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THE ∘ OPERATION AND ∗ OPERATION OF COHEN-MACAULAY BIPARTITE GRAPHS

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Abstract. Let G be a finite simple graph with the vertex set V and let I_G be its edge ideal in the polynomial ring $S = \mathbb{K}[V]$. We compute the depth and the Castelnuovo-Mumford regularity of S/I_G when $G = G_1 \circ G_2$ or $G = G_1 * G_2$ is a graph obtained from Cohen-Macaulay bipartite graphs G_1, G_2 by the $\circ$ operation or $*$ operation, respectively.

Keywords: regularity; depth; $\circ$ operation; $*$ operation; Cohen-Macaulay bipartite graph

MSC 2020: 13C15, 13A15, 13D02, 05E40

1. Introduction

Let G be a simple graph on the vertex set $V(G)$ with no isolated vertices. Let $\mathbb{K}$ be a field and $S := \mathbb{K}[V(G)] = \mathbb{K}[v\colon\ v \in V(G)]$ a polynomial ring with $\deg(v) = 1$. The edge ideal of G, denoted by I_G, is defined as $I_G := (uv\colon\ \{u, v\} \in E(G))$, where $E(G)$ is the set of all edges of G. In the last two decades many authors have established connections between the combinatorial properties of G and the algebraic properties of I_G, see e.g. [6], [7], [8], [15], [16], [23], [24], [25], [26].

For any two monomial ideals I and J, it is known that $\operatorname{reg}(I + J) \leqslant \operatorname{reg}(I) + \operatorname{reg}(J) - 1$, see [9] and [14]. Suppose we are given a finite simple graph G and its subgraphs G_1 and G_2 with the edge sets $E(G)$, $E(G_1)$, and $E(G_2)$, respectively, such that $E(G) = E(G_1) \cup E(G_2)$. Then for the edge ideal the above inequalities imply $\operatorname{reg}(I_G) \leqslant \operatorname{reg}(I_{G_1}) + \operatorname{reg}(I_{G_2}) - 1$. Herzog et al. in [12] introduced the concept of splitting graphs of a graph. Recall that a graph $G' = (V(G'), E(G'))$ is said to be a splitting graph of another graph $G = (V(G), E(G))$, if there exists

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a surjective map $\alpha\colon V(G') \to V(G)$ such that $\{\alpha(u), \alpha(v)\}$ is an edge of G for all edges of G for all edges $\{u, v\}$ of G' and the map $E(G') \to E(G)$, $e \mapsto \alpha(e)$ is bijective. We denote by G' a splitting graph of G and by α a splitting map of G. They proved in [12], Theorem 1.3 and Proposition 1.6, that $\operatorname{reg}(I_G) \leqslant \operatorname{reg}(I_{G'})$ if $\operatorname{reg}(I_G) = \theta(G)+1$, where $\theta(G)$ is the induced matching number of G, or the splitting map $\alpha\colon V(G') \to V(G)$ is special. They also proved that for path graphs and cycle graphs of even length, $\operatorname{depth}(S_{G'}/I_{G'}) \geqslant \operatorname{depth}(S_G/I_G)$, where S_G and $S_{G'}$ are the polynomial rings over a field $\mathbb{K}$ in the variables corresponding to $V(G)$ and $V(G')$, respectively.

In this paper, we are interested in whether formulas for the regularity and depth of S_G/I_G can be accurately determined in terms of this information of S_{G_1}/I_{G_1} and S_{G_2}/I_{G_2} for a splitting graph $G' := G_1 \sqcup G_2$ of graph G and subgraphs of G'. So far, there are fewer results in this direction, and it turns out to be a very hard problem.

The regularity of the edge ideal I_G is often related to the matching of the graph G. Let $\theta(G)$ and $\beta(G)$ be the induced matching number and the minimum matching number of G, respectively. Then

$$\theta(G) \leqslant \operatorname{reg}(S/I_G) \leqslant \beta(G)$$

by [8] and [15]. The left inequality becomes an equality in the following cases:

(a) G is a chordal graph;
(b) G is a weakly chordal graph;
(c) G is a sequentially Cohen-Macaulay bipartite graph;
(d) G is an unmixed bipartite graph;
(e) G is a very well-covered graph;
(f) G is a C_5-free vertex decomposable graph;
(g) G is a (C_4, C_5)-free vertex decomposable graph, see [2], [8], [14], [17], [18], [21], [23].

For the depth of the edge ideal I_G, it is also related to the invariants of the graph G. Let $u, v \in V(G)$. The distance between u and v, denoted by $d(u, v)$, is the length of the shortest path between u and v. If G is connected, then the diameter of G is $d(G) = \max\{d(u, v)\colon u, v \in V(G)\}$. Fouli and Morey in [7] showed that if a graph G has p connected components, then

$$\operatorname{depth}(S/I_G) \geqslant \sum_{i=1}^{p} \left\lceil \frac{d_i + 1}{3} \right\rceil,$$

where $\lceil \frac{1}{3}(d_i + 1) \rceil$ is the smallest integer greater or equal than $\frac{1}{3}(d_i + 1)$ and d_i is the diameter of the ith connected component of G. The second author of this paper in [24] proved that if G is a path, then $\operatorname{depth}(S/I_G)$ can reach this lower bound.

Morey et al. in [19] showed that for a connected bipartite graph G with n vertices,

$$\operatorname{depth}(S/I_G) \geqslant \left\lfloor \frac{n}{2} \right\rfloor,$$

where $\lfloor \frac{1}{2}n \rfloor$ is the largest integer less or equal than $\frac{1}{2}n$. The second author of this paper in [24] and [25] gave some exact formulas for the depth and regularity of the edge ideals of path graphs and cycle graphs, respectively.

Bolognini et al. introduced in [3] a family of simple graphs that can be obtained from some graphs by the $\circ$ operation or the $*$ operation, see Definition 4.1. If $G = (G_1, f_1) \circ (G_2, f_2)$, where f_i is a leaf of G_i and v_i is its neighbor vertex in G_i for $i = 1, 2$, we set $G' = (G_1 \setminus f_1) \sqcup (G_2 \setminus f_1)$ and define a map $\alpha' \colon V(G') \to V(G)$ by $\alpha'(v_1) = \alpha'(v_2) = v$ and $\alpha'(u) = u$ with $u \in V(G') \setminus \{v_1, v_2\}$. It is clear that G' is a splitting graph of G and G' is an induced subgraph of $G_1 \sqcup G_2$. If $G = (G_1, f_1) * (G_2, f_2)$, where f_i is a leaf of G_i for $i = 1, 2$, we set $G' = G_1 \sqcup G_2$ and define a map $\alpha \colon V(G') \to V(G)$ by $\alpha(f_1) = \alpha(f_2) = f$ and $\alpha(u) = u$ with $u \in V(G') \setminus \{f_1, f_2\}$. It is also clear that G' is a splitting graph of G. The first three authors of this article in [26] studied two families of simple graphs obtained from some fan graphs by the $*$ operation and the $\circ$ operation, respectively. For such two graphs, we gave some formulas for the depth and regularity of S/I_G.

In this article, we are interested in algebraic properties of depth and regularity of S/I_G when G is a graph obtained from two Cohen-Macaulay (abbreviated as C-M) bipartite graphs by the $\circ$ operation or the $*$ operation. The results are as follows:

Theorem 1.1. *Let $G = (G_1, u_1) \circ (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_G(u_i) = \{v_i\}$. Then*

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - s,$$

where $s = 1$ if $\deg_{G_i}(v_i) = 1$ for all $i \in [2]$, otherwise $s = 2$. And if $t = |\{i \colon \theta(G_i \setminus v_i) \neq \theta(G_i)\}|$, then

$$\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - t.$$

Theorem 1.2. *Let $G = (G_1, u_1) * (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_{G_i}(u_i) = \{v_i\}$. Then*

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.$$

If $t = |\{i \colon \theta(G_i \setminus v_i) \neq \theta(G_i)\}|$, then

$$\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - s,$$

where $s = 0$ if $t \leqslant 1$, and $s = 1$ if $t = 2$.

The article is organized as follows. In Section 2, we will review some basic definitions and terminology that we will need later. In Section 3, we will study the depth and regularity of a bipartite graph obtained from a C-M bipartite graph by deleting its one leaf. We give some exact formulas for the depth and regularity of the edge ideal of such a graph. In Section 4, we will study some graphs that are derived from C-M bipartite graphs by the $\circ$ operation or the $*$ operation. For such graphs, we give some exact formulas for the depth and regularity of their edge ideals.

2. Preliminary

In this section, we gather together the necessary definitions and basic facts that will be used throughout this paper. However, for more details, we refer the reader to [4], [11], [22].

Let $G = (V(G), E(G))$ be a finite simple (no loops, no multiple edges) graph, where $V(G)$ and $E(G)$ are the vertex and edge sets of G, respectively. Sometimes for short we denote $V(G)$ and $E(G)$ by V and E, respectively. A graph $H = (V(H), E(H))$ is said to be a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The *neighborhood* of a vertex v in G is defined as $N_G(v) = \{u\colon \{u, v\} \in E(G)\}$ and its degree, denoted by $\deg_G(v)$, is $|N_G(v)|$. If $|N_G(v)| = 1$, then v is called a *leaf*. Set $N_G[v] = N_G(v) \cup \{v\}$. For $A \subset V(G)$, $G|_A$ denotes the *induced subgraph* of G on the set A, i.e., for $i, j \in A$, $\{i, j\} \in E(G|_A)$ if and only if $\{i, j\} \in E(G)$. For $W \subseteq V(G)$, we denote by $G \setminus W$ the induced subgraph of G on $V(G) \setminus W$. For a vertex $v \in V(G)$, we denote by $G \setminus v$ for simplicity the induced subgraph of G on the set $V(G) \setminus \{v\}$.

A subset $M \subset E(G)$ is a *matching* of G if $e \cap e' = \emptyset$ for any pair of edges $e, e' \in M$. A matching is a *maximal matching* if it has the greatest possible number of edges. The *minimum matching number* of G, denoted by $\beta(G)$, is the minimum size of a maximal matching in G. An *induced matching* in G is a matching M such that the induced subgraph of G over the vertices of M does not contain any edge other than those already in M. The *induced matching number* of G, denoted by $\theta(G)$, is the maximum size of an induced matching in G.

A *walk* of length $(n-1)$ in a graph G is an alternating sequence of vertices and edges $w = \{v_1, z_1, v_2, \ldots, v_{n-1}, z_{n-1}, v_n\}$, where $z_i = \{v_i, v_{i+1}\}$ is the edge joining v_i and v_{i+1}. A walk is closed if $v_1 = v_n$. A walk may also be denoted by $\{v_1, \ldots, v_n\}$, the edges being evident by context. A *cycle* of length n is a closed walk, in which the points $v_1, \ldots, v_n$ are distinct. We denote the graph consisting of a cycle with n vertices by C_n. A *path* is a walk with all the points distinct. For simplicity, a *path* with n vertices, denoted by P_n, is a walk with the vertex set $[n]$ and edge set $\{\{1, 2\}, \{2, 3\}, \ldots, \{n-1, n\}\}$, where $[n]$ denotes the set $\{1, \ldots, n\}$, and the length of P_n is defined to be $n-1$. Any graph isomorphic to P_n is also called a path.

In the sequel, let S_+ be the unique graded maximal ideal of the standard graded algebra S. The local cohomology modules of a finitely generated graded S-module M with respect to S_+ are denoted by $H^i_{S_+}(M)$ for $i \in \mathbb{Z}$.

Definition 2.1. Let M be a finitely generated graded S-module.

The *depth* of M is defined as

$$\operatorname{depth}(M) := \min\{i\colon\ H^i_{S_+}(M) \neq 0\}.$$

(1) For $i = 0, \ldots, \dim(M)$, the ith *a-invariant* of M is defined as

$$a_i(M) := \max\{t\colon\ (H^i_{S_+}(M))_t \neq 0\}$$

with the convention that $\max \emptyset = -\infty$.

(2) The *Castelnuovo-Mumford regularity* of M is defined as

$$\operatorname{reg}(M) := \max\{a_i(M) + i\colon\ 0 \leqslant i \leqslant \dim(M)\}.$$

A graph G is called *Cohen-Macaulay* (abbreviated as C-M) if the quotient ring S/I_G is Cohen-Macaulay, i.e., $\operatorname{depth}(S/I_G) = \dim(S/I_G)$. For a proper nonzero homogeneous ideal I in S, it is known that $\operatorname{reg}(S/I) = \operatorname{reg}(I) - 1$.

The following lemmas are often used to compute the depth and regularity of a module. In particular, since the facts in Lemma 2.2 are well-known, they will be used implicitly in this paper.

Lemma 2.2. *Let M, N be two finitely generated graded S-modules. Then*

(1) $\operatorname{depth}(M \oplus N) = \min\{\operatorname{depth}(M), \operatorname{depth}(N)\}$, *and*

(2) $\operatorname{reg}(M \oplus N) = \max\{\operatorname{reg}(M), \operatorname{reg}(N)\}$.

Lemma 2.3 ([13], Lemmas 2.1 and 3.1)**.** *Let $0 \to M \to N \to P \to 0$ be an exact sequence of finitely generated graded S-modules. Then we have*

(1) $\operatorname{depth}(M) \geqslant \min\{\operatorname{depth}(N), \operatorname{depth}(P)+1\}$, *and equality holds if* $\operatorname{depth}(N) \neq \operatorname{depth}(P)$,

(2) $\operatorname{reg}(M) \leqslant \max\{\operatorname{reg}(N), \operatorname{reg}(P)+1\}$, *and equality holds if* $\operatorname{reg}(N) \neq \operatorname{reg}(P)$.

Lemma 2.4 ([13], Lemmas 2.2 and 3.2)**.** *Let $S_1 = \mathbb{K}[x_1, \ldots, x_m]$ and $S_2 = \mathbb{K}[x_{m+1}, \ldots, x_n]$ be two polynomial rings over $\mathbb{K}$, let $I \subset S_1$ and $J \subset S_2$ be two nonzero homogeneous ideals. Let $S = S_1 \otimes_{\mathbb{K}} S_2$. Then we have*

(1) $\operatorname{reg}(S/(I+J)) = \operatorname{reg}(S_1/I) + \operatorname{reg}(S_2/J)$,

(2) $\operatorname{depth}(S/(I+J)) = \operatorname{depth}(S_1/I) + \operatorname{depth}(S_2/J)$.

For a subset $A \subset V(G)$, let $(A) = (v\colon\ v \in A)$ be an ideal of $S = \mathbb{K}[V]$ generated by the elements in A. The following lemma is very important for the whole paper.

Lemma 2.5 ([26], Lemma 1.5)**.** *Let $G = (V, E)$ be a connected simple graph. Let $J = (N_G(v)) + I_{G\setminus N_G[v]}$ and $K = (v) + I_{G\setminus v}$, where $v \in V$. Then*
(1) $J + K = (N_G[v]) + I_{G\setminus N_G[v]}$;
(2) $I_G = J \cap K$;
(3) $\operatorname{depth}(S/J) = \operatorname{depth}(S/(J + K)) + 1$;
(4) $\operatorname{reg}(S/J) = \operatorname{reg}(S/(J + K))$.

From Lemmas 2.2, 2.3 and 2.5, we get the following lemma.

Lemma 2.6. *Let J and K be as in Lemma* 2.5.
(1) *If* $\operatorname{reg}(S_G/K) > \operatorname{reg}(S_G/J)$, *then* $\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_G/K)$.
(2) *If* $\operatorname{depth}(S_G/K) = \operatorname{depth}(S_G/J)$, *then* $\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_G/K) = \operatorname{depth}(S_G/J)$.

P r o o f. The expected result follows from the following short exact sequence:

$$0 \to \frac{S_G}{J \cap K} \to \frac{S_G}{J} \oplus \frac{S_G}{K} \to \frac{S_G}{J + K} \to 0.$$

□

A graph G is called *bipartite* if there exists a *bipartition* $V(G) = V_1 \sqcup V_2$ with $V_1 \cap V_2 = \emptyset$ such that each edge of G is of the form $\{i, j\}$ with $i \in V_1$ and $j \in V_2$. In [10], Herzog and Hibi classified all C-M bipartite graphs. We state their result.

Theorem 2.7 ([10], Theorem 3.4)**.** *Let $G = (V(G), E(G))$ be a bipartite graph with bipartition $V(G) = \{x_1, x_2, \ldots, x_n\} \sqcup \{y_1, y_2, \ldots, y_m\}$. Then G is C-M if and only if $n = m$, and there exists a labeling such that*
(1) $\{x_i, y_i\} \in E(G)$ *for* $i \in [n]$,
(2) *if* $\{x_i, y_j\} \in E(G)$, *then* $i \leqslant j$, *and*
(3) *if* $\{x_i, y_j\} \in E(G)$ *and* $\{x_j, y_k\} \in E(G)$ *with* $i < j < k$, *then* $\{x_i, y_k\} \in E(G)$.

By Theorem 2.7, the vertices y_1 and x_n must be of degree one, and their neighborhood points are x_1 and y_n, respectively. Let $N_G(y_n) = \{x_{i_1}, \ldots, x_{i_s}, x_n\}$ for some $x_{i_j} \in \{x_1, \ldots, x_n\}$. Francisco et al. in [5] showed:

Lemma 2.8 ([5], Lemma 3.4)**.** *Let G be a C-M bipartite graph with bipartition $V(G) = \{x_1, \ldots, x_n\} \sqcup \{y_1, \ldots, y_n\}$. Then*
(1) $G \setminus \{x_n, y_n\}$ *is a C-M bipartite graph,*
(2) $G \setminus \{x_{i_1}, y_{i_1}, \ldots, x_{i_s}, y_{i_s}, x_n, y_n\}$ *is a C-M bipartite graph.*

Lemma 2.9. *Let $G = (V(G), E(G))$ be a C-M bipartite graph without isolated vertices. Then*
(1) $\operatorname{depth}(S/I_G) = \frac{1}{2}|V(G)|$,
(2) $\operatorname{reg}(S/I_G) = \theta(G)$, *where $\theta(G)$ is the induced matching number of G.*

P r o o f. (1) Since the bipartite graph G is C-M, all of the associated primes of I_G have the same height. Moreover, each associated prime corresponds to a vertex cover of G. Since $\{x_1, \ldots, x_n\}$ is a vertex cover of G, all associated primes have height n. By Theorem 2.7, G is a graph with $2n$ vertices. So $\frac{1}{2}|V(G)| = n = \dim(S/I_G) = \operatorname{depth}(S/I_G)$, where the last equality also uses the fact that G is C-M.

(2) It is a direct consequence of [21], Corollary 3.4. □

Lemma 2.10 ([24], Theorem 3.3, Corollary 3.3)**.** *Let $n \geqslant 2$ be an integer and P_n be a path with n vertices. Then*

$$\operatorname{depth}(S/I_G) = \left\lceil \frac{n}{3} \right\rceil \quad \textit{and} \quad \operatorname{reg}(S/I_G) = \left\lfloor \frac{n+1}{3} \right\rfloor.$$

3. Study of bipartite graphs

In this section, we will study the depth and regularity of a bipartite graph obtained from a Cohen-Macaulay bipartite graph by deleting its one leaf. By repeatedly applying Lemma 2.6, we give some exact formulas for the depth and regularity of the edge ideal of such a graph.

Lemma 3.1. *Let G be a C-M bipartite graph with a leaf u. Let $N_G(u) = \{v\}$ with $\deg_G(v) \geqslant 2$. Then*

$$\operatorname{depth}(S_{G\setminus u}/I_{G\setminus u}) = \operatorname{depth}(S_G/I_G) - 1.$$

P r o o f. Let $V(G) = \{x_1, x_2, \ldots, x_n\} \sqcup \{y_1, y_2, \ldots, y_n\}$ and $u = x_i$ for some $i \in [n]$, by symmetry. Then $v = y_i$ with $\deg_G(y_i) \geqslant 2$. Thus, $n \geqslant 2$. We prove the claimed formula by induction on n. If $n = 2$, then G and $G \setminus u$ are paths with 4 and 3 vertices, respectively. This case is covered by Lemma 2.10.

In the following, we assume that $n \geqslant 3$. Let $H = G \setminus u$ and $N_H(v) = \{x_{i_1}, \ldots, x_{i_s}\}$ with $1 \leqslant i_1 < i_2 < \ldots < i_s < i$, then $\deg_H(y_{i_1}) = 1$ by Theorem 2.7 (2). Let $N_H(x_{i_1}) = \{y_{j_1}, y_{j_2}, \ldots, y_{j_t}\}$ with $j_1 = i_1 < j_2 < \ldots < j_t \leqslant n$. Set $J = (N_H(x_{i_1})) + I_{H\setminus N_H[x_{i_1}]}$, $K = (x_{i_1}) + I_{H\setminus x_{i_1}}$. We distinguish between the following two cases:

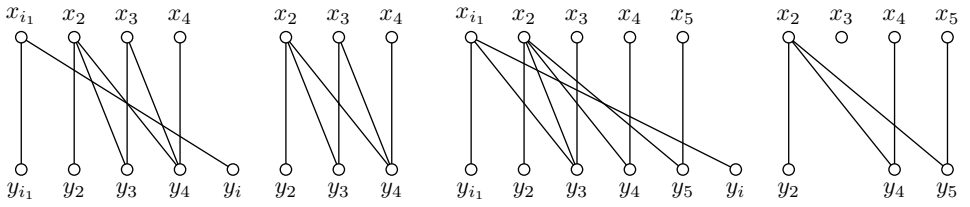
(1) If $|N_H(v)| = 1$, then $s = 1$ and $H \setminus x_{i_1}$ is the disjoint union of $G \setminus \{u, v, x_{i_1}, y_{i_1}\}$ and the isolated set $\{y_{i_1}, v\}$. Thus, by Lemma 2.9 (1), we have

$$\operatorname{depth}(S_H/K) = 2 + (n-2) = n.$$

Meanwhile, $H \setminus N_H[x_{i_1}]$ has one of the following forms. Figure 1 will be helpful in understanding the arguments.

(a) $H \setminus N_H[x_{i_1}] = G \setminus \{u, v, x_{i_1}, y_{i_1}\}$;

(b) $H \setminus N_H[x_{i_1}]$ is the disjoint union of $G \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_t}, y_{j_t}\}$ and the isolated set $\{x_{j_2}, \ldots, x_{j_t}\} \setminus \{u\}$.



(a) $H \setminus N_H[x_{i_1}]$ (b) $H \setminus N_H[x_{i_1}]$

Figure 1.

By Lemma 2.8 (2), we get that both $G \setminus \{u, v, x_{i_1}, y_{i_1}\}$ and $G \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_t}, y_{j_t}\}$ are C-M bipartite graphs. We consider the following two subcases:

(i) If $H \setminus N_H[x_{i_1}]$ is of form (a), then we get by Lemma 2.9 (1) that

$$\operatorname{depth}(S_H/J) = 1 + \operatorname{depth}(S_{H \setminus N_H[x_{i_1}]}/I_{H \setminus N_H[x_{i_1}]}) = 1 + (n-2) = n-1.$$

(ii) If $H \setminus N_H[x_{i_1}]$ is of form (b), then we also have

$$\operatorname{depth}(S_H/J) = 1 + \operatorname{depth}(S_{H \setminus N_H[x_{i_1}]}/I_{H \setminus N_H[x_{i_1}]}) = 1 + (t-2) + (n-t) = n-1.$$

(2) If $|N_H(v)| \geqslant 2$, then $H \setminus x_{i_1}$ is the disjoint union of $G \setminus \{x_{i_1}, y_{i_1}, u\}$ and an isolated vertex y_{i_1}. Note that $G \setminus \{x_{i_1}, y_{i_1}, u\}$ can be viewed as $G_1 \setminus u$, where $G_1 = G \setminus \{x_{i_1}, y_{i_1}\}$. In this case, let $H' = G_1 \setminus u$, then $|N_{H'}(v)| = |N_H(v)| - 1$. Thus, by induction and Lemma 2.9 (1), we have

$$\begin{aligned}\operatorname{depth}(S_H/K) &= 1 + \operatorname{depth}(S_{G \setminus \{x_{i_1}, y_{i_1}, u\}}/I_{G \setminus \{x_{i_1}, y_{i_1}, u\}}) \\ &= 1 + \operatorname{depth}(S_{G_1}/I_{G_1}) - 1 = \operatorname{depth}(S_{G_1}/I_{G_1}) = n-1.\end{aligned}$$

At the same time, $H \setminus N_H[x_{i_1}]$ has one of the following forms. Figure 2 will be helpful in understanding the arguments.

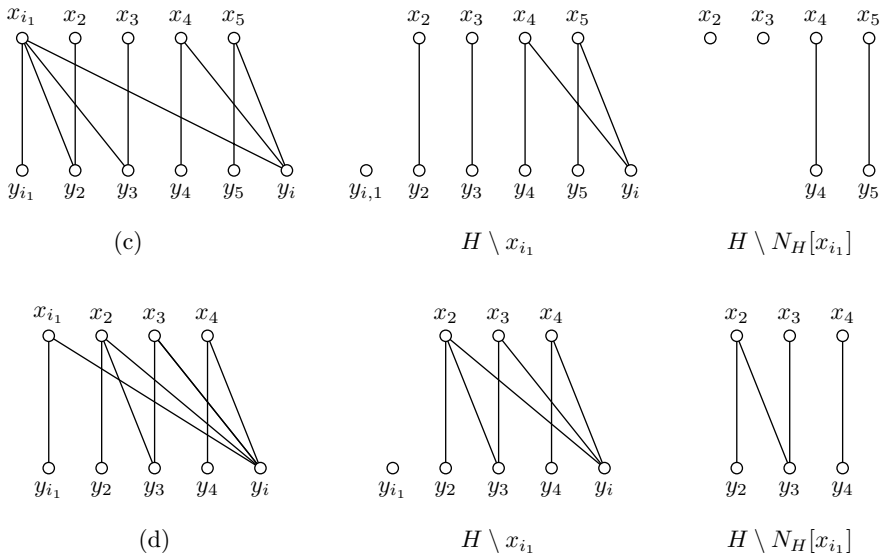


Figure 2.

(c) $H\setminus N_H[x_{i_1}]$ is the disjoint union of $G\setminus\{x_{j_1},\dots,x_{j_t},y_{j_1},\dots,y_{j_t}\}$ and the isolated set $\{x_{j_2},\dots,x_{j_t}\}\setminus\{u\}$;

(d) $H \setminus N_H[x_{i_1}] = G \setminus \{u, v, x_{i_1}, y_{i_1}\}$.

As shown in (a) and (b) of case (1) above, we can get

$$\operatorname{depth}(S_H/J) = 1 + \operatorname{depth}(S_{H\setminus N_H[x_{i_1}]}/I_{H\setminus N_H[x_{i_1}]}) = 1 + (n-2) = n-1.$$

Therefore, the desired result can be obtained from Lemma 2.6 by replacing G in Lemma 2.6 with H. □

Remark 3.2. Let G be a C-M bipartite graph with a leaf u. Let $N_G(u) = \{v\}$ with $\deg_G(v) = 1$. Then $G \setminus u$ is the disjoint union of $G \setminus \{u, v\}$ and an isolated vertex v. So $\operatorname{depth}(S_{G\setminus u}/I_{G\setminus u}) = 1 + \operatorname{depth}(S_{G\setminus\{u,v\}}/I_{G\setminus\{u,v\}}) = \operatorname{depth}(S_G/I_G)$.

Lemma 3.3 ([1], Lemma 3.5)**.** *Let G be a simple graph and H be its induced subgraph. Then* $\operatorname{reg}(I_H) \leqslant \operatorname{reg}(I_G)$.

Lemma 3.4. *Let $G = (V, E)$ be a C-M bipartite graph with a leaf u. Let $N_G(u) = \{v\}$, $J = (N_G(v)) + I_{G\setminus N_G[v]}$ and $K = (v) + I_{G\setminus v}$. Then*

(1) $\operatorname{reg}(S_G/J) \leqslant \operatorname{reg}(S_G/I_G) - 1$;

(2) $\operatorname{reg}(S_G/K) \leqslant \operatorname{reg}(S_G/I_G)$.

P r o o f. It is clear that $\operatorname{reg}(S_G/K) \leqslant \operatorname{reg}(S_G/I_G)$ and $\operatorname{reg}(S_G/J) \leqslant \operatorname{reg}(S_G/I_G)$ by Lemma 3.3, since $\operatorname{reg}(S_G/K) = \operatorname{reg}(S_{G\setminus v}/I_{G\setminus v})$, $\operatorname{reg}(S_G/J) = \operatorname{reg}(S_{G\setminus N_G[v]}/I_{G\setminus N_G[v]})$ and both $G \setminus v$ and $G \setminus N_G[v]$ are induced subgraphs of G.

Let $V = X \sqcup Y$ with $X = \{x_1, x_2, \ldots, x_n\}$, $Y = \{y_1, y_2, \ldots, y_n\}$ and $u = x_l$. Suppose $N_G(u) = \{v\}$ and $N_G(v) = \{x_{i_1}, \ldots, x_{i_t}\}$ with $1 \leqslant i_1 < i_2 < \ldots < i_t = l$. Two cases are discussed below:

(i) If $N_G(v) = X$, then $G \setminus N_G[v] = Y \setminus \{v\}$ consists of isolated points. Hence, $I_{G\setminus N_G[v]} = 0$, which implies that $\operatorname{reg}(S_G/J) = \operatorname{reg}(S_{G\setminus N_G[v]}/I_{G\setminus N_G[v]}) = 0$.

(ii) If $N_G(v) \subsetneq X$, then $G \setminus N_G[v]$ is the disjoint union of a graph H and the isolated set $\{y_{i_1}, y_{i_2}, \ldots, y_{i_{t-1}}\}$, where $H = G \setminus \{x_{i_1}, \ldots, x_{i_t}, y_{i_1}, \ldots, y_{i_t}\}$. So by Lemma 2.9 (2) we have

$$\operatorname{reg}(S_G/J) = \operatorname{reg}(S_{G\setminus N_G[v]}/I_{G\setminus N_G[v]}) = \operatorname{reg}(S_H/I_H) = \theta(H),$$

since H is a C-M bipartite graph by Lemma 2.8 (2). Let $M = \{e_1, e_2, \ldots, e_{\theta(H)}\}$ be an induced matching of H and $e = \{u, v\}$. Since $V(H) \cap N_G[v] = \emptyset$ and u is a leaf with $N_G(u) = \{v\}$, we have $M \cap \{e\} = \emptyset$, which implies $M \sqcup \{e\}$ is an induced matching of G. Hence, in this case, $\operatorname{reg}(S_G/I_G) \geqslant \theta(H) + 1$, establishing the claim. □

Lemma 3.5. *Let $G = (V, E)$ be a C-M bipartite graph with a leaf u. Let $N_G(u) = \{v\}$ with $\deg_G(v) \geqslant 2$. Suppose $V = X \sqcup Y$, where $X = \{x_1, x_2, \ldots, x_n\}$ and $Y = \{y_1, y_2, \ldots, y_n\}$. Let $u = y_{i_1}$, $N_G(v) = \{y_{i_1}, y_{i_2}, \ldots, y_{i_t}\}$ with $1 \leqslant i_1 < i_2 < \ldots < i_t \leqslant n$ and $w = y_{i_t}$. Suppose also that $J = (N_G(w)) + I_{G\setminus N_G[w]}$, $K = (w) + I_{G\setminus w}$. If $\theta(G \setminus v) = \theta(G) - 1$, then*

(1) $\operatorname{reg}(S_G/J) \leqslant \operatorname{reg}(S_G/I_G) - 2$;

(2) $\operatorname{reg}(S_G/K) = \operatorname{reg}(S_G/I_G)$.

Proof. Since $N_G(v) = \{y_{i_1}, y_{i_2}, \ldots, y_{i_t}\}$ with $1 \leqslant i_1 < i_2 < \ldots < i_t \leqslant n$, we get that x_{i_t} is a leaf and $N_G(x_{i_t}) = \{w\}$. Let $N_G(w) = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$ with $1 \leqslant j_1 < j_2 < \ldots < j_m = i_t$. It follows that $\operatorname{reg}(S_G/K) = \operatorname{reg}(S_{G\setminus w}/I_{G\setminus w})$ and $\operatorname{reg}(S_G/J) \leqslant \operatorname{reg}(S_G/I_G) - 1$ from Lemma 3.4.

(1) We do a proof by contradiction, so suppose that $\operatorname{reg}(S_G/J) \geqslant \operatorname{reg}(S_G/I_G) - 1$. By the previous paragraph, this implies that $\operatorname{reg}(S_G/J) = \operatorname{reg}(S_G/I_G) - 1$. By Lemma 2.9, we have

$$\operatorname{reg}(S_G/J) = \theta(G) - 1. \tag{†}$$

We distinguish between the following two cases:

(i) If $N_G(w) = X$, then $G \setminus N_G[w] = Y \setminus \{w\}$ consists of isolated points. It follows that $\operatorname{reg}(S_G/J) = \operatorname{reg}(S_{G\setminus N_G[w]}/I_{G\setminus N_G[w]}) = 0$, which implies $\theta(G) = 1$ by formula (†). Note that since $\deg_G(v) \geqslant 2$, we have $\operatorname{reg}(S_{G\setminus\{u,v\}}/I_{G\setminus\{u,v\}}) \geqslant 1$. On the other hand, since $G\setminus\{u, v\}$ is a C-M bipartite graph by Lemma 2.8 (1), it follows from Lemma 2.9 (2) that $\operatorname{reg}(S_{G\setminus\{u,v\}}/I_{G\setminus\{u,v\}}) = \theta(G\setminus\{u, v\}) = \theta(G\setminus v) = \theta(G) - 1$. Thus, $\theta(G) \geqslant 2$, a contradiction to $\theta(G) = 1$.

(ii) If $N_G(w) \subsetneq X$, then $G \setminus N_G[w]$ is the disjoint union of H and the isolated set $\{y_{j_1}, y_{j_2}, \ldots, y_{j_{m-1}}\}$, where $H = G \setminus \{x_{j_1}, \ldots, x_{j_m}, y_{j_1}, \ldots, y_{j_m}\}$ is a C-M bipartite graph by Lemma 2.8 (2). So by Lemma 2.9 (2) we have

$$\operatorname{reg}(S_G/J) = \operatorname{reg}(S_{G\setminus N_G[u]}/I_{G\setminus N_G[u]}) = \operatorname{reg}(S_H/I_H) = \theta(H).$$

It follows that $\theta(H) = \theta(G) - 1$ by formula (†). Let $M = \{e_1, e_2, \ldots, e_{\theta(G)-1}\}$ be an induced matching of H and $e = \{x_{i_t}, w\}$. Since $V(H) \cap N_G[w] = \emptyset$ and x_{i_t} is a leaf with $N_G(x_{i_t}) = \{w\}$, we have $M \cap \{e\} = \emptyset$, which implies that $M \sqcup \{e\}$ is an induced matching of $G \setminus v$. Note that the size of $M \sqcup \{e\}$ is $\theta(G)$, which contradicts $\theta(G \setminus v) = \theta(G) - 1$.

Therefore, in both cases we always have $\operatorname{reg}(S_G/J) \leqslant \operatorname{reg}(S_G/I_G) - 2$.

(2) Let $M = \{e_1, e_2, \ldots, e_{\theta(G)}\}$ be any induced matching of G. We claim that: $\{u, v\} \in M$.

Indeed, if $\{u, v\} \notin M$, then there exists some $i \in [\theta(G)]$ such that v is a vertex of e_i, since $\deg_G(u) = 1$. Otherwise, every element in $N_G(v)$ is not incident on e_i for each $i \in [\theta(G)]$, so $M = \{e_1, e_2, \ldots, e_{\theta(G)}\}$ is also an induced matching of $G \setminus v$. This implies that $\theta(G \setminus v) = \theta(G)$, which is contradicted by $\theta(G \setminus v) = \theta(G) - 1$. Let $e_i = \{v, y_{i_l}\}$, then $\{x_{i_l}\} \cap e_j = \emptyset$ for any $e_j \in M$ with $j \neq i$. Suppose for a contradiction that $\{x_{i_l}\} \cap e_j \neq \emptyset$ for some $e_j \in M$ with $j \neq i$, then we choose edge $e = \{x_{i_l}, y_{i_l}\}$, thus $e \cap e_i = \{y_{i_l}\}$ and $e \cap e_j = \{x_{i_l}\}$, which contradicts M being an induced matching of G. This implies that x_{i_l} cannot be a vertex of any edge in the set $M \setminus \{e_i\}$. Substituting edge $\{x_{i_l}, y_{i_l}\}$ for e_i yields an induced match of $G \setminus v$. Consequently, $\theta(G \setminus v) = \theta(G)$, which contradicts with $\theta(G \setminus v) = \theta(G) - 1$.

Note that $w = y_{i_t}$ and $N_G(v) = \{y_{i_1}, y_{i_2}, \ldots, y_{i_t}\}$, thus $w \in N_G(v)$. Since $\{u, v\} \in M$, w cannot be a vertex of any edge in M by the definition of induced matching. This implies that $\theta(G \setminus w) = \theta(G)$. It follows that $\operatorname{reg}(S_G/K) = \operatorname{reg}(S_{G\setminus w}/I_{G\setminus w}) = \operatorname{reg}(S_{G\setminus\{w,x_{i_t}\}}/I_{G\setminus\{w,x_{i_t}\}}) = \theta(G\setminus\{w, x_{i_t}\}) = \theta(G\setminus w) = \theta(G) = \operatorname{reg}(S_G/I_G)$ by Lemmas 2.8 (1) and 2.9 (2). □

Remark 3.6. Let $G = (V, E)$ be a C-M bipartite graph with a leaf u. Suppose $N_G(u) = \{v\}$ with $\deg_G(v) \geqslant 2$, and $N_G(v) = \{y_{j_1}, y_{j_2}, \ldots, y_{j_t}\}$, where $u = y_{j_1}$ with $1 \leqslant j_1 < j_2 < \ldots < j_t$. If $\theta(G \setminus v) < \theta(G)$, then we can obtain $\operatorname{reg}(S_G/I_G) \geqslant 2$ by similar arguments as subcase (ii) in the proof of Lemma 3.4 and $\theta(G' \setminus v) < \theta(G')$ by the proof of Lemma 3.5 (2), where $G' = G \setminus \{x_{j_t}, y_{j_t}\}$.

Theorem 3.7. *Let $G = (V, E)$ be a C-M bipartite graph with a leaf u. Let $N_G(u) = \{v\}$. Then*

$$\operatorname{reg}(S_{G\setminus u}/I_{G\setminus u}) = \operatorname{reg}(S_G/I_G) - s, \quad \text{where } s = \begin{cases} 0 & \text{if } \theta(G \setminus v) = \theta(G), \\ 1 & \text{otherwise.} \end{cases}$$

P r o o f. Let $V = X \sqcup Y$ with $X = \{x_1, \ldots, x_n\}$, $Y = \{y_1, \ldots, y_n\}$ and $u = y_l$. Assume that $N_G(u) = \{v\}$ and $N_G(v) = \{y_{i_1}, y_{i_2}, \ldots, y_{i_t}\}$ with $l = i_1 < i_2 < \ldots < i_t \leqslant n$. Let $H = G \setminus u$.

(1) $\theta(G \setminus v) = \theta(G)$. In this case, we suppose $J = (N_H(v)) + I_{H \setminus N_H[v]}$, $K = (v) + I_{H \setminus v}$. Then $I_H = J \cap K$ and $H \setminus \{x_{i_1}, x_{i_2}, y_{i_2}, \ldots, x_{i_t}, y_{i_t}\} = G \setminus \{x_{i_1}, y_{i_1}, \ldots, x_{i_t}, y_{i_t}\}$ is a C-M bipartite graph by Lemma 2.8 (2). Thus, by Lemma 3.4 (1), we have

$$\operatorname{reg}(S_H/J) = \operatorname{reg}(S_{H \setminus N_H[v]}/I_{H \setminus N_H[v]}) = \operatorname{reg}(S_{G \setminus N_G[v]}/I_{G \setminus N_G[v]}) \leqslant \operatorname{reg}(S_G/I_G) - 1.$$

Meanwhile, $H \setminus v = G \setminus \{u, v\}$ is a C-M bipartite graph by Lemma 2.8 (2). Thus, $\operatorname{reg}(S_H/K) = \operatorname{reg}(S_{H \setminus v}/I_{H \setminus v}) = \operatorname{reg}(S_{G \setminus \{u,v\}}/I_{G \setminus \{u,v\}}) = \theta(G \setminus \{u, v\}) = \theta(G \setminus v) = \operatorname{reg}(S_G/I_G)$. Therefore, it follows from Lemma 2.6 that $\operatorname{reg}(S_{G \setminus u}/I_{G \setminus u}) = \operatorname{reg}(S_G/I_G)$ by replacing G in Lemma 2.6 with H.

(2) $\theta(G \setminus v) \neq \theta(G)$. We prove the statement by induction on $\deg_G(v)$.

(i) If $\deg_G(v) = 1$, then H is the disjoint union of $G \setminus \{u, v\}$ and isolated vertex v, and $G \setminus \{u, v\}$ is a C-M bipartite graph by Lemma 2.8 (1). Thus, $\operatorname{reg}(S_H/I_H) = \operatorname{reg}(S_{G \setminus \{u,v\}}/I_{G \setminus \{u,v\}}) = \theta(G \setminus \{u, v\}) = \theta(G \setminus v) = \theta(G) - 1 = \operatorname{reg}(S_G/I_G) - 1$.

(ii) If $\deg_G(v) \geqslant 2$. Let $w = y_{i_t}$ and $N_G(w) = \{x_{j_1}, x_{j_2}, \ldots, x_{j_m}\}$, where $1 \leqslant j_1 < j_2 < \ldots < j_m = i_t$. In this case, let $J = (N_H(w)) + I_{H \setminus N_H[w]}$, $K = (w) + I_{H \setminus w}$, then $I_H = J \cap K$. We divide into the following two cases for $H \setminus N_H[w]$:

(a) If $N_H[w] = X$, then $H \setminus N_H[w] = Y \setminus \{u, w\}$ consists of isolated points. Hence, $I_{H \setminus N_H[w]} = 0$, which implies $\operatorname{reg}(S_H/J) = \operatorname{reg}(S_{H \setminus N_H[u]}/I_{H \setminus N_H[u]}) = 0 \leqslant \operatorname{reg}(S_G/I_G) - 2$ by Remark 3.6.

(b) If $N_H[w] \subsetneq X$, then $H \setminus N_H[w]$ is the disjoint union of H' and isolated set $\{y_{j_1}, y_{j_2}, \ldots, y_{j_{t-1}}\} \setminus u$, where $H' = H \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_m}, y_{j_m}\}$. By Lemma 2.8 (2), we have $H \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_m}, y_{j_m}\} = G \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_m}, y_{j_m}\}$ is a C-M bipartite graph. Thus, $\operatorname{reg}(S_H/J) = \operatorname{reg}(S_{H \setminus N_H[w]}/I_{H \setminus N_H[w]}) = \operatorname{reg}(S_{G \setminus N_G[w]}/I_{G \setminus N_G[w]}) \leqslant \operatorname{reg}(S_G/I_G) - 2$ by Lemma 3.5 (1).

In order to compute $\operatorname{reg}(S_H/K)$, we apply induction on $\deg_G(v)$.

If $\deg_G(v) = 2$, $H \setminus w$ is the disjoint union of $G \setminus \{x_{i_1}, y_{i_1}, x_{i_t}, y_{i_t}\}$ and isolated set $\{x_{i_1}, x_{i_t}\}$, then

$$\begin{aligned}\operatorname{reg}(S_H/K) = {} & \operatorname{reg}(S_{H \setminus w}/I_{H \setminus w}) = \operatorname{reg}(S_{G \setminus \{x_{i_1}, y_{i_1}, x_{i_t}, y_{i_t}\}}/I_{G \setminus \{x_{i_1}, y_{i_1}, x_{i_t}, y_{i_t}\}}) \\ & = \theta(G \setminus \{x_{i_1}, y_{i_1}, x_{i_t}, y_{i_t}\}) = \theta(G) - 1 = \operatorname{reg}(S_G/I_G) - 1.\end{aligned}$$

Now assume that $\deg_G(v) \geqslant 3$. Let $G' = G \setminus \{x_{i_t}, y_{i_t}\}$, then G' is a C-M bipartite graph with a leaf $x_{i_{t-1}}$ and $\deg_{G'}(v) = \deg_G(v) - 1$. Meanwhile, $H \setminus w$ is the disjoint

union of $G' \setminus u$ and isolated point x_{i_t}. Thus,

$$\begin{aligned}\operatorname{reg}(S_H/K) &= \operatorname{reg}(S_{H\setminus w}/I_{H\setminus w}) = \operatorname{reg}(S_{G'\setminus u}/I_{G'\setminus u}) = \operatorname{reg}(S_{G'}/I_{G'}) - 1\\ &= \theta(G \setminus \{x_{i_t}, y_{i_t}\}) - 1 = \operatorname{reg}(S_G/I_G) - 1,\end{aligned}$$

where the last equality holds by Lemma 3.5 (2). Note that $I_H = J \cap K$. We obtain $\operatorname{reg}(S_{G\setminus u}/I_{G\setminus u}) = \operatorname{reg}(S_G/I_G) - s$ by Lemma 2.6 replacing G with H. □

The following example shows that both cases in Theorem 3.7 can happen.

Example 3.8. Suppose that G is a C-M bipartite graph as in Figure 3. Then $\operatorname{reg}(S_G/I_G) = 2$ by Lemma 2.9 (2). Moreover, $\operatorname{reg}(S_{G\setminus u}/I_{G\setminus u}) = 2$ if $u = x_3$, and $\operatorname{reg}(S_{G\setminus u}/I_{G\setminus u}) = 1$ if $u = y_1$.

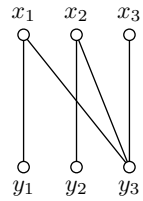


Figure 3. C-M bipartite graph G.

4. THE ∘ OPERATION AND THE ∗ OPERATION

In this section, we will study some graphs obtained from Cohen-Macaulay bipartite graphs by the ∘ operation or the ∗ operation. The main task of this section is to give some exact formulas for the depth and regularity of the edge ideals of such graphs. We start by recalling from [3], [20] the two aforementioned special gluing operations.

Definition 4.1. For $i = 1, 2$, let G_i be a graph with a leaf u_i. Furthermore, let $N_G(u_i) = \{v_i\}$ with $\deg_{G_i}(v_i) \geqslant 2$.

(1) Let G be a graph obtained from G_1 and G_2 by first removing the leaves u_1, u_2, and then identifying the vertices v_1 and v_2. In this case, we say that G is obtained from G_1 and G_2 by the ∘ *operation* and write $G = (G_1, u_1) \circ (G_2, u_2)$ or simply $G = G_1 \circ G_2$. If v_1 and v_2 are identified as the vertex v in G, then we also write $G = G_1 \circ_v G_2$.

(2) Let H be the graph obtained from G_1 and G_2 by identifying the vertices u_1 and u_2. In this case, we say that H is obtained from G_1 and G_2 by the ∗ *operation* and write $H = (G_1, u_1) * (G_2, u_2)$ or simply $H = G_1 * G_2$. If we denote the identified vertex in H by u, then we also write $H = G_1 *_u G_2$.

Example 4.2. Below are examples of the $\circ$ operation or the $*$ operation of two C-M bipartite graphs G_1 and G_2.

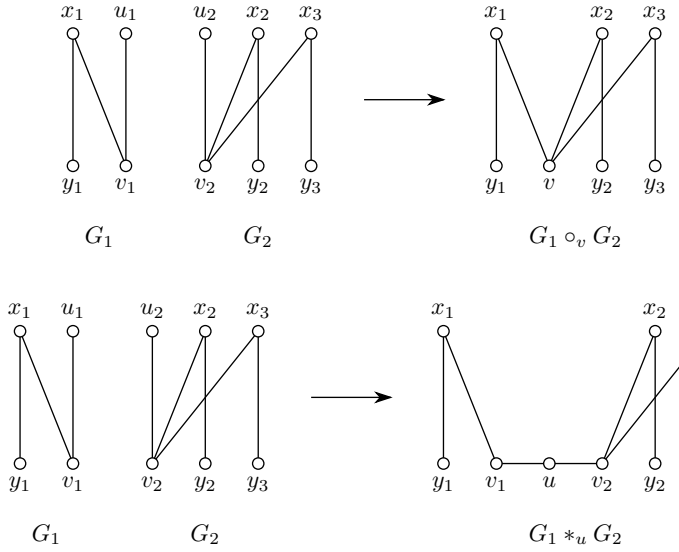


Figure 4.

For two graphs G_1 and G_2, let their clique sum $G_1 \cup_v G_2$ be a union of graphs G_1 and G_2 such that $V(G_1) \cap V(G_2) = \{v\}$.

Theorem 4.3. *Let $G = (G_1, u_1) \circ (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_G(u_i) = \{v_i\}$. Then*

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - s,$$

where

$$s = \begin{cases} 1 & \text{if } \deg_{G_i}(v_i) = 1 \text{ for all } i \in [2], \\ 2 & \text{otherwise.} \end{cases}$$

P r o o f. Let $V(G_i) = \{x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}\} \sqcup \{y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}\}$ for $i \in [2]$. By symmetry, we can assume that every $u_i = x_{i,j_i}$ for some $j_i \in [n_i]$, and $v_i = y_{i,j_i}$ is the only neighbor point of x_{i,j_i} in G_i. Suppose y_{1,j_1} and y_{2,j_2} are identified as v in G by the $\circ$ operation. We distinguish three cases:

(I) If $\deg_{G_i}(v_i) = 1$ for all $i \in [2]$, then G is the disjoint union of $G_1 \setminus \{u_1, v_1\}$, $G_2 \setminus \{u_2, v_2\}$ and an isolated vertex v. Thus, by Lemmas 2.8 (1) and 2.9 (1), we have

$$\begin{aligned} \operatorname{depth}(S_G/I_G) &= 1 + [\operatorname{depth}(S_{G_1}/I_{G_1}) - 1] + [\operatorname{depth}(S_{G_2}/I_{G_2}) - 1] \\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1. \end{aligned}$$

(II) If $\deg_{G_1}(v_1) \geqslant 2$ and $\deg_{G_2}(v_2) = 1$, then $G = (G_1 \setminus u_1) \sqcup (G_2 \setminus \{u_2, v_2\})$. Thus, by Lemmas 2.8 (1), 2.9 (1) and 3.1, we have

$$\begin{aligned}\operatorname{depth}(S_G/I_G) &= \operatorname{depth}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) + \operatorname{depth}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2.\end{aligned}$$

(III) If $\deg_{G_i}(v_i) \geqslant 2$ for all $i \in [2]$, we will prove the statement by induction on n_2. When $n_2 = 2$, let $N_{G_2}(v_2) = \{x_{2,1}, x_{2,2}\}$, where $x_{2,2} = u_2$. In this case, $G = G_1 \cup_{x_{2,1}} P_2$ is the clique sum of G_1 and a path P_2 with vertex set $\{x_{2,1}, y_{2,1}\}$. Set $J = (N_G(y_{2,1})) + I_{G\setminus N_G[y_{2,1}]}$, $K = (y_{2,1}) + I_{G\setminus y_{2,1}}$. Note that $G\setminus N_G[y_{2,1}] = G_1\setminus u_1$ and $G \setminus y_{2,1} = G_1$, we obtain $\operatorname{depth}(S_G/J) = 1 + \operatorname{depth}(S_{G\setminus N_G[y_{2,1}]}/I_{G\setminus N_G[y_{2,1}]}) = 1 + \operatorname{depth}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) = \operatorname{depth}(S_{G_1}/I_{G_1})$ by Lemma 3.1, and $\operatorname{depth}(S_G/K) = \operatorname{depth}(S_{G\setminus y_{2,1}}/I_{G\setminus y_{2,1}}) = \operatorname{depth}(S_{G_1}/I_{G_1})$. By Lemma 2.6, the desired depth results hold.

In the following, we assume that $n_2 \geqslant 3$. Let $N_{G_2}(v_2) = \{x_{2,l_1}, x_{2,l_2}, \ldots, x_{2,l_s}\}$ with $1 \leqslant l_1 < l_2 < \ldots < l_s = j_2$, then $\deg_{G_2}(y_{2,l_1}) = 1$ by Theorem 2.7 (2). In this case, let $N_G(x_{2,l_1}) = \{y_{2,k_1}, y_{2,k_2}, \ldots, y_{2,k_t}\}$ with $l_1 = k_1 < k_2 < \ldots < k_t \leqslant n_2$ and $k_s = j_2$ for some $s \in [t]$. Set $J = (N_G(x_{2,l_1})) + I_{G\setminus N_G[x_{2,l_1}]}$, $K = (x_{2,l_1}) + I_{G\setminus x_{2,l_1}}$. We distinguish between the following two cases:

(A1) If $|N_{G_2}(v)| = 2$, then $s = 2$ and $G \setminus x_{2,l_1}$ is the disjoint union of $G_1 \setminus u_1$, $G_2 \setminus \{x_{2,l_1}, y_{2,l_1}, x_{2,j_2}, y_{2,j_2}\}$ and an isolated vertex y_{2,l_1}. So by Lemmas 2.9 (1) and 3.1, we have

$$\begin{aligned}\operatorname{depth}(S_G/K) &= 1 + \operatorname{depth}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) + \operatorname{depth}(S_{H_1}/I_{H_1})\\ &= 1 + [\operatorname{depth}(S_{G_1}/I_{G_1}) - 1] + [\operatorname{depth}(S_{G_2}/I_{G_2}) - 2]\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2,\end{aligned}$$

where $H_1 = G_2 \setminus \{x_{2,l_1}, y_{2,l_1}, x_{2,j_2}, y_{2,j_2}\}$.

Meanwhile, $G \setminus N_G[x_{2,l_1}]$ takes one of the following forms:

(a) $G \setminus N_G[x_{2,l_1}] = H \sqcup H_1$, where $H = G_1 \setminus \{u_1, v_1\}$;

(b) $G\setminus N_G[x_{2,l_1}]$ is the disjoint union of H, H_2 and the isolated set $\{x_{2,k_2}, \ldots, x_{2,k_t}\}\setminus \{u_2\}$, where $H_2 = G_2 \setminus \{x_{2,k_1}, \ldots, x_{2,k_t}, y_{2,k_1}, \ldots, y_{2,k_t}\}$.

Note that H, H_1 and H_2 are C-M bipartite graphs. We consider two subcases:

(i) If $G \setminus N_G[x_{2,l_1}]$ is of form (a), then we get by Lemma 2.9 (1) that

$$\begin{aligned}\operatorname{depth}(S_G/J) &= 1 + \operatorname{depth}(S_H/I_H) + \operatorname{depth}(S_{H_1}/I_{H_1})\\ &= 1 + [\operatorname{depth}(S_{G_1}/I_{G_1}) - 1] + [\operatorname{depth}(S_{G_2}/I_{G_2}) - 2]\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2.\end{aligned}$$

(ii) If $G \setminus N_G[x_{2,l_1}]$ is of form (b), then we also have

$$\begin{aligned}\operatorname{depth}(S_G/J) &= 1 + \operatorname{depth}(S_H/I_H) + \operatorname{depth}(S_{H_2}/I_{H_2}) + (t-2)\\ &= 1 + [\operatorname{depth}(S_{G_1}/I_{G_1}) - 1] + (\operatorname{depth}(S_{G_2}/I_{G_2}) - t) + (t-2)\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2.\end{aligned}$$

(A2) If $|N_{G_2}(v)| \geqslant 3$, then $G \setminus x_{2,l_1}$ is the disjoint union of G' and an isolated vertex y_{2,l_1}, where $G' = (G_1, u_1) \circ (G_2', u_2)$ and $G_2' = G_2 \setminus \{x_{2,l_1}, y_{2,l_1}\}$. In this case, we have $|N_{G_2'}(v)| = |N_{G_2}(v)| - 1$. Thus, by induction and Lemma 2.9 (1), we have

$$\begin{aligned}\operatorname{depth}(S_H/K) &= 1 + \operatorname{depth}(S_{G'}/I_{G'}) = 1 + [\operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2'}/I_{G_2'}) - 2]\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + [\operatorname{depth}(S_{G_2}/I_{G_2}) - 1] - 1\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2.\end{aligned}$$

At the same time, $G \setminus N_G[x_{2,l_1}]$ has one of the following forms:

(c) $G \setminus N_G[x_{2,l_1}] = H \sqcup H_1$, where $H = G_1 \setminus \{u_1, v_1\}$;

(d) $G \setminus N_G[x_{2,l_1}]$ is the disjoint union of H, H_2 and isolated set $\{x_{2,k_2}, \ldots, x_{2,k_t}\} \setminus \{u_2\}$, where $H_2 = G_2 \setminus \{x_{2,k_1}, y_{2,k_1}, x_{2,k_2}, y_{2,k_2}, \ldots, x_{2,k_t}, y_{2,k_t}\}$.

Applying the analysis as (a) and (b) of case (1) above, we obtain

$$\operatorname{depth}(S_G/J) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 2.$$

In summary, the desired result follows from Lemma 2.6. □

The following are two examples that satisfy the two conditions in Theorem 4.3, respectively.

Example 4.4. Let $G = G_1 \circ G_2$. In Figure 5, G is the disjoint union of an isolated vertex v and a path of length 3 with vertex set $\{x_{1,1}, y_{1,1}, y_{1,3}, x_{1,3}\}$, thus $\operatorname{depth}(S_G/I_G) = 3$ by Lemma 2.10. In Figure 6, let $J = (x_{N_G(x_{2,1})}) + I_{G\setminus N_G[x_{2,1}]}$, $K = (x_{2,1}) + I_{G\setminus x_{2,1}}$. In this case,

$$\operatorname{depth}(S_G/J) = 1 + \operatorname{depth}(S_{G\setminus N_G[x_{2,1}]}/I_{G\setminus N_G[x_{2,1}]}) = 3$$

and $\operatorname{depth}(S_G/K) = 3$ according to Lemma 2.10. It follows from Lemma 2.6 that $\operatorname{depth}(S_G/I_G) = 3$.

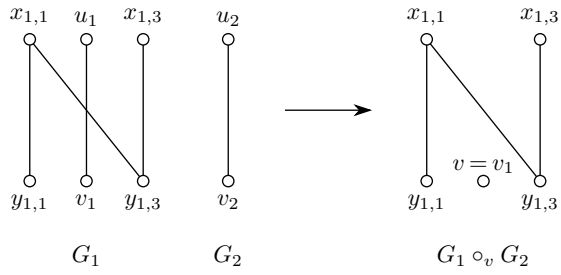


Figure 5.

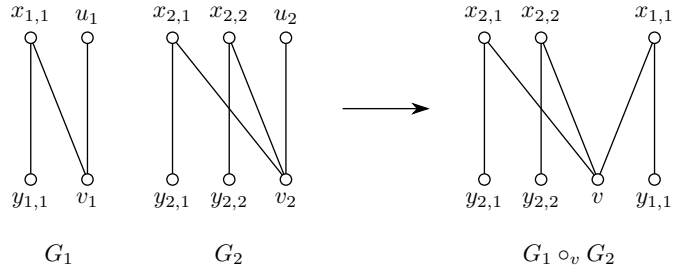


Figure 6.

Theorem 4.5. *Let $G = (G_1, u_1) \circ (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_{G_i}(u_i) = \{v_i\}$. If $t = |\{i\colon\ \theta(G_i \setminus v_i) \neq \theta(G_i)\}|$, then*

$$\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - t.$$

P r o o f. By definition, $t = 0, 1$, or 2. Let V_i be the vertex set of G_i and $V_i = X_i \sqcup Y_i$ be a bipartition of V_i with $X_i = \{x_{i,1}, x_{i,2}, \ldots, x_{i,n_i}\}$, $Y_i = \{y_{i,1}, y_{i,2}, \ldots, y_{i,n_i}\}$ for $i \in [2]$. By symmetry, let $N_{G_1}(v_1) = \{x_{1,i_1}, x_{1,i_2}, \ldots, x_{1,i_m}\}$ and $N_{G_2}(v_2) = \{y_{2,j_1}, y_{2,j_2}, \ldots, y_{2,j_t}\}$, where $u_1 = x_{1,i_m}$, $u_2 = y_{2,j_1}$, $1 \leqslant i_1 < i_2 < \ldots < i_m \leqslant n_1$ and $1 \leqslant j_1 < j_2 < \ldots < j_t \leqslant n_2$. Suppose that v_1 and v_2 are identified as v in G by the $\circ$ operation and $N_{G_2}(y_{2,j_t}) = \{x_{2,h_1}, x_{2,h_2}, \ldots, x_{2,h_s}\}$ with $1 \leqslant h_1 < h_2 < \ldots < h_s = j_t$. We distinguish the following two cases:

(I) If $t = 2$, then $\theta(G_i \setminus v_i) \neq \theta(G_i)$ for all $i \in [2]$. Now we prove the formulas for the regularity of S_G/I_G by induction on $\deg_{G_2}(v_2)$. If $\deg_{G_2}(v_2) = 1$, then $G = (G_1, u_1) \circ (G_2, u_2)$ is the disjoint union of $G_1 \setminus u_1$ and $G_2 \setminus \{u_2, v_2\}$. If $G_2 \setminus \{u_2, v_2\} = \emptyset$, then the desired result follows from Theorem 3.7. Now, we assume that $G_2 \setminus \{u_2, v_2\} \neq \emptyset$. In this case, $G_2 \setminus \{u_2, v_2\}$ is a C-M bipartite graph by Lemma 2.8 (1). It follows from

Lemmas 2.4 (1), 2.9 (2) and 3.7 that

$$\begin{aligned}\operatorname{reg}(S_G/I_G) &= \operatorname{reg}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) + \operatorname{reg}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\ &= (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + \theta(G_2\setminus\{u_2,v_2\})\\ &= (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + \theta(G_2\setminus v_2)\\ &= (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + (\theta(G_2) - 1)\\ &= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 2.\end{aligned}$$

Assume that $\deg_{G_2}(v_2) \geqslant 2$ and the regularity statement holds for $\deg_{G_2}(v_2) - 1$. In this case, let $w = y_{2,j_t}$, then x_{2,j_t} is a leaf of G_2. Choose $J = (N_G(w)) + I_{G\setminus N_G[w]}$, $K = (w) + I_{G\setminus w}$. Thus, $G\setminus w$ is the disjoint union of $G_1 \circ (G_2\setminus\{x_{2,j_t}, w\})$ and isolated vertex x_{2,j_t}. Let $G_2' = G_2\setminus\{x_{2,j_t}, w\}$, then $\deg_{G_2'}(v_2) = \deg_{G_2}(v_2) - 1$ and $\theta(G_2'\setminus v_2) \neq \theta(G_2')$ by Remark 3.6. By the induction hypothesis, we have

$$\begin{aligned}\operatorname{reg}(S_G/K) &= \operatorname{reg}(S_{G\setminus w}/I_{G\setminus w}) = \operatorname{reg}(S_{G_1\circ G_2'}/I_{G_1\circ G_2'})\\ &= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2'}/I_{G_2'}) - 2\\ &= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 2,\end{aligned}$$

where the last equality holds because of $t = 2$ and Lemma 3.5 (2).

In order to compute $\operatorname{reg}(S_G/J)$, we consider the induced subgraph $G\setminus N_G[w]$ of G. We distinguish the following two cases:

(a) If $N_{G_2}(w) = X_2$, then $G\setminus N_G[w]$ is the disjoint union of $G_1\setminus\{u_1, v_1\}$ and isolated set $Y_2\setminus\{u_2, w\}$, and $G_1\setminus\{u_1, v_1\}$ is a C-M bipartite graph by Lemma 2.8 (1). It follows that

$$\begin{aligned}\operatorname{reg}(S_G/J) &= \operatorname{reg}(S_{G\setminus N_G[w]}/I_{G\setminus N_G[w]}) = \operatorname{reg}(S_{G_1\setminus\{u_1,v_1\}}/I_{G_1\setminus\{u_1,v_1\}})\\ &= \theta(G_1\setminus\{u_1,v_1\}) = \theta(G_1\setminus v_1) = \operatorname{reg}(S_{G_1}/I_{G_1}) - 1.\end{aligned}$$

(b) If $N_{G_2}(w) \subsetneq X_2$, then $G\setminus N_G[w]$ is the disjoint union of $G_1\setminus\{u_1, v_1\}$, H and the isolated set $\{y_{2,h_1}, y_{2,h_2}, \ldots, y_{2,h_{s-1}}\}\setminus\{u_2\}$, where

$$H = G_2\setminus\{x_{2,h_1}, y_{2,h_1}, \ldots, x_{2,h_s}, y_{2,h_s}\}.$$

By Lemmas 2.9 (2) and 3.5 (1), we obtain

$$\begin{aligned}\operatorname{reg}(S_G/J) &= \operatorname{reg}(S_{G\setminus N_G[w]}/I_{G\setminus N_G[w]}) = \operatorname{reg}(S_{G_1\setminus\{u_1,v_1\}}/I_{G_1\setminus\{u_1,v_1\}}) + \operatorname{reg}(S_H/I_H)\\ &= \theta(G_1\setminus v_1) + \operatorname{reg}(S_H/I_H) = (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + \operatorname{reg}(S_H/I_H)\\ &\leqslant (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + (\operatorname{reg}(S_{G_2}/I_{G_2}) - 2)\\ &= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 3.\end{aligned}$$

where the penultimate inequality holds by Lemma 3.5 (1). Applying Lemma 2.6, we get the desired regularity results.

(II) If $t \leqslant 1$, then, in this case, we choose $J = (N_G(v)) + I_{G\setminus N_G[v]}$, $K = (v) + I_{G\setminus v}$. Thus, $G\setminus N_G[v]$ is the disjoint union of G_1', G_2' and the isolated set $\{y_{1,i_1}, \ldots, y_{1,i_{m-1}}, x_{2,j_2}, \ldots, x_{2,j_t}\}$, where $G_1' = G_1 \setminus \{x_{1,i_1}, y_{1,i_1}, \ldots, x_{1,i_m}, y_{1,i_m}\}$, $G_2' = G_2 \setminus \{x_{2,j_1}, y_{2,j_1}, \ldots, x_{2,j_t}, y_{2,j_t}\}$. By Lemmas 2.8 (2) and 2.9 (2), we get

$$\begin{aligned}\operatorname{reg}(S_G/J) &= \operatorname{reg}(S_{G\setminus N_G[v]}/I_{G\setminus N_G[v]}) = \operatorname{reg}(S_{G_1'}/I_{G_1'}) + \operatorname{reg}(S_{G_2'}/I_{G_2'})\\ &\leqslant (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + (\operatorname{reg}(S_{G_2}/I_{G_2}) - 1)\\ &= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 2,\end{aligned}$$

where the penultimate inequality holds by Lemma 3.4. On the other hand, $G \setminus v$ is the disjoint union of $G_1 \setminus \{u_1, v_1\}$ and $G_2 \setminus \{u_2, v_2\}$. By Lemmas 2.8 (1) and 2.9 (2), we get

$$\begin{aligned}\operatorname{reg}(S_G/K) &= \operatorname{reg}(S_{G\setminus v}/I_{G\setminus v})\\ &= \operatorname{reg}(S_{G_1\setminus\{u_1,v_1\}}/I_{G_1\setminus\{u_1,v_1\}}) + \operatorname{reg}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\ &= \theta(G_1 \setminus v_1) + \theta(G_2 \setminus v_2),\end{aligned}$$

Thus, if $t = 0$, then $\theta(G_i \setminus v_i) = \theta(G_i)$ for all $i \in [2]$. Thus, $\operatorname{reg}(S/K) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2})$. If $t = 1$, then $\theta(G_1 \setminus v_1) = \theta(G_1)$ and $\theta(G_1 \setminus v_1) \neq \theta(G_1)$, or vice versa. Thus, $\operatorname{reg}(S/K) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 1$.

Again using Lemma 2.6, we get $\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - t$. □

Now, we study the depth and regularity of the quotient ring of the edge ideal of a graph, which is obtained by the $*$ operation.

Theorem 4.6. *Let $G = (G_1, u_1) * (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_{G_i}(u_i) = \{v_i\}$. Let $t = |\{i\colon \theta(G_i \setminus v_i) \neq \theta(G_i)\}|$. Then*

$$\operatorname{reg}(S_G/I_G) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - s,$$

where

$$s = \begin{cases} 0 & \text{if } t \leqslant 1,\\ 1 & \text{if } t = 2.\end{cases}$$

Proof. By definition, $t = 0, 1$, or 2. Suppose u_1 and u_2 are identified as u in G by the $*$ operation. Let $N_{G_2}(v_2) = \{y_{j_1}, y_{j_2}, \ldots, y_{j_m}\}$ with $u_2 = y_{j_1}$, where $1 \leqslant j_1 < \ldots < j_m \leqslant n_2$. Then $G \setminus N_G[v_2]$ is the disjoint union of $G_1 \setminus u_1$, $G_2 \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_m}, y_{j_m}\}$ and isolated set $\{x_{j_2}, \ldots, x_{j_m}\}$, and $G \setminus v_2$ is the disjoint union of G_1 and $G_2 \setminus \{u_2, v_2\}$. We distinguish the following two cases:

(1) If $t \leqslant 1$, then $\theta(G_i \setminus v_i) = \theta(G_i)$ for some $i \in [2]$. By symmetry, we assume $\theta(G_2 \setminus v_2) = \theta(G_2)$. In this case, we choose $J = (N_G(v_2)) + I_{G\setminus N_G[v_2]}$, $K = (v_2) + I_{G\setminus v_2}$. Let $H = G_2 \setminus \{x_{j_1}, y_{j_1}, \ldots, x_{j_m}, y_{j_m}\}$. Then by Lemmas 3.3, 3.4 (1) and 2.9 (2), we have

$$\begin{aligned}
\operatorname{reg}(S_G/J) &= \operatorname{reg}(S_{G\setminus N_G[v_2]}/I_{G\setminus N_G[v_2]}) = \operatorname{reg}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) + \operatorname{reg}(S_H/I_H)\\
&\leqslant \operatorname{reg}(S_{G_1}/I_{G_1}) + (\operatorname{reg}(S_{G_2}/I_{G_2}) - 1)\\
&= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 1,\\
\operatorname{reg}(S_G/K) &= \operatorname{reg}(S_{G\setminus v_2}/I_{G\setminus v_2}) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\
&= \operatorname{reg}(S_{G_1}/I_{G_1}) + \theta(G_2 \setminus v_2) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}).
\end{aligned}$$

Applying Lemma 2.6, we obtain the expected results.

(2) If $t = 2$, then $\theta(G_i \setminus v_i) \neq \theta(G_i)$ for all $i \in [2]$. In this case, we choose $J = (N_G(v_2)) + I_{G\setminus N_G[v_2]}$, $K = (v_2) + I_{G\setminus v_2}$. Thus, by Theorem 3.7, we have

$$\begin{aligned}
\operatorname{reg}(S_G/J) &= \operatorname{reg}(S_{G\setminus N_G[v_2]}/I_{G\setminus N_G[v_2]}) = \operatorname{reg}(S_{G_1\setminus u_1}/I_{G_1\setminus u_1}) + \operatorname{reg}(S_H/I_H)\\
&= (\operatorname{reg}(S_{G_1}/I_{G_1}) - 1) + (\operatorname{reg}(S_{G_2}/I_{G_2}) - 1)\\
&= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 2,
\end{aligned}$$

where the penultimate equality holds by the proof of Lemma 3.5 (2). Meanwhile, by Lemmas 2.8 (1) and 2.9 (2), we have

$$\begin{aligned}
\operatorname{reg}(S_G/K) &= \operatorname{reg}(S_{G\setminus v_2}/I_{G\setminus v_2}) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\
&= \operatorname{reg}(S_{G_1}/I_{G_1}) + \theta(G_2 \setminus v_2) = \operatorname{reg}(S_{G_1}/I_{G_1}) + \theta(G_2) - 1\\
&= \operatorname{reg}(S_{G_1}/I_{G_1}) + \operatorname{reg}(S_{G_2}/I_{G_2}) - 1.
\end{aligned}$$

Again applying Lemma 2.6, we obtain the desired result. □

Lemma 4.7. *Let $G = G_1 \cup_u P_2$ be the clique sum of a C-M bipartite graph G_1 and a path P_2 with vertex set $\{u, v_2\}$, where u is a leaf of G_1 and $N_{G_1}(u) = \{v_1\}$. Then*

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}).$$

P r o o f. If $\deg_{G_1}(v_1) = 1$, then G is the disjoint union of $G_1 \setminus \{u_1, v_1\}$ and a path of length 3 with vertex set $\{v_1, v_2, u\}$. Thus, by Lemmas 2.9 (1) and 2.10, we get $\operatorname{depth}(S_G/I_G) = 1 + \frac{1}{2}(|V(G_1)| - 2) = \operatorname{depth}(S_{G_1}/I_{G_1})$ since $G_1 \setminus \{u, v_1\}$ is a C-M bipartite graph by Lemma 2.8 (1).

If $\deg_{G_1}(v_1) \geqslant 2$, then we choose $J = (N_G(v_2)) + I_{G\setminus N_G[v_2]}$, $K = (v_2) + I_{G\setminus v_2}$. In this case, $G \setminus v_2 = G_1$ and $G \setminus N_G[v_2] = G_1 \setminus u$. Then by Lemma 3.1, we obtain that

$$\begin{aligned}\operatorname{depth}(S_G/J) &= 1 + \operatorname{depth}(S_{G\setminus N_G[v_2]}/I_{G\setminus N_G[v_2]}) = 1 + \operatorname{depth}(S_{G_1\setminus u}/I_{G_1\setminus u})\\ &= 1 + (\operatorname{depth}(S_{G_1}/I_{G_1}) - 1) = \operatorname{depth}(S_{G_1}/I_{G_1}),\\ \operatorname{depth}(S_G/K) &= \operatorname{depth}(S_{G\setminus v_2}/I_{G\setminus v_2}) = \operatorname{depth}(S_{G_1}/I_{G_1}).\end{aligned}$$

Using Lemma 2.6, we get $\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1})$. □

Theorem 4.8. *Let $G = (G_1, u_1) * (G_2, u_2)$, where each G_i is a C-M bipartite graph with a leaf u_i. Let $N_{G_i}(u_i) = \{v_i\}$. Then*

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.$$

P r o o f. For any $i \in [2]$, let $V(G_i) = X_i \sqcup Y_i$ with $X_i = \{x_{i,1}, \ldots, x_{i,n_i}\}$, $Y_i = \{y_{i,1}, \ldots, y_{i,n_i}\}$ and $u_i = x_{i,j_i}$ for some $j_i \in [n_i]$. Then $N_{G_i}(u_i) = \{y_{i,j_i}\}$. Let $v_i = y_{i,j_i}$ and $N_{G_1}(v_1) = \{x_{1,k_1}, \ldots, x_{1,k_t}\}$ with $1 \leqslant k_1 < \ldots < k_t = j_1$. We distinguish the following two cases:

(1) If $\deg_{G_i}(v_i) = 1$ for some $i \in [2]$, then we assume $\deg_{G_2}(v_2) = 1$ by symmetry. Then $G_2 = P_2 \sqcup (G_2 \setminus \{u_2, v_2\})$, which implies $G = (G_1 \cup_{u_1} P_2) \sqcup (G_2 \setminus \{u_2, v_2\})$. By Lemmas 2.4 (2), 2.9 (1), and 4.7, we have

$$\begin{aligned}\operatorname{depth}(S_G/I_G) &= \operatorname{depth}(S_{G_1\cup_{u_1} P_2}/I_{G_1\cup_{u_1} P_2}) + \operatorname{depth}(S_{G_2\setminus\{u_2,v_2\}}/I_{G_2\setminus\{u_2,v_2\}})\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.\end{aligned}$$

(2) If $\deg_{G_i}(v_i) \geqslant 2$ for all $i \in [2]$, then we choose $J = (N_G(v_1)) + I_{G\setminus N_G[v_1]}$, $K = (v_1) + I_{G\setminus v_1}$. In this case, $G \setminus v_1$ is the disjoint union of G_2 and $G_1 \setminus \{u_1, v_1\}$. Thus, by Lemma 2.9 (1), we get

$$\begin{aligned}\operatorname{depth}(S_G/K) &= \operatorname{depth}(S_{G_2}/I_{G_2}) + \operatorname{depth}(S_{G_1\setminus\{u_1,v_1\}}/I_{G_1\setminus\{u_1,v_1\}})\\ &= \operatorname{depth}(S_{G_2}/I_{G_2}) + \operatorname{depth}(S_{G_1}/I_{G_1}) - 1.\end{aligned}$$

In order to compute the depth of S_G/J, we distinguish the following two cases:

(i) If $N_G(v_1) = X_1$, then $G \setminus N_G[v_1]$ is the disjoint union of $G_2 \setminus u_2$ and isolated set $Y_1 \setminus v_1$. By Lemmas 2.4 (2), 2.9 (1) and 3.1, we have

$$\begin{aligned}\operatorname{depth}(S_G/J) &= 1 + \operatorname{depth}(S_{G\setminus N_G[v_1]}/I_{G\setminus N_G[v_1]})\\ &= 1 + \operatorname{depth}(S_{G_2\setminus u_2}/I_{G_2\setminus u_2}) + (n_1 - 1)\\ &= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.\end{aligned}$$

(ii) If $N_G(v_1) \subsetneq X_1$, then $G \setminus N_G[v_1]$ is the disjoint union of $G_1 \setminus \{x_{1,k_1}, y_{1,k_1}, \ldots, x_{1,k_t}, y_{1,k_t}\}$, $G_2 \setminus u_2$, and isolated set $\{y_{1,k_1}, y_{1,k_2}, \ldots, y_{1,k_{t-1}}\}$. Note that $H = G_1 \setminus \{x_{1,k_1}, y_{1,k_1}, \ldots, x_{1,k_t}, y_{1,k_t}\}$ is a C-M bipartite graph by Lemma 2.8 (2). Thus, by Lemmas 2.9 (1) and 3.1, we have

$$\begin{aligned}
\operatorname{depth}(S_G/J) &= 1 + \operatorname{depth}(S_{G\setminus N_G[v_1]}/I_{G\setminus N_G[v_1]}) \\
&= 1 + \operatorname{depth}(S_H/I_H) + \operatorname{depth}(S_{G_2\setminus u_2}/I_{G_2\setminus u_2}) + (t-1) \\
&= 1 + \frac{|V(G_1)| - 2t}{2} + [\operatorname{depth}(S_{G_2}/I_{G_2}) - 1] + (t-1) \\
&= \frac{|V(G_1)|}{2} + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1 \\
&= \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.
\end{aligned}$$

Applying Lemma 2.6, we obtain that

$$\operatorname{depth}(S_G/I_G) = \operatorname{depth}(S_{G_1}/I_{G_1}) + \operatorname{depth}(S_{G_2}/I_{G_2}) - 1.$$

We finish the proof. □

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