \to \binom{A}{B} \to 0 \quad \text{and} \quad \xi : 0 \to \binom{0}{Y} \xrightarrow{\binom{0}{f}} \binom{A}{C} \to \binom{A}{B} \to 0 $$

are split for any $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$. There is a split exact sequence $0 \to X \xrightarrow{m} M \to A \to 0$ in $\mathcal{A}$, thus there exists $m'\colon M \to X$ such that $m'm = 1_X$. It follows that $(Tm'\varphi)n = (Tm')Tm = T(m'm) = 1_{TX}$, i.e., $\binom{m'}{Tm'\varphi}\binom{m}{n} = 1$, which implies that the sequence ζ is split, as desired. Similarly, it is easy to see that ξ is split since the exact sequence $0 \to Y \to C \to D \to 0$ is split.

Next we claim that ${}^{\perp}\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle \subseteq \binom{{}^{\perp}\mathcal{X}}{{}_{\perp}\mathcal{Y}}$. We only need to show that ${}^{\perp}\mathbf{h}(\mathcal{X},0) \subseteq \binom{{}^{\perp}\mathcal{X}}{\mathcal{B}}$ and ${}^{\perp}\mathbf{h}(0,\mathcal{Y}) \subseteq \binom{\mathcal{A}}{{}_{\perp}\mathcal{Y}}$. Let $\binom{A}{B}$ be an object and ${}^{\perp}\mathbf{h}(\mathcal{X},0)$. Assume that $\varepsilon\colon\ 0 \to X \to M \to A \to 0$ is an exact sequence in $\mathcal{A}$ with $X \in \mathcal{X}$. It follows from T being $\mathcal{X}-$exact that $0 \to TX \to TM \to TA \to 0$ is exact in $\mathcal{B}$. Consider the following pullback diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TX & \longrightarrow & N & \longrightarrow & B & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & TX & \longrightarrow & TM & \longrightarrow & TA & \longrightarrow & 0. \end{array}$$

There is an exact sequence $\varrho\colon\ 0 \to \binom{X}{TX} \to \binom{M}{N} \to \binom{A}{B} \to 0$ in $(\mathcal{B}\downarrow T)$. Hence, the sequence ϱ is split by hypothesis, so the sequence ε is split, i.e., $A \in {}^{\perp}\mathcal{X}$. Consequently, ${}^{\perp}\mathbf{h}(\mathcal{X},0) \subseteq \binom{{}^{\perp}\mathcal{X}}{\mathcal{B}}$.

Let $\binom{A}{B}$ be an object in ${}^{\perp}\mathbf{h}(0,\mathcal{X})$ and $\sigma\colon\ 0 \to Y \to D \to B \to 0$ be an exact sequence in $\mathcal{B}$ with $Y \in \mathcal{Y}$. Then there is an exact sequence $\theta\colon\ 0 \to \binom{0}{Y} \to \binom{A}{D} \to \binom{A}{B} \to 0$ and θ is split by hypothesis. So the sequence σ is split, i.e., $B \in {}^{\perp}\mathcal{Y}$. Consequently, ${}^{\perp}\mathbf{h}(0,\mathcal{Y}) \subseteq \binom{\mathcal{A}}{{}_{\perp}\mathcal{Y}}$. □

Lemma 3.6. *If $\mathcal{B}$ has enough projective objects, then $\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle = \binom{0}{\mathcal{P}}^{\perp}$, where $\mathcal{P}$ is the class of projective objects in $\mathcal{B}$.*

P r o o f. Let $\mathcal{B}$ have enough projective objects and $\mathcal{P}$ be the class of projective objects in $\mathcal{B}$. Assume that $\binom{A}{B}_{\varphi} \in \binom{0}{\mathcal{P}}^{\perp}$ and there is an epimorphism $\sigma\colon\ P \to TA$ with P is a projective object in $\mathcal{B}$. Then one has the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \xrightarrow{\binom{1}{0}} & B\oplus P & \xrightarrow{(0,1)} & P & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle(\varphi,\sigma)} & & \downarrow & & \\ 0 & \longrightarrow & TA & = & TA & \longrightarrow & 0 & \longrightarrow & 0. \end{array}$$

Clearly, we have an exact sequence $\xi\colon\ 0 \to \binom{A}{B} \to \binom{A}{B\oplus P} \to \binom{0}{P} \to 0$ in $(\mathcal{B}\downarrow T)$ and ξ is split by hypothesis. It follows that there is a morphism $\binom{f}{g}\colon\ P \to P\oplus B$ such that $(0,1)\binom{f}{g} = 1_P$ and $(\varphi,\sigma)\binom{f}{g} = 0$, which implies $g = 1_P$ and $\varphi f = -\sigma$. So φ is an epimorphism, i.e., $\binom{A}{B}_{\varphi} \in \langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$ by Proposition 3.3.

Conversely, let $\binom{A}{B}_{\varphi} \in \langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$. Consider the exact sequence

$$\sigma\colon\ 0 \to \binom{A}{B}_{\varphi} \xrightarrow{\binom{1}{\pi}} \binom{A}{M}_{\phi} \xrightarrow{\binom{1}{p}} \binom{0}{P} \to 0$$

with P being projective. Since P is projective, there exists a map $\pi'\colon\ M \to B$ such that $\pi'\pi = 1_B$. Hence, we have $(\phi - \varphi\pi')\pi = 0$. By the universal property of the cokernel, we get a map $\alpha\colon\ P \to TA$ such that $\phi - \varphi\pi' = \alpha p$. Since φ is an

epimorphism by Proposition 3.3 and P is projective, there exists a map $\beta\colon P \to B$ such that $\alpha = \varphi\beta$. Setting $q = \pi' + \beta p$, we have

$$\varphi q = \varphi(\pi' + \beta p) = \varphi\pi' + \varphi\beta p = \varphi\pi' + \alpha p = \phi, \quad q\pi = (\pi' + \beta p)\pi = \pi'\pi + \beta p\pi = 1_B.$$

Thus, $\binom{1}{q}\binom{1}{\pi} = 1$, and so ξ is split. □

Proposition 3.7. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $\mathcal{X}$ be a subclass of $\mathcal{A}$ with $0 \in \mathcal{X}$, $\mathcal{Y}$ be a subclass of $\mathcal{B}$ with $0 \in \mathcal{Y}$. If $\mathcal{B}$ has enough projective objects, then*

$$\langle \mathbf{h}(\mathcal{X}^\perp, \mathcal{Y}^\perp)\rangle = \binom{\mathcal{X}}{\mathcal{Y}}^\perp \cap \binom{0}{\mathcal{P}}^\perp,$$

where $\mathcal{P}$ is the class of projective objects in $\mathcal{B}$.

P r o o f. Let $\mathcal{B}$ have enough projective objects and $\mathcal{P}$ be the class of projective objects in $\mathcal{B}$. At first, we claim that $\binom{\mathcal{X}}{\mathcal{Y}}^\perp \cap \binom{0}{\mathcal{P}}^\perp \subseteq \langle \mathbf{h}(\mathcal{X}^\perp, \mathcal{Y}^\perp)\rangle$. Assume that $\binom{A}{B}_\varphi$ is an object in $\binom{\mathcal{X}}{\mathcal{Y}}^\perp \cap \binom{0}{\mathcal{P}}^\perp$. It follows from Proposition 3.3 and Lemma 3.6 that φ is an epimorphism. Note that both ${}^\perp\mathcal{X}$ and ${}^\perp\mathcal{Y}$ are closed under extensions, and so it is sufficient to show $A \in \mathcal{X}^\perp$ and $\ker\varphi \in \mathcal{Y}^\perp$ by Proposition 3.3.

Let $\zeta\colon 0 \to A \to M \to X \to 0$ be an exact sequence in $\mathcal{A}$ with $X \in \mathcal{X}$. Consequently, there is an exact sequence $\xi\colon 0 \to \binom{A}{B} \to \binom{M}{B} \to \binom{X}{0} \to 0$ in $(\mathcal{B} \downarrow T)$. By hypothesis, the sequence ξ is split since $\binom{X}{0} \in \binom{\mathcal{X}}{\mathcal{Y}}$. So the sequence ζ is split and $A \in \mathcal{X}^\perp$, as desired.

Let $0 \to \ker\varphi \to M' \xrightarrow{\tau} Y \to 0$ be an exact sequence in $\mathcal{B}$ with $Y \in \mathcal{Y}$. Then we have a pushout diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \ker\varphi & \longrightarrow & M' & \xrightarrow{\tau} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle k} & & \downarrow{\scriptstyle k'} & & \| & & \\
0 & \longrightarrow & B & \xrightarrow{\pi} & L & \xrightarrow{\tau'} & Y & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi'} & & & & \\
 & & TA & = & TA & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}$$

which induces an exact sequence

$$\xi'\colon 0 \to \binom{A}{B}_\varphi \xrightarrow{\binom{1}{\pi}} \binom{A}{L}_{\varphi'} \xrightarrow{\binom{1}{\tau'}} \binom{0}{Y} \to 0$$

in $(\mathcal{B}\downarrow T)$. Since ξ' is split by hypothesis, there exists $\varsigma\colon Y\to L$ such that $\tau'\varsigma=1$ and $\varphi'\varsigma=0$. By the universal property of the kernel, there is a morphism $g\colon Y\to M$ such that $\varsigma=k'g$. Thus, we have $\tau g=\tau'k'g=\tau'\varsigma=1$, which means the first row in the above diagram is split. So $\ker\varphi\in\mathcal{Y}^{\perp}$, as required.

Nextly, we claim that $\langle\mathbf{h}(\mathcal{X}^{\perp},\mathcal{Y}^{\perp})\rangle\subseteq\binom{\mathcal{X}}{\mathcal{Y}}^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}$. By Remark 3.2 and Lemma 3.6, we only need to show that $\mathbf{h}({}^{\perp}\mathcal{X},0)\subseteq\binom{\mathcal{X}}{\mathcal{Y}}^{\perp}$ and $\mathbf{h}(0,{}^{\perp}\mathcal{Y})\subseteq\binom{\mathcal{X}}{\mathcal{Y}}^{\perp}$. Let $A\in\mathcal{X}^{\perp}$ and $\varepsilon\colon 0\to\binom{A}{TA}\xrightarrow{\binom{m}{n}}\binom{M}{N}_{\phi}\to\binom{X}{Y}\to 0$ be an exact sequence in $(\mathcal{B}\downarrow T)$ with $X\in\mathcal{X}$ and $Y\in\mathcal{Y}$. Note that the exact sequence $0\to A\to M\to X\to 0$ in $\mathcal{A}$ is split. Thus, there exists $m'\colon M\to A$ such that $m'm=1_A$ and $(Tm')\phi n=(Tm')Tm=1_{TA}$. It follows that the sequence ε is split. So $\mathbf{h}({}^{\perp}\mathcal{X},0)\subseteq\binom{\mathcal{X}}{\mathcal{Y}}^{\perp}$. Similarly, one can show that $\mathbf{h}(0,{}^{\perp}\mathcal{Y})\subseteq\binom{\mathcal{X}}{\mathcal{Y}}^{\perp}$. □

Corollary 3.8. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories with enough projective objects and enough injective objects. If $T\colon\mathcal{A}\to\mathcal{B}$ is an exact functor, then $\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$ is a Frobenius category if and only if $\mathcal{A}$ and $\mathcal{B}$ are Frobenius and T preserves injective objects.*

P r o o f. Let $\mathcal{P}$ (or $\mathcal{Q}$) be the class of projective objects in $\mathcal{A}$ (or $\mathcal{B}$) and $\mathcal{I}$ (or $\mathcal{J}$) the class of injective objects in $\mathcal{A}$ (or $\mathcal{B}$). Then one has

$$\mathbf{h}(\mathcal{A},0)^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}=\binom{\mathcal{A}}{T\mathcal{A}}^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}=\langle\mathbf{h}(\mathcal{A}^{\perp},(T\mathcal{A})^{\perp})\rangle=\langle\mathbf{h}(\mathcal{I},(T\mathcal{A})^{\perp})\rangle,$$
$$\mathbf{h}(0,\mathcal{B})^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}=\binom{0}{\mathcal{B}}^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}=\langle\mathbf{h}(0^{\perp},\mathcal{B}^{\perp})\rangle=\langle\mathbf{h}(\mathcal{A},\mathcal{J})\rangle.$$

Thus, one gets

$$\begin{aligned}\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle^{\perp}&=\mathbf{h}(\mathcal{A},\mathcal{B})^{\perp}=\mathbf{h}(\mathcal{A},0)^{\perp}\cap\mathbf{h}(0,\mathcal{B})^{\perp}\\&=\mathbf{h}(\mathcal{A},0)^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}\cap\mathbf{h}(0,\mathcal{B})^{\perp}\\&=\left(\mathbf{h}(\mathcal{A},0)^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}\right)\cap\left(\mathbf{h}(0,\mathcal{B})^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}\right)\\&=\langle\mathbf{h}(\mathcal{I},T\mathcal{A}^{\perp})\rangle\cap\langle\mathbf{h}(\mathcal{A},\mathcal{J})\rangle=\langle\mathbf{h}(\mathcal{I},\mathcal{J})\rangle.\end{aligned}$$

Moreover, the class of projective objects in the exact category $\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$ is $\binom{\mathcal{P}}{\mathcal{Q}}\cap\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$. Therefore, $\langle\mathbf{h}(\mathcal{A},\mathcal{B})\rangle$ is Frobenius if and only if $T(\mathcal{I})\subset\mathcal{Q}=\mathcal{J}$ and $\mathcal{I}=\mathcal{P}$, as desired. □

4. Complete cotorsion pairs

In this section, we characterize when complete hereditary cotorsion pairs in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce complete hereditary cotorsion pairs in $(\mathcal{B}\downarrow T)$.

Lemma 4.1. *Let $\mathcal{A}$ and $\mathcal{B}$ be abelian categories, $\mathcal{X}$ be a subclass of $\mathcal{A}$ with $0\in\mathcal{X}$, $\mathcal{Y}$ be a subclass of $\mathcal{B}$ with $0\in\mathcal{Y}$. The following hold for a comma category $(\mathcal{B}\downarrow T)$:*

(1) $\binom{\mathcal{X}}{\mathcal{Y}}$ *is resolving in $(\mathcal{B}\downarrow T)$ if and only if $\mathcal{X}$ and $\mathcal{Y}$ are resolving in $\mathcal{A}$ and $\mathcal{B}$, respectively.*

(2) *If $T\colon \mathcal{A}\to\mathcal{B}$ is $\mathcal{X}$-exact and $\mathcal{X}$, $\mathcal{Y}$ are closed under extensions in $\mathcal{A}$ and $\mathcal{B}$, respectively, then $\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ is coresolving in $(\mathcal{B}\downarrow T)$ if and only if $\mathcal{X}$ and $\mathcal{Y}$ are coresolving in $\mathcal{A}$ and $\mathcal{B}$, respectively.*

P r o o f. The proof of (1) is straightforward.

(2) Assume that T is $\mathcal{X}$-exact. For the "only if" part, we assume that $\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ is coresolving in $(\mathcal{B}\downarrow T)$. Let $0\to X'\to X\to X''\to 0$ be an exact sequence in $\mathcal{A}$ with $X'\in\mathcal{X}$. It follows that the sequence $0\to\mathbf{h}(X',0)\to\mathbf{h}(X,0)\to\mathbf{h}(X'',0)\to 0$ is exact in $(\mathcal{B}\downarrow T)$. Thus, we get that $\mathbf{h}(X'',0)\in\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ if and only if $\mathbf{h}(X,0)\in\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ since $\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ is coresolving. Hence, $X\in\mathcal{X}$ if and only if $X''\in\mathcal{X}$. So $\mathcal{X}$ is coresolving in $\mathcal{A}$. Similarly, one can show that $\mathcal{Y}$ is coresolving in $\mathcal{B}$.

For the "if" part, let $\mathcal{X}$ and $\mathcal{Y}$ be coresolving in $\mathcal{A}$ and $\mathcal{B}$, respectively. Then $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions in $\mathcal{A}$ and $\mathcal{B}$, respectively. Let $0\to\binom{X'}{Y'}_{\varphi'}\to\binom{X}{Y}_{\varphi}\to\binom{X''}{Y''}_{\varphi''}\to 0$ be an exact sequence in $(\mathcal{B}\downarrow T)$ with $\binom{X'}{Y'}_{\varphi'}\in\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$. Note that T is $\mathcal{X}$-exact, so we have the following commutative diagram of exact sequences:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & Y' & \longrightarrow & Y & \longrightarrow & Y'' & \longrightarrow & 0\\
 & & \downarrow{\scriptstyle\varphi'} & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\varphi''} & & \\
0 & \longrightarrow & TX' & \longrightarrow & TX & \longrightarrow & TX'' & \longrightarrow & 0.
\end{array}$$

Note that φ' is an epimorphism and $\ker\varphi'\in\mathcal{Y}$ by Proposition 3.3. Thus, we have that $\operatorname{coker}\varphi\cong\operatorname{coker}\varphi''$ and $0\to\ker\varphi'\to\ker\varphi\to\ker\varphi''\to 0$ is exact in $\mathcal{B}$. Therefore, we get that φ is surjective if and only if φ'' is surjective. Since $\mathcal{Y}$ is coresolving in $\mathcal{B}$ by hypothesis, $\ker\varphi\in\mathcal{Y}$ if and only if $\ker\varphi''\in\mathcal{Y}$. Note that $0\to X'\to X\to X''\to 0$ is an exact sequence in $\mathcal{A}$ and $\mathcal{X}$ is coresolving in $\mathcal{A}$. It follows that $X\in\mathcal{X}$ if and only if $X''\in\mathcal{X}$. So $\binom{X}{Y}_{\varphi}\in\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ if and only if $\binom{X''}{Y''}_{\varphi'}\in\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$ by Proposition 3.3. □

Theorem 4.2. *Let $\mathcal{A}$ and $\mathcal{B}$ both have enough projective objects and enough injective objects, $\mathcal{X}$ be a subclass of $\mathcal{A}$ with $0\in\mathcal{X}$, $\mathcal{Y}$ be a subclass of $\mathcal{B}$ with $0\in\mathcal{Y}$. If $T\colon\mathcal{A}\to\mathcal{B}$ is $\mathcal{X}$-exact, then $({}^{\perp}\mathcal{X},\mathcal{X})$ is a (hereditary) cotorsion pair in $\mathcal{A}$ and*

$({}^{\perp}\mathcal{Y},\mathcal{Y})$) is a (hereditary) cotorsion pair in $\mathcal{B}$ if and only if $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}},\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle)$ is a (hereditary) cotorsion pair in $(\mathcal{B}\downarrow T)$ and $\mathcal{X}$ and $\mathcal{Y}$ are closed under extensions.

P r o o f. ($\Rightarrow$): Let $T\colon \mathcal{A}\to\mathcal{B}$ be $\mathcal{X}$-exact, $({}^{\perp}\mathcal{X},\mathcal{X})$ be a cotorsion pair in $\mathcal{A}$ and $({}^{\perp}\mathcal{Y},\mathcal{Y})$ a cotorsion pair in $\mathcal{B}$. Then all projective objects belong to ${}^{\perp}\mathcal{Y}$. By Proposition 3.7, one has

$$\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle=\langle\mathbf{h}(({}^{\perp}\mathcal{X})^{\perp},({}^{\perp}\mathcal{Y})^{\perp})\rangle=\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}^{\perp}\cap\binom{0}{\mathcal{P}}^{\perp}=\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}^{\perp}.$$

It follows from Proposition 3.5 that ${}^{\perp}\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle=\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$, so $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}},\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle)$ is a cotorsion pair in $(\mathcal{B}\downarrow T)$.

($\Leftarrow$): Let $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}},\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle)$ be a cotorsion pair in $(\mathcal{B}\downarrow T)$ and $\mathcal{X}$, $\mathcal{Y}$ be closed under extensions. It is sufficient to show that $({}^{\perp}\mathcal{X})^{\perp}\subseteq\mathcal{X}$ and $({}^{\perp}\mathcal{Y})^{\perp}\subseteq\mathcal{Y}$. Assume that $X\in({}^{\perp}\mathcal{X})^{\perp}$ and $Y\in({}^{\perp}\mathcal{Y})^{\perp}$. Then for any $M\in{}^{\perp}\mathcal{X}$ and $N\in{}^{\perp}\mathcal{Y}$, it is clear that the exact sequences

$$0\to\binom{X}{TX}\to\binom{A}{B}\to\binom{M}{N}\to 0\quad\text{and}\quad 0\to\binom{0}{Y}\to\binom{M}{C}\to\binom{M}{N}\to 0$$

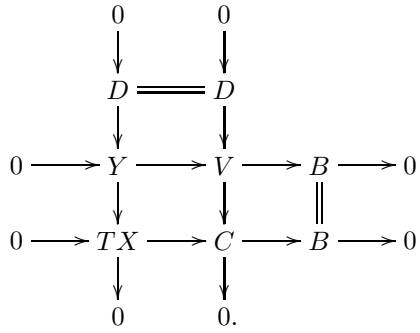
are split. This implies that $\binom{X}{TX},\binom{0}{Y}\in\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}^{\perp}=\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle$. Therefore, $X\in\mathcal{X}$, $Y\in\mathcal{Y}$ by Proposition 3.3. □

Proposition 4.3. *Let $T\colon\mathcal{A}\to\mathcal{B}$ be $\mathcal{X}$-exact and $\mathcal{A}$ and $\mathcal{B}$ both have enough projective objects and enough injective objects, $\mathcal{X}$ be a subclass of $\mathcal{A}$ with $0\in\mathcal{X}$, $\mathcal{Y}$ be a subclass of $\mathcal{B}$ with $0\in\mathcal{Y}$. If $({}^{\perp}\mathcal{X},\mathcal{X})$ and $({}^{\perp}\mathcal{Y},\mathcal{Y})$ are complete cotorsion pairs in $\mathcal{A}$ and $\mathcal{B}$, respectively, then so is $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}},\langle\mathbf{h}(\mathcal{X},\mathcal{Y})\rangle)$. Moreover, the converse holds if $T({}^{\perp}\mathcal{X}\cap\mathcal{X})\subseteq{}^{\perp}\mathcal{Y}$ and $\mathcal{X}$, $\mathcal{Y}$ are closed under extensions.*

P r o o f. Assume that $({}^{\perp}\mathcal{X},\mathcal{X})$ is a complete cotorsion pair in $\mathcal{A}$ and $({}^{\perp}\mathcal{Y},\mathcal{Y})$ is a complete cotorsion pair in $\mathcal{B}$. For any $\binom{A}{B}\in(\mathcal{B}\downarrow T)$ there exists an exact sequence $0\to X\to U\to A\to 0$ in $\mathcal{A}$ with $U\in{}^{\perp}\mathcal{X}$ and $X\in\mathcal{X}$. Since T is $\mathcal{X}$-exact, there is a pullback diagram

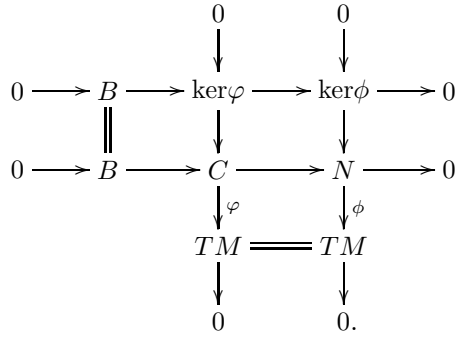
$$\begin{array}{ccccccccc}
0 & \longrightarrow & TX & \longrightarrow & C & \longrightarrow & B & \longrightarrow & 0\\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TX & \longrightarrow & TU & \longrightarrow & TA & \longrightarrow & 0.
\end{array}$$

Furthermore, we have an exact sequence $0 \to D \to V \to C \to 0$ in $\mathcal{A}$ with $V \in {}^{\perp}\mathcal{Y}$ and $D \in \mathcal{Y}$. Consider the following pullback diagram:



Thus, we get an exact sequence $0 \to \binom{X}{Y} \to \binom{U}{V} \to \binom{A}{B} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{U}{V} \in \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$ and $\binom{X}{Y} \in \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$. Note that $\mathcal{A}$ and $\mathcal{B}$ have enough projective objects, so is $(\mathcal{B} \downarrow T)$. It follows that $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}, \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle)$ is a complete cotorsion pair in $(\mathcal{B} \downarrow T)$.

Conversely, assume that $(\binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}, \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle)$ is a complete cotorsion pair in $(\mathcal{B} \downarrow T)$, $T({}^{\perp}\mathcal{X} \cap \mathcal{X}) \subseteq {}^{\perp}\mathcal{Y}$ and $\mathcal{X}, \mathcal{Y}$ are closed under extensions. Then for any $A \in \mathcal{A}$ we have an exact sequence $0 \to \binom{A}{TA} \to \binom{X}{Y} \to \binom{U}{V} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{X}{Y} \in \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$ and $\binom{U}{V} \in \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$. Thus, we have an exact sequence $0 \to A \to X \to U \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $U \in {}^{\perp}\mathcal{X}$ which means that $({}^{\perp}\mathcal{X}, \mathcal{X})$ is a complete cotorsion pair in $\mathcal{A}$. Furthermore, for any $B \in \mathcal{B}$, we have an exact sequence $0 \to \binom{0}{B} \to \binom{M}{C}_{\varphi} \to \binom{M}{N}_{\phi} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{M}{C}_{\varphi} \in \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$ and $\binom{M}{N}_{\phi} \in \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$. Thus, we obtain that $M \in {}^{\perp}\mathcal{X} \cap \mathcal{X}$, $\ker\varphi \in \mathcal{Y}$, $N \in {}^{\perp}\mathcal{Y}$ and φ is an epimorphism. Consequently, we have the following commutative diagram of exact rows and columns:



By hypotheses, $TM \in {}^{\perp}\mathcal{Y}$. It follows that the middle column splits, which induces that the right column is split. Thus, $\ker\phi$ is a direct summand of N, i.e., $\ker\phi \in {}^{\perp}\mathcal{Y}$. It follows from the exact sequence $0 \to B \to \ker\varphi \to \ker\phi \to 0$ that $({}^{\perp}\mathcal{Y}, \mathcal{Y})$ is a complete cotorsion pair in $\mathcal{B}$. □

Remark 4.4. If we replace the condition "$T({}^{\perp}\mathcal{X}\cap\mathcal{X})\subseteq {}^{\perp}\mathcal{Y}$" by "$T({}^{\perp}\mathcal{X}\cap\mathcal{X})\subseteq \mathcal{Y}$" in Proposition 4.3, the result still holds. In the case $T({}^{\perp}\mathcal{X}\cap\mathcal{X})\subseteq\mathcal{Y}$ for any $B\in\mathcal{B}$, it is easy to check that the exact sequence $0\to B\to C\to N\to 0$ in the third commutative diagram in Proposition 4.3 satisfies that $C\in\mathcal{Y}$ and $N\in{}^{\perp}\mathcal{Y}$. This implies that $({}^{\perp}\mathcal{Y},\mathcal{Y})$ is a complete cotorsion pair in $\mathcal{B}$.

Let $\mathcal{L}$ be a class of objects in an abelian category $\mathcal{D}$. We denote by $\mathsf{Smd}(\mathcal{L})$ the class of direct summands of objects in $\mathcal{L}$.

Lemma 4.5. *Let $\mathcal{D}$ be an abelian category with enough projective objects. If $\mathcal{L}$ is special preeveloping in $\mathcal{D}$, then $({}^{\perp}\mathsf{Smd}(\mathcal{L}),\mathsf{Smd}(\mathcal{L}))$ is a complete cotorsion pair.*

P r o o f. Clearly, ${}^{\perp}\mathsf{Smd}(\mathcal{L})={}^{\perp}\mathcal{L}$. It is sufficient to show that $\mathsf{Smd}(\mathcal{L})$ is a special preenveloping class. We claim that $({}^{\perp}\mathsf{Smd}(\mathcal{L}))^{\perp}\subseteq\mathsf{Smd}(\mathcal{L})$. For any $X\in({}^{\perp}\mathsf{Smd}(\mathcal{L}))^{\perp}$, there is an exact sequence $\xi\colon\ 0\to X\to L\to\bar{L}\to 0$ with $\bar{L}\in{}^{\perp}\mathcal{L}$ and $L\in\mathcal{L}$ since $\mathcal{L}$ is special preenveloping. Thus, ξ is split as ${}^{\perp}\mathsf{Smd}(\mathcal{L})={}^{\perp}\mathcal{L}$. It follows that $X\in\mathsf{Smd}(\mathcal{L})$. Therefore, $({}^{\perp}\mathsf{Smd}(\mathcal{L}))^{\perp}=\mathsf{Smd}(\mathcal{L})$. □

5. Gorenstein injective objects

In this section, we characterize when special preenveloping classes in abelian categories $\mathcal{A}$ and $\mathcal{B}$ can induce special preenveloping classes in $(\mathcal{B}\downarrow T)$.

Definition 5.1. The left exact functor $T\colon\ \mathcal{A}\to\mathcal{B}$ is *cocompatible* if the following two conditions hold:

(C1) TE° is exact for each exact sequence E° of injective objects in $\mathcal{A}$.

(C2) $\mathrm{Hom}_{\mathcal{B}}(TI,J^{\circ})$ is exact for each totally acyclic complex of injective objects J° in $\mathcal{B}$ and each injective object I in $\mathcal{A}$.

Moreover, $T\colon\ \mathcal{A}\to\mathcal{B}$ is called *weak cocompatible* if T satisfies conditions (W1) and (C2), where

(W1) TE° is exact for each totally acyclic complex of injective objects E° in $\mathcal{A}$.

Proposition 5.2. *Let M be an R-S-bimodule and $T=\mathrm{Hom}_R(M,-)$. Then:*

(1) *If $\mathrm{pd}_R M$ is finite, then T satisfies* (C1).

(2) *If $\mathrm{fd}M_S$ is finite, then T satisfies* (C2).

P r o o f. (1) Let $E^{\circ}=\ldots\to E_1\overset{d_1}{\to}E_0\overset{d_0}{\to}Q_{-1}\to\ldots$ be an exact sequence of injective left R-modules and $\mathrm{pd}_R M=n$. By dimension shifting, we have $\mathrm{Ext}^1_R(M,\ker d_i)=\mathrm{Ext}^{n+1}_R(M,\ker d_{i-n})=0$ for any integer i. It follows that TE° is still exact.

(2) Let $\mathrm{fd}M_S$ be finite. For any injective left R-module I, $\mathrm{id}_R\mathrm{Hom}(M,I)$ is finite since $\mathrm{fd}M_S$ is finite and I is injective. Then $\mathrm{Ext}^1_R(\mathrm{Hom}(M,I),\ker\phi_i)=0$ for any totally acyclic complex of injective R-modules $J^\circ=\ldots\to J_1\overset{\phi_1}{\to}J_0\overset{\phi_0}{\to}J_{-1}\to\ldots$ □

Lemma 5.3. *Let G be an object in $\mathcal{B}$ and L an object in $\mathcal{A}$. The following statements hold:*

(1) *If $\mathbf{h}(0,G)$ is a Gorenstein injective object, then G is Gorenstein injective.*

(2) *If $\mathbf{h}(L,0)$ is a Gorenstein injective object, then L is Gorenstein injective.*

P r o o f. (1) Let $\mathbf{h}(0,G)$ be a Gorenstein injective object. There exists a totally acyclic complex

$$\mathbf{h}(I^\circ,E^\circ)=\ldots\to\mathbf{h}(I_1,E_1)\to\mathbf{h}(I_0,E_0)\to\mathbf{h}(I_{-1},E_{-1})\to\ldots$$

of injective objects in $(\mathcal{B}\downarrow T)$ with $Z_0(\mathbf{h}(I^\circ,E^\circ))=\mathbf{h}(0,G)$. By definition, $\mathbf{h}(I^\circ,E^\circ)$ is $\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(0,X),-)$-exact for each injective object X in $\mathcal{B}$. Now one has

$$\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(0,X),\mathbf{h}(I^\circ,E^\circ))\simeq\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}(\mathbf{q}(\mathbf{h}(0,X)),(I^\circ,E^\circ))=\mathrm{Hom}_{\mathcal{B}}(X,E^\circ),$$

which implies that $E^\circ=\ldots\to E_1\to E_0\to E_{-1}\to\ldots$ is $\mathrm{Hom}_{\mathcal{B}}(X,-)$-exact. Clearly, $G=Z_0(E^\circ)$ and G is Gorenstein injective.

(2) Let $\mathbf{h}(L,0)$ be a Gorenstein injective object. Then there is a totally acyclic complex

$$\mathbf{h}(J^\circ,F^\circ)=\ldots\to\mathbf{h}(J_1,F_1)\to\mathbf{h}(J_0,F_0)\to\mathbf{h}(J_{-1},F_{-1})\to\ldots$$

of injective objects in $(\mathcal{B}\downarrow T)$ with $Z_0(\mathbf{h}(J^\circ,F^\circ))=\mathbf{h}(L,0)$. By definition, $\mathbf{h}(J^\circ,F^\circ)$ is $\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(Y,0),-)$-exact for each injective object Y in $\mathcal{A}$. Then we have

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(Y,0),\mathbf{h}(J^\circ,F^\circ))&\simeq\mathrm{Hom}_{\mathcal{A}}\times\mathcal{B}(\mathbf{q}(\mathbf{h}(Y,0)),(J^\circ,F^\circ))\\&\simeq\mathrm{Hom}_{\mathcal{A}}(Y,J^\circ)\times\mathrm{Hom}_{\mathcal{B}}(TY,F^\circ)\end{aligned}$$

which implies that $J^\circ=\ldots\to J_1\to J_0\to J_{-1}\to\ldots$ is $\mathrm{Hom}_{\mathcal{A}}(Y,-)$-exact. Clearly, $L=Z_0(J^\circ)$ and L is Gorenstein injective. □

Lemma 5.4. *For the comma category $(\mathcal{B}\downarrow T)$, one has that*

(1) *T satisfies* (W1) *if and only if $\mathbf{h}(G,0)$ is a Gorenstein injective object for each Gorenstein injective object G in $\mathcal{A}$,*

(2) *T satisfies* (C2) *if and only if $\mathbf{h}(0,L)$ is a Gorenstein injective object for each Gorenstein injective object L in $\mathcal{B}$.*

P r o o f. (1) ($\Leftarrow$): Assume that $E^\circ = \ldots \to E_1 \overset{d_1}{\to} E_0 \overset{d_0}{\to} E_{-1} \to \ldots$ is a totally acyclic complex of injective objects in $\mathcal{A}$ and $G_0 = Z_0(E^\circ)$. There is an exact sequence $0 \to G_1 \to E_1 \to G_0 \to 0$ with E_1 injective and G_1 Gorenstein injective. Consequently, $\mathbf{h}(G_0, 0)$ is Gorenstein injective in $(\mathcal{B} \downarrow T)$ by hypothesis. Thus, there is an exact sequence $0 \to \binom{G_A}{G_B} \to \binom{E_A}{E_B} \to \binom{G_0}{TG_0} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{E_A}{E_B}$ injective and $\binom{G_A}{G_B}$ Gorenstein injective. Since E_A is injective, we have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_A & \longrightarrow & E_A & \longrightarrow & G_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & G_1 & \longrightarrow & E_1 & \longrightarrow & G_0 & \longrightarrow & 0. \end{array}$$

Therefore, we obtain the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G_B & \longrightarrow & E_B & \longrightarrow & TG_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & TG_A & \longrightarrow & TE_A & \longrightarrow & TG_0 & & \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & TG_1 & \longrightarrow & TE_1 & \overset{Tf}{\longrightarrow} & TG_0. & & \end{array}$$

which implies that Tf is surjective, as required.

($\Rightarrow$): Let G be a Gorenstein injective object in $\mathcal{A}$. Then there is a totally acyclic complex of injective objects $I^\circ = \ldots \to I_1 \overset{d_1}{\to} I_0 \overset{d_0}{\to} I_{-1} \to \ldots$ in $\mathcal{A}$ such that $G = \ker d_0$. By hypothesis,

$$\mathbf{h}(I^\circ, 0) = \ldots \longrightarrow \mathbf{h}(I_1, 0) \xrightarrow{\mathbf{h}(d_1, 0)} \mathbf{h}(I_0, 0) \xrightarrow{\mathbf{h}(d_0, 0)} \mathbf{h}(I_{-1}, 0) \longrightarrow \cdots$$

is an exact sequence of injective objects in $(\mathcal{B} \downarrow T)$. Then we have

$$\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(J, F), \mathbf{h}(I^\circ, 0)) \simeq \mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}(\mathbf{q}(\mathbf{h}(J, F)), (I^\circ, 0)) = \mathrm{Hom}_{\mathcal{A}}(J, I^\circ)$$

for each injective object $\mathbf{h}(J, F)$ in $(\mathcal{B} \downarrow T)$. Thus, $\mathbf{h}(I^\circ, 0)$ is totally acyclic. Then $\mathbf{h}(G, 0)$ is a Gorenstein injective object by $\mathbf{h}(G, 0) = \ker \mathbf{h}(\mathbf{d_0}, \mathbf{0})$.

(2) Let $F^\circ = \ldots \to F_1 \overset{\tau_1}{\to} F_0 \overset{\tau_0}{\to} F_{-1} \to \ldots$ be a totally acyclic complex of injective objects in $\mathcal{B}$. Then there exists an exact sequence

$$\mathbf{h}(0, F^\circ) = \ldots \longrightarrow \mathbf{h}(0, F_1) \xrightarrow{\mathbf{h}(0, \tau_1)} \mathbf{h}(0, F_0) \xrightarrow{\mathbf{h}(0, \tau_0)} \mathbf{h}(0, F_{-1}) \longrightarrow \cdots$$

in $(\mathcal{B}\downarrow T)$ with each term injective. Then one has

$$\begin{aligned}\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(X,Y),\mathbf{h}(0,F^\circ)) &\simeq \mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}(\mathbf{q}(\mathbf{h}(X,Y)),(0,F^\circ))\\ &= \mathrm{Hom}_{\mathcal{B}}(Y\oplus TX,F^\circ)\end{aligned}$$

for each injective object $\mathbf{h}(X,Y)$ in $(\mathcal{B}\downarrow T)$. It follows that $\mathrm{Hom}_{\mathcal{B}}(TX,F^\circ)$ is exact for each injective object X in $\mathcal{A}$ if and only if $\mathbf{h}(0,F^\circ)$ is a totally acyclic complex of injective objects in $(\mathcal{B}\downarrow T)$ for each totally acyclic complex of injective objects F° in $\mathcal{B}$. □

Proposition 5.5. *Let $(\mathcal{B}\downarrow T)$ be a comma category. Then $\langle\mathbf{h}(\mathcal{GI}_{\mathcal{A}},\mathcal{GI}_{\mathcal{B}})\rangle\subseteq\mathcal{GI}_{(\mathcal{B}\downarrow T)}$ if and only if T is weak cocompatible. Moreover, if T is cocompatible, then $\mathcal{GI}_{(\mathcal{B}\downarrow T)}=\langle\mathbf{h}(\mathcal{GI}_{\mathcal{A}},\mathcal{GI}_{\mathcal{B}})\rangle$.*

P r o o f. By Lemmas 5.3 and 5.4, $\langle\mathbf{h}(\mathcal{GI}_{\mathcal{A}},\mathcal{GI}_{\mathcal{B}})\rangle\subseteq\mathcal{GI}_{(\mathcal{B}\downarrow T)}$ is satisfied. It is sufficient to show $\mathcal{GI}_{(\mathcal{B}\downarrow T)}\subseteq\langle\mathbf{h}(\mathcal{GI}_{\mathcal{A}},\mathcal{GI}_{\mathcal{B}})\rangle$ provided that T is cocompatible.

Assume that $\binom{H}{G}_{\varphi}\in\mathcal{GI}_{(\mathcal{B}\downarrow T)}$ and there is a totally acyclic complex

$$\mathbf{h}(I^\circ,E^\circ)=\ldots\longrightarrow\mathbf{h}(I_1,E_1)\xrightarrow{\mathbf{h}(d_1,\delta_1)}\mathbf{h}(I_0,E_0)\xrightarrow{\mathbf{h}(d_0,\delta_0)}\mathbf{h}(P_{-1},E_{-1})\longrightarrow\cdots$$

of injective objects in $(\mathcal{B}\downarrow T)$ with $\binom{H}{G}_{\varphi}=\ker\mathbf{h}(d_0,\delta_0)$. Then I° is an exact sequence of injective objects in $\mathcal{A}$, and TI° is exact since T is cocompatible. Thus, we have the following exact commutative diagram

$$\begin{array}{ccccccccc}0&\longrightarrow&G_i&\longrightarrow&E_i\oplus TI_i&\longrightarrow&G_{i-1}&\longrightarrow&0\\&&\downarrow{\scriptstyle\varphi_i}&&\downarrow{\scriptstyle(0,1)}&&\downarrow{\scriptstyle\varphi_{i-1}}&&\\0&\longrightarrow&TH_i&\longrightarrow&TI_i&\longrightarrow&TH_{i-1}&\longrightarrow&0,\end{array}$$

where $G_i=\ker\delta_i$, $\binom{H_i}{G_i}_{\varphi_i}=\ker\mathbf{h}(d_i,\delta_i)$ and φ_i is canonical induced. In particular, $G_0=G$, $H_0=H$ and $\varphi_0=\varphi$. It follows that each φ_i is surjective and there is an exact sequence $0\to\ker\varphi_i\to E_i\to\ker\varphi_{i-1}\to 0$. Therefore, we have an exact sequence $E^\circ=\ldots\to E_1\xrightarrow{\tau_1}E_0\xrightarrow{\tau_0}P_{-1}\to\ldots$ with each term injective in $\mathcal{B}$ and $\ker\tau_0=\ker\varphi$. For each injective object J in $\mathcal{B}$, one has

$$\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(0,J),\mathbf{h}(I^\circ,E^\circ))\simeq\mathrm{Hom}_{\mathcal{A}\times\mathcal{B}}(\mathbf{q}(\mathbf{h}(0,J),(I^\circ,E^\circ)))=\mathrm{Hom}_{\mathcal{B}}(J,E^\circ).$$

By definition, $\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(0,J),\mathbf{h}(I^\circ,E^\circ))$ is exact. It follows that E° is totally acyclic. This implies that $\ker\varphi$ is a Gorenstein injective object in $\mathcal{B}$.

Let C be an injective object in $\mathcal{A}$. Applying $\mathrm{Hom}_{(\mathcal{B}\downarrow T)}(\mathbf{h}(C,0),-)$ to the totally acyclic complex of injective objects $\mathbf{h}(I^\circ,E^\circ)$ in $(\mathcal{B}\downarrow T)$, we obtain that $I^\circ=\ldots\to I_1\to I_0\to I_{-1}\to\ldots$ is $\mathrm{Hom}_{\mathcal{B}}(C,-)$-exact and the proof is similar to the above. So $H=Z_0(I^\circ)$ is Gorenstein injective. □

Theorem 5.6. *Let $\mathcal{A}$ and $\mathcal{B}$ both have enough projective objects and enough injective objects.*

(1) *Assume that $\mathcal{X}$ is a subclass of $\mathcal{A}$ with $0 \in \mathcal{X}$, $\mathcal{Y}$ is a subclass of $\mathcal{B}$ with $0 \in \mathcal{Y}$ and $T\colon \mathcal{A} \to \mathcal{B}$ is a $\mathcal{X}$-exact functor. If $\mathcal{X}$ and $\mathcal{Y}$ are special prenveloping, then $\langle \mathbf{h}(\mathcal{X}, \mathcal{Y})\rangle$ is also special prenveloping in $(\mathcal{B} \downarrow T)$. Moreover, the converse holds when $T({}^{\perp}\mathcal{X} \cap \mathcal{X}) \subseteq {}^{\perp}\mathcal{Y}$ and $\mathcal{X}, \mathcal{Y}$ are closed under extensions.*

(2) *If $T\colon \mathcal{A} \to \mathcal{B}$ is a cocompatible functor, then $\mathcal{GI}_{\mathcal{A}}$ and $\mathcal{GI}_{\mathcal{B}}$ are special preenveloping in $\mathcal{A}$ and $\mathcal{B}$, respectively, if and only if $\mathcal{GI}_{(\mathcal{B}\downarrow T)}$ is special preenveloping in $(\mathcal{B} \downarrow T)$.*

P r o o f. At first, we claim that T is $\mathsf{Smd}(\mathcal{X})$-exact. For each exact sequence $0 \to X \to A_1 \to A_0 \to 0$ in $\mathcal{A}$ with $X \in \mathsf{Smd}(\mathcal{X})$, there exists M such that $X \oplus M \in \mathcal{X}$. Then there is an induced exact sequence $0 \to X \oplus M \to A_1 \oplus M \to A_0 \to 0$. Consequently, we obtain the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & TX \oplus TM & \longrightarrow & TA_1 \oplus TM & \longrightarrow & TA_0 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & TX & \longrightarrow & TA_1 & \longrightarrow & TA_0 & \longrightarrow & 0, \end{array}$$

where the top row is exact since T is $\mathcal{X}$-exact. It follows that the bottom row is exact.

Since $({}^{\perp}\mathsf{Smd}(\mathcal{X}), \mathsf{Smd}(\mathcal{X}))$ and $({}^{\perp}\mathsf{Smd}(\mathcal{Y}), \mathsf{Smd}(\mathcal{Y}))$ are complete cotorsion pairs by Lemma 4.5, the pair $\left(\binom{{}^{\perp}\mathsf{Smd}(\mathcal{X})}{{}^{\perp}\mathsf{Smd}(\mathcal{Y})}, \langle \mathbf{h}(\mathsf{Smd}(\mathcal{X}), \mathsf{Smd}(\mathcal{Y}))\rangle\right)$ is a complete cotorsion pair in $(\mathcal{B} \downarrow T)$ by Proposition 4.3. Thus, for each object $\binom{A}{B} \in (\mathcal{B} \downarrow T)$ there is an exact sequence $0 \to \binom{A}{B} \to \binom{\overline{X}}{M}_{\bar\varphi} \to \binom{\overline{C}}{\overline{D}}_{\overline\psi} \to 0$ with $\binom{\overline{X}}{M}_{\bar\varphi} \in \langle \mathbf{h}(\mathsf{Smd}(\mathcal{X}), \mathsf{Smd}(\mathcal{Y}))\rangle$ and $\binom{\overline{C}}{\overline{D}}_{\overline\psi} \in \binom{{}^{\perp}\mathsf{Smd}(\mathcal{X})}{{}^{\perp}\mathsf{Smd}(\mathcal{Y})}$.

Since $\mathcal{X}$ is special preenveloping, we have an exact sequence $0 \to \overline{X} \to X \to K \to 0$ in $\mathcal{A}$ with $X \in \mathcal{X}$ and $K \in {}^{\perp}\mathcal{X}$. It follows from ${}^{\perp}\mathcal{X} = {}^{\perp}\,\mathsf{Smd}(\mathcal{X})$ that $X \cong \overline{X} \oplus K$. Similarly, one can show that there exists an exact sequence $0 \to \ker\bar\varphi \to Y' \to U \to 0$ in $\mathcal{B}$ with $U \oplus \ker\bar\varphi \cong Y'$ and $U \in {}^{\perp}\mathcal{Y}$ by noting that $\ker\bar\varphi \in \mathsf{Smd}(\mathcal{Y})$ and $\mathcal{Y}$ is special preenveloping. For $TK \in \mathcal{B}$ there is an exact sequence $0 \to \overline{Y} \to N \xrightarrow{i} TK \to 0$ in $\mathcal{B}$ with $N \in {}^{\perp}\mathsf{Smd}(\mathcal{Y}) = {}^{\perp}\mathcal{Y}$ and $\overline{Y} \in \mathsf{Smd}(\mathcal{Y})$. Moreover, there is an exact sequence $0 \to \overline{Y} \to Y \to L \to 0$ in $\mathcal{B}$ with $Y \in \mathcal{Y}$ and $L \in {}^{\perp}\mathcal{Y}$, and so $Y \cong \overline{Y} \oplus L$. Therefore, we have an exact sequence

$$0 \to \binom{A}{B} \to \binom{X}{M}_{\varphi} \to \binom{C}{D}_{\psi} \to 0$$

in $(\mathcal{B} \downarrow T)$ with $X \cong \overline{X} \oplus K$, $C \cong \overline{C} \oplus K$, $M \cong \overline{M} \oplus U \oplus N \oplus L$, $D \cong \overline{D} \oplus U \oplus N \oplus L$,

$$\psi = \begin{pmatrix} \overline\psi & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix} \quad \text{and} \quad \varphi = \begin{pmatrix} \bar\varphi & 0 & 0 & 0 \\ 0 & 0 & i & 0 \end{pmatrix}.$$

Clearly, $\binom{C}{D}_{\psi} \in \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$. Notice that we have an exact sequence $0 \to \binom{0}{Y \oplus Y'} \to \binom{X}{M}_{\varphi} \to \binom{X}{TX} \to 0$ in $(\mathcal{B} \downarrow T)$. Thus, $\binom{X}{M}_{\varphi} \in \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$. So we obtain that $\langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$ is special preenveloping.

Conversely, assume that $\langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$ is special preenveloping. For any $\binom{A}{B} \in (\mathcal{B} \downarrow T)$, there is an exact sequence $\xi\colon\ 0 \to \binom{A}{B} \to \binom{X}{M}_{\varphi} \to \binom{C}{D}_{\psi} \to 0$ in $(\mathcal{B} \downarrow T)$ with $\binom{X}{M} \in \langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$ and $\binom{C}{D} \in {}^{\perp}\langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle$. By Proposition 3.5, ${}^{\perp}\langle \mathbf{h}(\mathcal{X}, \mathcal{Y}) \rangle = \binom{{}^{\perp}\mathcal{X}}{{}^{\perp}\mathcal{Y}}$. So the exact sequence $0 \to A \to X \to C \to 0$ implies that $\mathcal{X}$ is special preenveloping. Moreover, let $A = 0$ for the sequence ξ, then we have an exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \longrightarrow & \ker\varphi & \longrightarrow & \ker\psi & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & B & \xrightarrow{\simeq} & M & \longrightarrow & D & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \\
0 & \longrightarrow & 0 & \longrightarrow & TX & \longrightarrow & TC & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0. & &
\end{array}$$

Note that $M \in \mathcal{X} \cap {}^{\perp}\mathcal{X}$, $\ker\varphi \in \mathcal{Y}$, so the middle column is split. This implies that the right column is split. If follows that $\ker\phi \in {}^{\perp}\mathcal{Y}$ since $D \in {}^{\perp}\mathcal{Y}$. Thus, the exact sequence $0 \to B \to \ker\varphi \to \ker\phi \to 0$ yields the special $\mathcal{Y}$-preenvelop of B.

(2) Assume that $T\colon\ \mathcal{A} \to \mathcal{B}$ is a cocompatible functor. Note that ${}^{\perp}\mathcal{GI}_{\mathcal{A}} \cap \mathcal{GI}_{\mathcal{A}}$ is the class of injective objects in $\mathcal{A}$. It follows from (C2) in Definition 5.1 that $T({}^{\perp}\mathcal{GI}_{\mathcal{A}} \cap \mathcal{GI}_{\mathcal{A}}) \subseteq {}^{\perp}\mathcal{GI}_{\mathcal{B}}$. By (1) and Proposition 5.5, it suffices to show that T is $\mathcal{GI}_{\mathcal{A}}$-exact.

In fact, for each exact sequence $0 \to X \to A \to B \to 0$ in $\mathcal{A}$ with $X \in \mathcal{GI}_{\mathcal{A}}$, there is an exact sequence $0 \to X \to E_0 \to K_{-1} \to 0$ in $\mathcal{A}$ such that K_{-1} is a Gorenstein object and E_0 is an injective object. We choose an exact sequence $0 \to B \to J_0 \to C_{-1} \to 0$ in $\mathcal{A}$ with J_0 an injective object. Thus, we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X & \longrightarrow & A & \longrightarrow & B & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & E_0 & \longrightarrow & W_0 & \longrightarrow & J_0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & K_{-1} & \longrightarrow & Z_{-1} & \longrightarrow & C_{-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

where the second row is split. Note that $0 \to TX \to TE_0 \to TK_{-1} \to 0$ is an exact sequence in $\mathcal{B}$ by (C1) in Definition 5.1. Applying the functor T to the above diagram, we have the following commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TX & \longrightarrow & TA & \longrightarrow & TB & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TE_0 & \longrightarrow & TW_0 & \longrightarrow & TJ_0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & TK_{-1} & \longrightarrow & TZ_{-1} & \longrightarrow & TC_{-1} & & \\
 & & \downarrow & & & & & & \\
 & & 0. & & & & & &
\end{array}$$

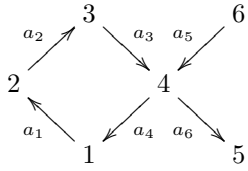
By the Snake lemma, we have that $0 \to TX \to TA \to TB \to 0$ is an exact sequence in $\mathcal{B}$. □

Suppose that $\Lambda = \left(\begin{smallmatrix} S & 0 \\ M & R \end{smallmatrix}\right)$ is a triangular matrix ring and ${}_RM_S$ an R-S-bimodule. Define $T \cong \mathrm{Hom}_R(M, -)\colon\ R\text{-Mod} \to S\text{-Mod}$, then we know that Λ-Mod is equivalent to the comma category $(S\text{-Mod} \downarrow T)$. By Proposition 5.2 and Theorem 5.6 (2), we have the following result.

Corollary 5.7. *Let $\Lambda = \left(\begin{smallmatrix} S & 0 \\ M & R \end{smallmatrix}\right)$ be a triangular matrix ring. Assume that $\mathrm{pd}_R M$ and $\mathrm{fd} M_S$ are finite. Then $\mathcal{GI}(\Lambda)$ is special preenveloping in Λ-Mod if and only if $\mathcal{GI}(R)$ and $\mathcal{GI}(S)$ are special preenveloping in R-Mod and S-Mod, respectively.*

Next, we provide an instance to illustrate the above result. We assume that the composition of arrows α and β with $\mathfrak{t}(\alpha) = \mathfrak{s}(\beta)$ in this example, where $\mathfrak{s}$ and $\mathfrak{t}$ are two functions defined on quiver $\mathcal{Q} = (\mathcal{Q}_0, \mathcal{Q}_1, \mathfrak{s}, \mathfrak{t})$ sending any arrow to its starting point and ending point, respectively.

Example 5.8. Let k be a field and $S = k\mathcal{Q}/\mathcal{I}$ a finite-dimensional k-algebra given by $\mathcal{Q} =$

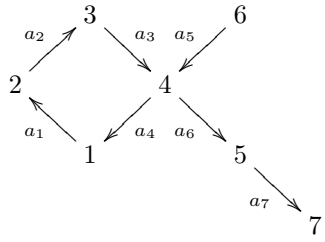


and $\mathcal{I} = \langle a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_5a_6 \rangle$. Let $R = k$ be the k-module with k-dimension one and $\Lambda = \left(\begin{smallmatrix} k & 0 \\ {}_kS(5)_S & S \end{smallmatrix}\right)$ (${}_kS(5)_S = S(5)_S$ be the simple module in mod-S corresponding to the vertex $5 \in \mathcal{Q}_0$). Then Λ is a finite-dimensional k-algebra, and we

obtain that $\Lambda = k\mathcal{Q}'/\mathcal{I}'$ by

$$\dim_k \varepsilon_i'(\mathrm{rad}\Lambda/\mathrm{rad}^2\Lambda)\varepsilon_j' = \sharp\{a\colon\ j \to i\colon\ a \in \mathcal{Q}_1'\}$$

(ε_i is the primitive idempotent of Λ corresponding to $i \in \mathcal{Q}_0$, and $\sharp N$ the number of all elements of the set N), where



and $\mathcal{I}' = \langle a_1a_2, a_2a_3, a_3a_4, a_4a_1, a_5a_6, a_6a_7\rangle$, see [1], Chapter II, Definition 3.1 and Theorem 3.7. Then the following statements hold:

▷ $T := \mathrm{Hom}_k(S(5), -)$ is cocompatible by Proposition 5.2 because $S(5)$ is a left projective k-module and right flat S-module, that is, $\mathrm{pd}_k S(5) = 0$ and $\mathrm{fd}S(5)_S = 0$.

▷ The Gorenstein injective left modules over S and Λ can be calculated by the dual of [3], Theorem 4.1 (or the dual of [9], Theorem 2.5) because S and Λ are gentle algebras; to be more precise,

$$\mathcal{GI}(S) = \left\{ {}_SS(1), {}_SS(2), {}_SS(3), {}_S\begin{pmatrix}4\\5\end{pmatrix} \right\} \cup \{{}_SE(i)\colon\ i \in \mathcal{Q}_0\};$$
$$\mathcal{GI}(\Lambda) = \left\{ {}_\Lambda S(1), {}_\Lambda S(2), {}_\Lambda S(3), {}_\Lambda\begin{pmatrix}4\\5\end{pmatrix} \right\} \cup \{{}_\Lambda E(i)\colon\ i \in \mathcal{Q}_0'\}.$$

▷ $\mathcal{GI}(R)$ $(= \mathcal{GI}(k)) = k$.

One can check that $\mathcal{GI}(R)$ and $\mathcal{GI}(S)$ are special preenveloping in R- mod and S- mod, respectively. Then $\mathcal{GI}(\Lambda)$ is special preenveloping in Λ- mod by Corallary 5.7, and vice verse. In fact, we can show the above conclusion by the Auslander-Reiten quiver $\Gamma(\Lambda\text{- mod})$ of Λ- mod which is shown in Figure 1. (The modules enclosed with solid lines and dashed lines are indecomposable injective and non-injective Gorenstein-injective modules, respectively.)

Notice that the composition of two arrows α and β we defined is $\alpha\beta$ ($\mathfrak{t}(\alpha) = \mathfrak{s}(\beta)$), then the representations of indecomposable left Λ-modules and the Auslander-Reiten quiver $\Gamma(\Lambda\text{- mod})$ of Λ- mod are opposite to that of right Λ-modules and mod-Λ, respectively.

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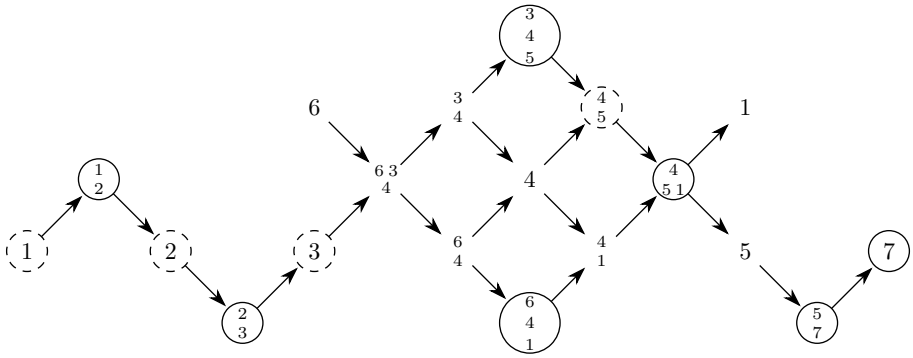


Figure 1. The Auslander-Reiten quiver of Λ in Example 5.8.

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