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TURÁN NUMBER OF TWO VERTEX-DISJOINT COPIES OF CLIQUES

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Abstract. The Turán number of a given graph H, denoted by $\mathrm{ex}(n, H)$, is the maximum number of edges in an H-free graph on n vertices. Applying a well-known result of Hajnal and Szemerédi, we determine the Turán number $\mathrm{ex}(n, K_p \cup K_q)$ of a vertex-disjoint union of cliques K_p and K_q for all values of n.

Keywords: clique; Hajnal and Szemerédi theorem; Turán number; extremal graph

MSC 2020: 05C35, 05D05

1. Introduction

Given two graphs G and H, G is called *H-free* if it does not contain a copy of H as a subgraph. The *Turán number* $\mathrm{ex}(n, H)$ of H is the maximum number of edges in all H-free graphs on n vertices. Determining $\mathrm{ex}(n, H)$ is one of the most important problems in extremal graph theory and the Turán graph plays a key role in nondegenerate graph. For two integers n and q with $n \geqslant q \geqslant 2$, the *Turán graph* $T_q(n)$ is an n-vertex complete q-partite graph with parts of size $\lceil n/q \rceil$ or $\lfloor n/q \rfloor$. Let $t_q(n)$ denote the number of edges in $T_q(n)$. The systematic study of $\mathrm{ex}(n, H)$ was started by Turán (see [20]) by showing that $\mathrm{ex}(n, K_q) = t_{q-1}(n) = (1 - 1/(q-1) + o(1))\binom{n}{2}$ and the unique extremal graph is $T_{q-1}(n)$. This extended the Mantel theorem, which shows $\mathrm{ex}(n, K_3) = \lfloor n^2/4 \rfloor$. For a general graph H, the celebrated Erdős-Stone-Simonovits theorem (see [8], [9]) states that $\mathrm{ex}(n, H) = (1-1/(\chi(H)-1)+o(1))\binom{n}{2}$, where $\chi(H)$ denotes the chromatic number of H.

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For a general graph H, $\mathrm{ex}(n, H)$ has been determined asymptotically. Until now, there are only a few graphs whose Turán numbers are determined exactly, including paths (see [1], [2]), cliques (see [5]), odd cycles (see [10], [12]), wheels (see [23]) and some other specific graphs, see [3], [17], [22], [24], [25]. For two graphs G and H, we use $G \cup H$ to denote the vertex-disjoint union of G and H, and let kH be the vertex-disjoint union of k copies of H. A natural problem is that if the Turán number of graphs G and H is determined, how about $\mathrm{ex}(n, G \cup H)$ or $\mathrm{ex}(n, kH)$. Recently, Hou, et al. in [16] have presented a general approach for determining the maximum number of edges in an n-vertex r-graph that does not contain t vertex-disjoint copies of nondegenerate hypergraph. For some results on the Turán number for disjoint copies of graphs, we refer the readers to [11], [14], [15], [21], [26], [27].

Let $M_k = kK_2$ denote the matching of size k. Erdős and Gallai in [7] in 1959 studied the Turán number of matchings and paths, showing that $\mathrm{ex}(n, M_k) = \max\left\{\binom{2k-1}{2}, \binom{k-1}{2} + (k-1)(n-k+1)\right\}$ and $\mathrm{ex}(n, P_k) \leqslant (k-2)n/2$. Determining the Turán number of vertex-disjoint copies of clique was studied by Erdős, see [6]. Later, Moon in [18] and Simonovits in [19] determined $\mathrm{ex}(n, kK_p)$ for a sufficiently large n by proving the following theorem.

Theorem 1.1 ([18], [19]). *For a sufficiently large n,*

$$\mathrm{ex}(n, kK_p) = (k-1)(n-1) + t_{p-1}(n-k+1) - \binom{k-1}{2},$$

and the join of a complete graph K_{k-1} and $T_{p-1}(n-k+1)$ is the unique extremal graph for kK_p.

Determining $\mathrm{ex}(n, kK_p)$ for small n remains open. Recently, Chen, et al. in [4] have determined $\mathrm{ex}(n, 2K_p)$ for all n.

Theorem 1.2 ([4]). *Let $n \geqslant 2p+2$ and $p \geqslant 2$ be integers. Then*

$$\mathrm{ex}(n, 2K_{p+1}) = \begin{cases} \binom{n}{2} - 3(n-2p-1), & n \leqslant 3p+1; \\ (n-1) + t_p(n-1), & n \geqslant 3p+2. \end{cases}$$

When $2p+2 \leqslant n \leqslant 3p+1$, the extremal graph for $2K_{p+1}$ is the complement of $(n-2p-1)K_3 \cup I$, where I is an independent set on $n - 3(n-2p-1)$ vertices. When $n \geqslant 3p+2$, the join of a vertex and $T_p(n-k+1)$ is the extremal graph for $2K_{p+1}$.

In this paper, we determine the Turán number of a vertex-disjoint union of two cliques with different order by showing the next theorem.

Theorem 1.3. *Let n, p, q be positive integers with $q > p \geqslant 3$ and $n \geqslant p + q$. Then*

$$\mathrm{ex}(n, K_p \cup K_q) = \begin{cases} \binom{n}{2} - 3(n-p-q+1), & n \leqslant p+q+\max\Big\{2p-q, \Big\lfloor \frac{p}{2} \Big\rfloor - 1\Big\}; \\ t_{q-1}(n), & n > p+q+\max\Big\{2p-q, \Big\lfloor \frac{p}{2} \Big\rfloor - 1\Big\}. \end{cases}$$

2. Proof of Theorem

In this section, we prove Theorem 1.3. First, we give some definitions and notations. We denote the vertex-disjoint union of xK_3 and yK_4 by $K(x, y)$ and m isolated vertices by $D(m)$. For $n = p+q+t$ and $0 \leqslant t \leqslant \max\{2p-q, \lfloor p/2 \rfloor - 1\}$, let $J_3(n)$ be the graph consisting of $(t+1)K_3$ and $D(p+q-2t-3)$. For a graph G, let $\overline{G}$ be the complement of G. The minimum degree and maximum degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. Let $[l] = \{1, \ldots, l\}$. An l-partition of G, denoted by $(V_1, V_2, \ldots, V_l)$, is a group of disjoint nonempty subsets $V_1, V_2, \ldots, V_l$ of G with $V(G) = \bigcup\limits_{i=1}^{l} V_i$. The partition is called *balanced* if $||V_i| - |V_j|| \leqslant 1$ for every $i, j \in [l]$. Now, we list some lemmas.

Theorem 2.1 ([13])**.** *For every graph G with $\Delta(G) < l$, G has a balanced partition $(V_1, V_2, \ldots, V_l)$ such that V_i is an independent set of G for every $i \in [l]$.*

Lemma 2.1. *Let p, q, n be intergers with $q > p \geqslant 2$ and $n \geqslant p + q$. If G is a $(K_p \cup K_q)$-free graph on n vertices, then*

$$\delta(G) \leqslant (n-1) - \Big\lfloor \frac{n-1}{q-1} \Big\rfloor.$$

P r o o f. For a contradiction, suppose that $\delta(G) \geqslant (n-1) - \lfloor (n-1)/(q-1) \rfloor + 1$. Then $\Delta(\overline{G}) \leqslant \lfloor (n-1)/(q-1) \rfloor - 1$. Set $l = \lfloor (n-1)/(q-1) \rfloor$. By Theorem 2.1, G has a balanced partition $(V_1, V_2, \ldots, V_l)$ such that $G[V_i]$ is a complete graph for $1 \leqslant i \leqslant l$. Since G is $(K_p \cup K_q)$-free, we have $|V_i| \leqslant q-1$ for all $i \in [l]$, and

$$|V(G)| = \sum_{i=1}^{l} |V_i| \leqslant (q-1)l = (q-1)\Big\lfloor \frac{n-1}{q-1} \Big\rfloor \leqslant n-1,$$

a contraction. □

Lemma 2.2. *Assume that p, q, t are integers with $q > p \geqslant 3$ and $0 \leqslant t \leqslant \max\{2p-q, \lfloor p/2 \rfloor - 1\}$. Suppose that G is a $(K_p \cup K_q)$-free graph on $n = p+q+t$ vertices. If G has $\mathrm{ex}(n, K_p \cup K_q)$ edges, then $e(\overline{G}) = 3(t+1)$. Moreover, $\overline{G}$ is the disjoint union of $K(x,y)$ and $D(n-3x-4y)$ for some integers x and y when $t \leqslant \max\{2p-q,\ \lfloor p/2 \rfloor - 1\} - 1$.*

P r o o f. Recall that $J_3(n) = (t+1)K_3 \cup D(p+q-2t-3)$. Then

$$e(\overline{J_3(n)}) = \binom{n}{2} - 3(t+1),$$

and $\overline{J_3(n)}$ has $t+1$ independent sets $I_1, \ldots, I_{t+1}$ such that $|I_i| = 3$ for every $i \in [t+1]$. If $\overline{J_3(n)}$ contains a copy of $K_p \cup K_q$, then such a copy contains at most two vertices of each I_i, implying $2(t+1) + p + q - 2t - 3 \geqslant p + q$, a contradiction. So $\overline{J_3(n)}$ is $(K_p \cup K_q)$-free, which means

$$e(G) = \mathrm{ex}(n, K_p \cup K_q) \geqslant \binom{n}{2} - 3(t+1), \tag{2.1}$$

and then

$$e(\overline{G}) \leqslant 3(t+1). \tag{2.2}$$

We finish the proof by induction on t. For the base case $t = 0$, it follows from (2.2) that $e(\overline{G}) \leqslant 3$. If $e(\overline{G}) \leqslant 1$, then we can find a copy of $K_p \cup K_q$ in G easily, a contradiction. If $e(\overline{G}) = 2$, then there are two possible cases of $\overline{G}$, see Figure 1. In both cases, we can find a copy of K_p in G on vertices of the left part of the bipartition. Similarly, we can find a copy of K_q in G on vertices of the right part of the bipartition. These two parts of the bipartitions have no common vertices. Thus, we can find a copy of $K_p \cup K_q$ in G, a contradiction. Hence, $e(\overline{G}) = 3$, and the possible five cases of $\overline{G}$ can be found in Figure 2. Using the same method as in the case "$e(\overline{G}) = 2$" above, we can find a copy of $K_p \cup K_q$ in G in Cases 1, 2, 3 and 4. Thus, $\overline{G}$ is a copy of $K_3 \cup D(n-3)$ in Case 5.

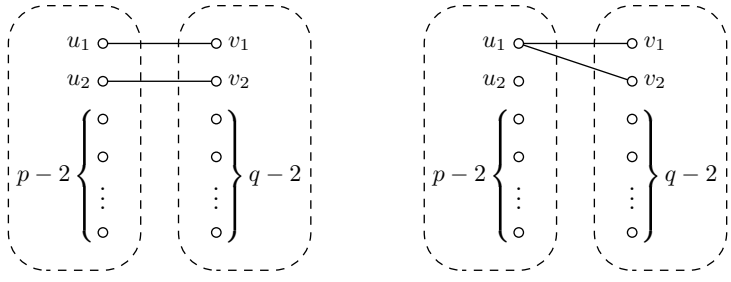


Figure 1. $t = 0$ and $e(\overline{G}) = 2$.

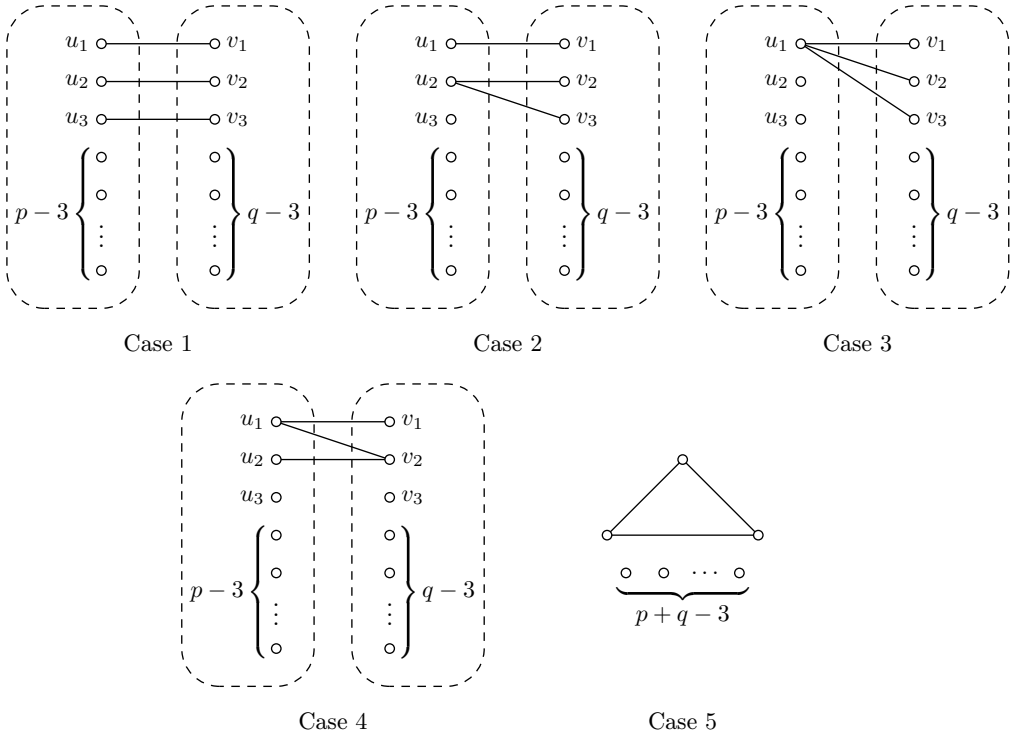


Figure 2. $t = 0$ and $e(\overline{G}) = 3$.

Assume that the theorem holds for $t - 1 \geqslant 0$, and we will prove it for t. Clearly $\Delta(\overline{G}) \geqslant 1$. For any $v \in V(G)$, note that $G - v$ is $(K_p \cup K_q)$-free. By the induction hypothesis, we have

$$e(G - v) \leqslant \mathrm{ex}(n-1, K_p \cup K_q) = \binom{n-1}{2} - 3t,$$

which implies

$$e(\overline{G} - \{v\}) = e(\overline{G - \{v\}}) \geqslant 3t.$$

Note that

$$e(G) \geqslant \binom{n}{2} - 3(t+1). \tag{2.3}$$

If $q - 1 \leqslant n \leqslant 2(q-1)$, then

$$t_{q-1}(n) = \binom{n}{2} - (n - (q-1)) = \binom{n}{2} - (n - (q-1)) = \binom{n}{2} - (p+t+1). \tag{2.4}$$

If $2(q-1) \leqslant n \leqslant 3(q-1)$, then

$$t_{q-1}(n) = \binom{n}{2} - 3(n-2(q-1)) - [q-1-(n-2(q-1))] = \binom{n}{2} - (2p-q+2t+3). \tag{2.5}$$

Recall that $t \leqslant \max\{2p-q, \lfloor p/2 \rfloor - 1\}$. If $\lfloor p/2 \rfloor - 1 \geqslant 2p-q$, then $n \leqslant q-1+p+\lfloor p/2 \rfloor \leqslant 2(q-1)$. By (2.3) and (2.4),

$$e(G) - t_{q-1}(n) \geqslant p - 2t - 2 \geqslant p - 2\left\lfloor \frac{p}{2} \right\rfloor \geqslant 0.$$

If $2p-q > \lfloor p/2 \rfloor - 1$, then $n \leqslant 3p \leqslant 3(q-1)$. In this case, when $n < 2(q-1)$, i.e., $p+q+t < 2(q-1)$, we have

$$t < q-p-2. \tag{2.6}$$

By $2p-q > \lfloor p/2 \rfloor - 1$, we have

$$q < 2p - \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right). \tag{2.7}$$

So

$$\begin{aligned} e(G) - t_{q-1}(n) &\geqslant p - 2t - 2 \overset{(2.6)}{\geqslant} p - 2(q-p-3) - 2 \overset{(2.7)}{\geqslant} 3p - 2\left(2p - \left\lfloor \frac{p}{2} \right\rfloor\right) + 4 \\ &= -p + 2\left\lfloor \frac{p}{2} \right\rfloor + 4 \geqslant 1. \end{aligned}$$

When $2(q-1) \leqslant n \leqslant p+q+(2p-q) = 3p$, by (2.3) and (2.5) we get

$$e(G) - t_{q-1}(n) \geqslant 2p - q - t \geqslant 0,$$

i.e., if $t \leqslant \max\{2p-q, \lfloor p/2 \rfloor - 1\} - 1$, we have $e(G) > t_{q-1}(n)$. If $t = \max\{2p-q, \lfloor p/2 \rfloor - 1\}$, then $e(G) \geqslant t_{q-1}(n)$. We conclude that $e(\overline{G}) = 3(t+1)$ if $e(G) = t_{q-1}(n)$. Now we assume

$$e(G) > t_{q-1}(n). \tag{2.8}$$

If $\Delta(\overline{G}) \geqslant 4$, then there exists a vertex $v \in V(G)$ such that

$$e(\overline{G} - v) = e(\overline{G}) - d_{\overline{G}}(v) \leqslant 3(t+1) - 4 = 3t - 1,$$

a contradiction.

If $\Delta(\overline{G}) = 1$, then $\overline{G}$ is a copy of $M_k \cup D(p+q+t-2k)$ for some $k \geqslant 1$, where M_k is a matching of size k. Take a maximum independent set I of G which consists of one vertex of each M_k and $D(p+q+t-2k)$. By (2.8) and the Turán theorem, $|I| \geqslant q$. Take an independent set I' of G which consists of another vertex of each M_k. If $k \geqslant p$, then I' and I are two disjoint idependent sets of $\overline{G}$ and we are done. Otherwise, we can choose $p-k$ isolated vertices of $D(p+q+t-2k)$ in I, and add them to I'. The resulting set I' is disjoint with I, and $|I'| = p$, $|I| = q+t$, this is a contradiction.

Suppose that $2 \leqslant \Delta(\overline{G}) \leqslant 3$. Now our aim is to find two disjoint independent sets I_1 and I_2 in $\overline{G}$ with size p and q, respectively. First, let I_1 and I_2 be two empty sets and then we will put vertices in I_1 and I_2 by considering the following cases.

Case 1: $\Delta(\overline{G}) = 3$. Let $v \in V(\overline{G})$ and $N_{\overline{G}}(v) = \{v_1, v_2, v_3\}$. Then $G - v$ is $(K_p \cup K_q)$-free and $e(\overline{G} - v) \leqslant 3t$ by (2.2). By induction, we know $e(\overline{G} - v) \geqslant 3t$. Thus, $e(\overline{G} - v) = 3t$, and applying induction again $\overline{G} - v$ is a copy of $K(x, y) \cup D(n-3x-4y-1)$. Let $\overline{G} - v = A_1 \cup \ldots \cup A_x \cup B_1 \cup \ldots \cup B_y \cup C$, where A_i is a copy of K_3 for $1 \leqslant i \leqslant x$, B_j is a copy of K_4 for $1 \leqslant j \leqslant y$, and C is a copy of $D(n-3x-4y-1)$. Note that

$$x + 2y = t. \tag{2.9}$$

Now we divide our proof into the following three cases.

Subcase 1.1: $N_{\overline{G}}(v) \subseteq C$. First, choosing two vertices from each A_i with $i \in [x]$, we put one in I_1 and the other in I_2. Do the same operation for each B_j, $j \in [y]$. Then, we put v in I_1 and $N_{\overline{G}}(v)$ in I_2. Thus, I_1 and I_2 are two vertex-disjoint independent sets with $|I_1| = x+y+1$ and $|I_2| = x+y+3$, and some isolated vertices of C left which can be put in I_1 and I_2. Let S denote the set of such isolated vertices. Then $|S| = n - (3x+4y+1+3)$ and $|S| + |I_1| + |I_2| = n - x - 2y = p + q$.

Now, we choose $p - (x+y+1)$ vertices from S and add them to I_1, and put the remaining vertices in S to I_2. Then we obtain vertex-disjoint independent sets $I_1^{'}$ and $I_2^{'}$ with $|I_1^{'}| = p$, $|I_2^{'}| = q$, a contradiction. This is possible. Indeed, if $q - p = 1$, then $p+q$ is odd. By $|I_1| + |I_2| = 2x + 2y + 4$ is even, we have $|S| \geqslant 1$. If $q - p \geqslant 2$, then by (2.9),

$$|S| \geqslant p + q - 2t - 4 \geqslant p + q - 2\max\Big\{2p - q, \Big\lfloor \frac{p}{2} \Big\rfloor - 1\Big\} - 4 \geqslant q - p.$$

In both of the above cases, we can find our desired $I_1^{'}$ and $I_2^{'}$ easily.

Subcase 1.2: Either $N_{\overline{G}}(v) \subseteq A_i$ for some $i \in [x]$ or $N_{\overline{G}}(v) \subseteq B_j$ for some $j \in [y]$. If $N_{\overline{G}}(v) \subset A_i$ for some i, then $\overline{G}$ is a copy of $K(x-1, y+1) \cup D(n-3x-4y-1)$, and we are done. By symmetry, let $N_{\overline{G}}(v) \subset B_1$. Then there is a vertex u in B_1 with $uv \notin E(\overline{G})$. First, we put u and v in I_1 and put a vertex of $N_{\overline{G}}(v)$ in I_2. Then,

choose two vertices from A_i (or B_j) for each $i \in [x]$ (or $2 \leqslant j \leqslant y$), and put one in I_1 and the other in I_2. Then $|I_1| = x + y + 1$ and $|I_2| = x + y$. Let S denote the set of isolated vertices in $\overline{G}$, which can be put in I_1 or I_2. By (2.9),

$$\begin{aligned} |S| &= n - (3x + 4y + 1) \geqslant p + q + t - (3t + 1) \\ &\geqslant p + q - 2\max\Big\{2p - q, \Big\lfloor \frac{p}{2} \Big\rfloor - 1\Big\} - 1 \geqslant q - p, \end{aligned}$$

and

$$|S| + |I_1| + |I_2| = n - x - 2y = p + q.$$

So we can get two vertex-disjoint independent sets I_1 and I_2 of size p, q, respectively, a contradiction.

Subcase 1.3: $N_{\overline{G}}(v) \cap K(x, y) \neq \emptyset$, $N(v) \not\subseteq A_i$ and $N(v) \not\subseteq B_j$ for all $i \in [x]$, $j \in [y]$. Let $c = |N_{\overline{G}}(v) \cap C|$. Then $0 \leqslant c \leqslant 2$. For each $i \in [x]$ (or $j \in [y]$), we can find a vertex $a_i \in A_i$ (or $b_j \in B_j$) with $a_i v \notin E(\overline{G})$ (or $b_j v \notin E(\overline{G})$), and then add v and these a_i, b_j to I_1. We choose one vertex outside I_1 in each A_i and B_j and c vertices in $N_{\overline{G}}(v) \cap C$, and add these vertices to I_2. Thus, I_1 and I_2 are two vertex-disjoint independent sets with $|I_1| = x + y + 1$ and $|I_2| = x + y + c$. Let S be the set of remaining isolated vertices. Then by (2.9),

$$|S| + |I_1| + |I_2| = n - x - 2y = p + q.$$

Note that $||I_1| - |I_2|| = 1$. By putting vertices of S into I_1, I_2, we can easily find I_1' and I_2' with size p and q, respectively.

Case 2: $\Delta(\overline{G}) = 2$. For each $v \in V(\overline{G})$, $\overline{G}$ consists of a union of cycles, paths, and isolated vertices. Let t_0 be the number of triangles, t_1 be the number of odd cycles with length at least five, t_2 be the number of odd paths, t_3 be the number of isolated vertices. Then

$$e(\overline{G}) \geqslant 3t_0 + 5t_1. \tag{2.10}$$

Hence, we can put such vertices of $\overline{G}$ into I_1 and I_2 that

$$|I_1| + |I_2| = n - t_0 - t_1 = p + q + t - t_0 - t_1. \tag{2.11}$$

If $t_2 + t_3 \geqslant q - p$, then $|I_1| + |I_2| \leqslant p + q - 1$, since otherwise we can get two vertex-disjoint independent sets I_1 and I_2 of sizes p, q, respectively. Combining $|I_1| + |I_2| \leqslant p + q - 1$ and (2.11), we have $t_0 + t_1 \geqslant t + 1$. This together with (2.2) and (2.10) yields $t_0 = t + 1$ and $t_1 = t_2 = 0$. Thus, $\overline{G}$ consists of $t + 1$ triangles and isolated vertices.

If $t_2 + t_3 \leqslant q - p - 1$. Take a maximum independent set $|I|$ of all cycles and paths of $\overline{G}$. If $|I| \geqslant q$, let $I_1 = I$ and I_2 be a maximum independent set of all cycles and paths of $\overline{G}$ except vertices of I_1. Therefore, $|I_2| = |I_1| - t_2 \geqslant p + 1$. If $|I| < q$, according to (2.8), we can take $q - |I|$ vertices from isolated vertices of $\overline{G}$ to form an independent set I_1 with I. Let I_2 be a maximum independent set of $\overline{G}$ except vertices of I_1, thus $|I_2| \geqslant |I_1| - t_2 - t_3 \geqslant p + 1$. These two cases contradict G being $(K_p \cup K_q)$-free. Thus, we finish the proof of the lemma. □

P r o o f of Theorem 1.3. Let G_n be an extremal graph for $K_p \cup K_q$ on n vertices. If $p + q \leqslant n \leqslant p + q + \max\{2p - q, \lfloor p/2 \rfloor - 1\}$, then by the proof of Lemma 2.3, we have $e(G_n) = \binom{n}{2} - 3(n - p - q + 1)$. Note that for each $v \in V(G_n)$, $G_n - \{v\}$ is also $(K_p \cup K_q)$-free. By Lemma 2.2,

$$e(G_n) - e(G_{n-1}) \leqslant (n-1) - \left\lfloor \frac{n-1}{q-1} \right\rfloor = t_{q-1}(n) - t_{q-1}(n-1). \tag{2.12}$$

Let $\alpha_n = e(G_n) - t_{q-1}(n) \geqslant 0$. By (2.12), we have $\alpha_n \leqslant \alpha_{n-1}$, then $\{\alpha_n\}$ is a decreasing sequence. First suppose that $2p - q \geqslant \lfloor p/2 \rfloor - 1$. For $n = p + q + (2p - q) = 3p$, if $n < 2(q-1)$, then by (2.4) we have

$$\alpha_n = (p + (2p - q) + 1) - 3(n - p - q + 1) = -3p + 2q - 2 \geqslant 0.$$

But for $n' = p + q + (2p - q + 1) = 3p + 1 \leqslant 2(q-1)$, by (2.4) and $2p - q \geqslant \lfloor p/2 \rfloor - 1$, we have $\alpha_{n'} \leqslant 0$. If $2(q-1) \leqslant n \leqslant 3(q-1)$, then by (2.5) we have

$$\alpha_n = (2p - q + 2(2p - q) + 3) - 3(n - p - q + 1) = 0.$$

Therefore, as $\{\alpha_n\}$ is a decreasing sequence and $\alpha_n \geqslant 0$, it can be inferred that $\alpha_n = 0$ for $n \geqslant 3q$, i.e., $e(G_n) = t_{q-1}(n)$ for $n \geqslant 3p$.

Suppose that $2p - q < \lfloor p/2 \rfloor - 1$. For $n = p + q + \lfloor p/2 \rfloor - 1 \leqslant 2(q-1)$, we have by (2.4)

$$\alpha_n = \left(p + \left(\left\lfloor \frac{p}{2} \right\rfloor - 1\right) + 1\right) - 3(n - p - q + 1) = p - 2\left\lfloor \frac{p}{2} \right\rfloor = 0 \quad \text{or} \quad 1.$$

But for $n' = p + q + \lfloor p/2 \rfloor$, by (2.4) or (2.5), we have $\alpha_{n'} \leqslant 0$. Therefore, by $\{\alpha_n\}$ is a decreasing sequence and $\alpha_n \geqslant 0$, it can be inferred that $\alpha_n = 0$ for $n \geqslant p+q+\lfloor p/2 \rfloor$, i.e., $e(G_n) = t_{q-1}(n)$ for $n \geqslant p + q + \lfloor p/2 \rfloor$. To sum up, when $n > p + q + \max\{2p - q, \lfloor p/2 \rfloor - 1\}$, $e(G_n) = t_{q-1}(n)$. □

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