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HIGHER-DIMENSIONAL AUSLANDER-REITEN SEQUENCES

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Abstract. Zhou and Zhu have shown that if $\mathscr{C}$ is an $(n+2)$-angulated category and $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$, then the quotient category $\mathscr{C}/\mathscr{X}$ is an n-abelian category. We show that if $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles, then $\mathscr{C}/\mathscr{X}$ has Auslander-Reiten n-exact sequences.

Keywords: $(n+2)$-angulated category; cluster tilting subcategory; n-abelian category; Auslander-Reiten $(n+2)$-angle; Auslander-Reiten n-exact sequence

MSC 2020: 16G70, 18G80, 18E10

1. Introduction

Since its introduction in the last seventies, as documented in [2], [3], the Auslander-Reiten theory of almost split sequences has emerged as a cornerstone in the representation theory of algebras, as highlighted in references [1], [4]. Subsequently, Happel in [8] extended this framework by introducing the analogous theory of Auslander-Reiten triangles in triangulated categories. This extension not only broadened the scope of the Auslander-Reiten theory but also facilitated its application in diverse mathematical domains, including algebraic topology and algebraic geometry, as evidenced by references [14], [15]. Since these advancements, the theory has undergone further independent development in both exact categories and triangulated categories, see [13], [18], [20].

Recently, Geiss, Keller and Oppermann in [7] introduced a new type of categories, called $(n+2)$-*angulated categories*, which generalize triangulated categories. We note that the case $n=1$ corresponds to triangulated categories. These categories appear

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for instance when considering certain n-cluster tilting subcategories of triangulated categories. Jasso in [12] introduced n-abelian categories which are an axiomatization of n-cluster tilting subcategories. The case $n = 1$ corresponds to the classical concepts of abelian categories. An important source of examples of n-abelian categories are n-cluster tilting subcategories. The notion of Auslander-Reiten $(n + 2)$-angles in a special $(n + 2)$-angulated category was first introduced by Iyama and Yoshino (see [11]), and then was generalized by Fedele (see [5]) to arbitrary $(n+2)$-angulated category. Fedele in [6] similarly introduced the concept of Auslander-Reiten n-exact sequences in n-abelian categories.

Cluster tilting theory provides a way of constructing n-abelian categories from $(n + 2)$-angulated categories. Zhou and Zhu in [23] showed that if $\mathscr{C}$ is an $(n + 2)$-angulated category and $\mathscr{X}$ is a cluster tilting subcategory of $\mathscr{C}$, then the quotient category $\mathscr{C}/\mathscr{X}$ is an n-abelian category. In other words, they established a connection between $(n + 2)$-angulated categories and n-abelian categories. In this paper, we discuss the relation between Auslander-Reiten sequences in these two categories. Our main result is the following.

Theorem 1.1 (see Theorem 3.4 for details)**.** *Let $\mathscr{C}$ be an $(n + 2)$-angulated category and $\mathscr{X}$ be a cluster tilting subcategory of $\mathscr{C}$. If $\mathscr{C}$ has Auslander-Reiten $(n + 2)$-angles, then $\mathscr{C}/\mathscr{X}$ has Auslander-Reiten n-exact sequences.*

This paper is organised as follows: In Section 2, we review some elementary definitions that we need to use, including n-abelian categories, Auslander-Reiten n-exact sequences, $(n+2)$-angulated categories and Auslander-Reiten $(n+2)$ angles. In Section 3, we prove our main result.

2. Preliminaries

Let $\mathscr{C}$ be an additive category and $f\colon A \to B$ be a morphism in $\mathscr{C}$. Recall that $g\colon B \to C$ is a *weak cokernel* of f if $gf = 0$ and for each morphism $h\colon B \to X$ such that $hf = 0$ there exists a morphism $s\colon C \to X$ (not necessarily unique) such that $sg = h$. If g is an epimorphism, we note that g is a *cokernel* of f. Dually, we can define *weak kernel*.

Definition 2.1 ([12], Definition 2.2)**.** Let $\mathscr{C}$ be an additive category and $a_n\colon A_n \to A_{n+1}$ be a morphism in $\mathscr{C}$. An *n-kernel* of a_n is a sequence

$$(a_0, a_1, \ldots, a_{n-1})\colon \quad A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-2}} A_{n-1} \xrightarrow{a_{n-1}} A_n \ ,$$

where the morphism a_{k-1} is a weak kernel of a_k for all $2 \leqslant k \leqslant n$ and a_0 is a kernel of a_1. In this case, the sequence

$$0 \longrightarrow A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1}$$

is called *left n-exact sequence*. We can define *right n-exact sequence* dually. A sequence is called *n-exact sequence* if it is both left n-exact sequence and right n-exact sequence.

Recall that throughout this paper, n is a positive integer. We start with the definition of n-abelian categories from [12].

Definition 2.2 ([12], Definition 3.1)**.** An additive category is called *n-abelian* if it satisfies some of the following axioms:

(A0) $\mathscr{C}$ has split idempotents.

(A1) Each morphism in $\mathscr{C}$ has an n-kernel and n-cokernel.

(A2) For any monomorphism $a_0\colon A_0 \to A_1$ and any n-cokernel $(a_1, \ldots, a_n)$ of a_0 in $\mathscr{C}$, the sequence

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1}$$

is an n-exact sequence.

(A2$^{\mathrm{op}}$) For any epimorphism $a'_n\colon A'_n \to A'_{n+1}$ and any n-kernel $(a'_0, \ldots, a'_{n-1})$ of a'_n in $\mathscr{C}$, the sequence

$$A'_0 \xrightarrow{a'_0} A'_1 \xrightarrow{a'_1} \cdots \xrightarrow{a'_{n-1}} A'_n \xrightarrow{a'_n} A'_{n+1}$$

is an n-exact sequence.

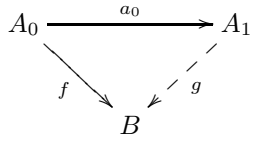
Now we give an example of n-abelian categories.

Example 2.3 ([12], Theorem 3.16)**.** Let $\mathcal{A}$ be an abelian category and $\mathscr{X}$ be an n-cluster tilting subcategory of $\mathcal{A}$. Then $\mathscr{X}$ is an n-abelian category.

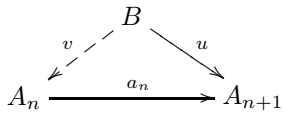
We recall the notion of Auslander-Reiten n-exact sequences in n-abelian categories.

Definition 2.4 ([1], Definition 1.1)**.** Let $\mathscr{C}$ be an additive category.

(1) A morphism $a_0\colon A_0 \to A_1$ is called *left almost split* in $\mathscr{C}$ if it is not a split monomorphism and for every $B \in \mathscr{C}$ and every morphism $f\colon A_0 \to B$ which is not split monomorphism, there exists a morphism $g\colon A_1 \to B$ such that $ga_0 = f$; this is reflected in the following commutative diagram:



(2) A morphism $a_n\colon\ A_n \to A_{n+1}$ is called *right almost split* in $\mathscr{C}$ if it is not a split epimorphism and for every $B \in \mathscr{C}$ and every morphism $u\colon\ B \to A_{n+1}$ which is not split epimorphism, there exists a morphism $v\colon\ B \to A_n$ such that $a_n v = u$; this is reflected in the following commutative diagram:



Let $\mathscr{C}$ be an additive category. We denote by $\mathrm{rad}_{\mathscr{C}}$ the *Jacobson radical* of $\mathscr{C}$, which is defined by the formula

$$\mathrm{rad}_{\mathscr{C}}(A,B) = \{f\colon\ A \to B\colon\ 1_A - gf \text{ is invertible for any } g\colon\ B \to A\},$$

where $A, B \in \mathscr{C}$.

Definition 2.5 ([6], Definition 4.3)**.** Let $\mathscr{C}$ be an n-abelian category. An n-exact sequence

$$0 \longrightarrow A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a'_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \longrightarrow 0$$

in $\mathscr{C}$ is called an *Auslander-Reiten n-exact sequence* if a_0 is left almost split, a_n is right almost split and $a_1, a_2, \ldots, a_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$ when $n \geqslant 2$.

Definition 2.6 ([12], Definition 3.11)**.** Let $\mathscr{C}$ be an additive category. We say that $P \in \mathscr{C}$ is a *projective object* if for any epimorphism $f\colon\ A \to B$ the sequence of abelian groups

$$\mathscr{C}(P,A) \xrightarrow{\mathscr{C}(P,f)} \mathscr{C}(P,B) \longrightarrow 0$$

is exact. We can define *injective object* dually.

Definition 2.7. Let $\mathcal{A}$ be an n-abelian category. If for any non-projective indecomposable object $A \in \mathcal{A}$ there exists an Auslander-Reiten n-exact sequence ending at A, and for any non-injective indecomposable object $B \in \mathcal{A}$ there exists an Auslander-Reiten n-exact sequence starting at B. Then we say that $\mathcal{A}$ has Auslander-Reiten n-exact sequences.

Example 2.8 ([9], Theorem 3.12)**.** Let $\mathcal{A}$ be an n-abelian category. If $\mathcal{A}$ is locally finite, then $\mathcal{A}$ has Auslander-Reiten n-exact sequences.

Now we recall the definition of $(n+2)$-angulated categories.

Definition 2.9 ([7], (Definition 2.1)**.** An $(n+2)$-*angulated category* is a triple $(\mathscr{C}, \Sigma, \Phi)$, where $\mathscr{C}$ is an additive category, Σ is an automorphism of $\mathscr{C}$, and Φ is a class of $(n+2)$-Σ-sequences (whose elements are called $(n+2)$-*angles*), which satisfies the following axioms:

(N1) (a) The class Φ is closed under isomorphisms, direct sums and direct summands.

(b) For any object $A \in \mathscr{C}$, the following trivial sequence

$$A \xrightarrow{1} A \to 0 \to 0 \to \ldots \to 0 \to \Sigma A$$

belongs to Φ.

(c) Each morphism $a_0 \colon A_0 \to A_1$ in $\mathscr{C}$ can be embedded in an $(n+2)$-angle, and a_0 is the first morphism.

(N2) An $(n+2)$-Σ-sequence belongs to Φ if and only if its left rotation belongs to Φ.

(N3) Given the solid part of the commutative diagram

$$\begin{array}{ccccccccccccc}
A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma A_0 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle \Sigma f_0} \\
B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & \cdots & \xrightarrow{b_{n-1}} & B_n & \xrightarrow{b_n} & B_{n+1} & \xrightarrow{b_{n+1}} & \Sigma B_0
\end{array}$$

with rows in Φ there exist the dotted morphisms such that the above diagram commutes, i.e., $(f_0, f_1, \ldots, f_{n+1})$ is a morphism of $(n+2)$-Σ-sequences.

(N4) In the situation of (N3), the morphisms $f_2, f_3, \ldots, f_{n+1}$ can be chosen so that

$$\begin{aligned}
A_1 \oplus B_0 &\xrightarrow{\begin{bmatrix} -a_1 & 0 \\ f_1 & b_0 \end{bmatrix}} A_2 \oplus B_1 \xrightarrow{\begin{bmatrix} -a_2 & 0 \\ f_2 & b_1 \end{bmatrix}} \cdots \\
\cdots &\xrightarrow{\begin{bmatrix} -a_{n+1} & 0 \\ f_{n+1} & b_n \end{bmatrix}} \Sigma A_0 \oplus B_{n+1} \xrightarrow{\begin{bmatrix} -\Sigma a_0 & 0 \\ \Sigma f_0 & b_{n+1} \end{bmatrix}} \Sigma A_1 \oplus \Sigma B_0
\end{aligned}$$

belongs to Φ.

Now we give an example of $(n+2)$-angulated categories.

Example 2.10 ([10], Definition 2.2)**.** Let Λ be a finite dimensional algebra over a field k. We say that Λ is an n-representation finite algebra if $\operatorname{gl.dim} \Lambda \leqslant n$ and the module category $\operatorname{mod} \Lambda$ has an n-cluster tilting object.

Assume that Λ is n-representation finite. Let $\mathscr{X}$ be an n-cluster tilting subcategory of $\operatorname{mod} \Lambda$. Then

$$\mathcal{T} = \{\mathscr{X}[dn] \colon d \in \mathbb{Z}\} \subseteq D^b \ (\operatorname{mod} \Lambda)$$

is an n-cluster tilting subcategory of $D^b \ (\operatorname{mod} \Lambda)$ and satisfies $\mathcal{T}[n] = \mathcal{T}$. By [7], Theorem 1, we know that $\mathcal{T}$ is an $(n+2)$-angulated category with an automorphism functor $[n]$.

Now we recall the definition of right $(n+2)$-angulated categories.

Definition 2.11 ([17], Definition 2.1)**.** A *right* $(n+2)$*-angulated category* is a triple $(\mathscr{C}, \Sigma, \Theta)$, where $\mathscr{C}$ is an additive category, Σ is an endofunctor of $\mathscr{C}$, and Θ is a class of $(n+2)$-Σ-sequences (whose elements are called *right* $(n+2)$*-angles*), which satisfies the following axioms:

(RN1) (a) The class Θ is closed under isomorphisms, direct sums and direct summands.

(b) For any object $A \in \mathscr{C}$, the trivial sequence

$$A \xrightarrow{1} A \to 0 \to 0 \to \ldots \to 0 \to \Sigma A$$

belongs to Θ.

(c) Each morphism $a_0\colon A_0 \to A_1$ in $\mathscr{C}$ can be embedded in a right $(n+2)$-angle, and a_0 is the first morphism.

(RN2) If an $(n+2)$-Σ-sequence belongs to Θ, then its left rotation belongs to Θ.

(RN3) Given the solid part of the commutative diagram

$$\begin{array}{ccccccccccc}
A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma A_0 \\
\downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle \Sigma f_0} \\
B_0 & \xrightarrow{b_0} & B_1 & \xrightarrow{b_1} & B_2 & \xrightarrow{b_2} & \cdots & \xrightarrow{b_{n-1}} & B_n & \xrightarrow{b_n} & B_{n+1} & \xrightarrow{b_{n+1}} & \Sigma B_0
\end{array}$$

with rows in Θ, there exists the dotted morphisms such that the above diagram commutes, i.e., $(f_0, f_1, \ldots, f_{n+1})$ is a morphism of $(n+2)$-Σ-sequences.

(RN4) Given a commutative diagram (see Diagram 1) with the first two rows and the second column in Θ, there exists morphisms

$$f_i\colon A_i \to B_i \quad (i = 2, 3, \ldots, n+1), \quad g_i\colon B_i \to C_i \quad (i = 2, 3, \ldots, n+1),$$
$$h_i\colon A_i \to C_{i-1} \quad (i = 3, 4, \ldots, n+1)$$

with the following two properties:

(a) The sequence $(1_{A_0}, f_1, \ldots, f_{n+1})$ is a morphism of $(n+2)$-Σ-sequences.

(b) The $(n+2)$-Σ-sequence

$$\begin{aligned}
A_2 &\xrightarrow{\left[\begin{smallmatrix} a_2 \\ f_2 \end{smallmatrix}\right]} A_3 \oplus B_2 \xrightarrow{\left[\begin{smallmatrix} -a_3 & 0 \\ f_3 & -b_2 \\ h_3 & g_2 \end{smallmatrix}\right]} A_4 \oplus B_3 \oplus C_2 \\
&\xrightarrow{\alpha_1} A_5 \oplus B_4 \oplus C_3 \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_{n-3}} A_{n+1} \oplus B_n \oplus C_{n-1} \\
&\xrightarrow{\beta} B_{n+1} \oplus C_n \xrightarrow{[\,g_{n+1}\ c_n\,]} C_{n+1} \xrightarrow{\Sigma a_1 \circ c_{n+1}} \Sigma A_2
\end{aligned}$$

belongs to Θ, and $c_{n+1}g_{n+1} = \Sigma a_0 \circ b_{n+1}$, where

$$\alpha_i = \begin{bmatrix} -a_{i+3} & 0 & 0 \\ (-1)^i f_{i+3} & -b_{i+2} & 0 \\ h_{i+3} & g_{i+2} & c_{i+1} \end{bmatrix}, \quad \beta = \begin{bmatrix} (-1)^n f_{n+1} & -b_n & 0 \\ h_{n+1} & g_n & c_{n-1} \end{bmatrix}.$$

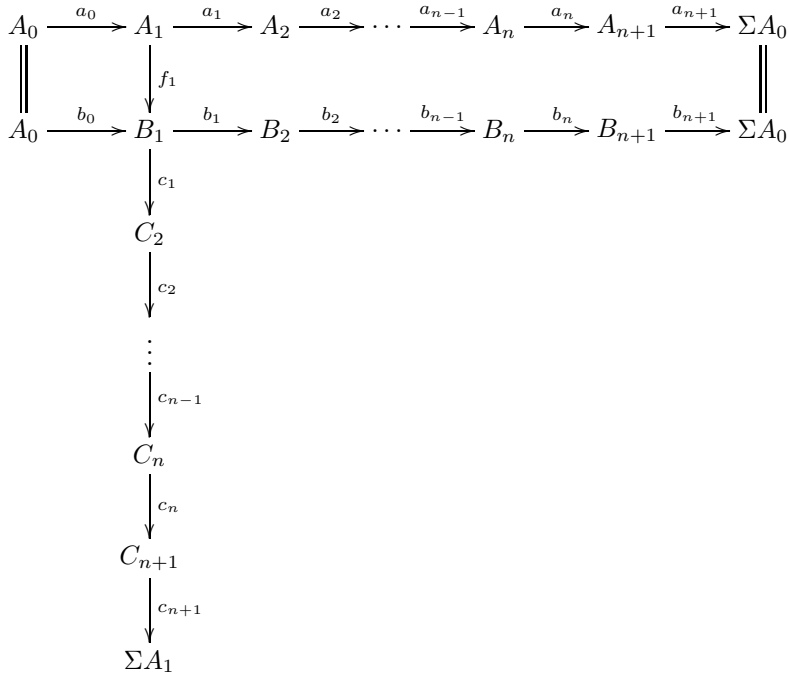


Diagram 1.

Definition 2.12 ([11], Definition 3.8, [5], Definition 5.1)**.** Let $\mathscr{C}$ be an $(n+2)$-angulated category. An $(n+2)$-angle

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma A_0 \tag{2.1}$$

in $\mathscr{C}$ is called an *Auslander-Reiten* $(n+2)$*-angle* if a_0 is left almost split, a_n is right almost split and $a_1, a_2, \ldots, a_{n-1}$ are in $\mathrm{rad}_{\mathscr{C}}$ when $n \geqslant 2$.

Remark 2.13 ([5], Remark 5.2)**.** Assume (2.1) as in Definition 2.12 is an Auslander-Reiten $(n+2)$-angle. Since a_0 is left almost split, $\mathrm{End}(A_0)$ is local and thus, A_0 is indecomposable. Similarly, since a_n is right almost split, $\mathrm{End}(A_{n+1})$ is local and thus, A_{n+1} is indecomposable.

Definition 2.14 ([21], Theorem 3.8)**.** Let $\mathscr{C}$ be an $(n+2)$-angulated category. If for any indecomposable object $C \in \mathscr{C}$ there exists an Auslander-Reiten $(n+2)$-angle ending at C, and an Auslander-Reiten $(n+2)$-angle starting at C, then we say that $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles.

Example 2.15 ([21], Theorem 3.8)**.** Let $\mathscr{C}$ be an $(n+2)$-angulated category. If $\mathscr{C}$ is locally finite, then $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles.

Let $\mathscr{C}$ be an additive category and $\mathscr{X}$ be a subcategory of $\mathscr{C}$. In this paper, when we say $\mathscr{X}$ is a subcategory of $\mathscr{C}$, it usually means that $\mathscr{X}$ is full, and is closed under isomorphisms, direct sums and direct summands. In the quotient category $\mathscr{C}/\mathscr{X}$, we denote by A the object in $\mathscr{C}$ and its image in $\mathscr{C}/\mathscr{X}$, and the morphisms are elements in $\mathscr{C}(A,B)/\mathscr{X}(A,B)$, where $\mathscr{X}(A,B)$ forms a subgroup of $\mathscr{C}(A,B)$, and these morphisms pass through objects within $\mathscr{X}$. We denote by $\overline{f}$ the image of the morphism $f\colon A \to B$ under the quotient functor $\mathscr{C} \to \mathscr{C}/\mathscr{X}$.

A morphism $f\colon A \to B$ in $\mathscr{C}$ is called *$\mathscr{X}$-monic* if for any object X in $\mathscr{X}$, the sequence

$$\mathscr{C}(B,X) \xrightarrow{\mathscr{C}(f,X)} \mathscr{C}(A,X) \longrightarrow 0$$

is exact. Additionally, the morphism f is called a *left $\mathscr{X}$-approximation* of A if B belongs to $\mathscr{X}$. The subcategory $\mathscr{X}$ is called *covariantly finite* of $\mathscr{C}$ if any object $A \in \mathscr{C}$ has a left $\mathscr{X}$-approximation. We can define *$\mathscr{X}$-epic, right $\mathscr{X}$-approximation* and *contravariantly finite* dually. If it is both covariantly finite and contravariantly finite, we call the subcategory $\mathscr{X}$ *functorially finite*.

Definition 2.16 ([22], Definition 5.1)**.** Let $\mathscr{C}$ be an $(n+2)$-angulated category. A subcategory $\mathscr{X}$ of $\mathscr{C}$ is called *strongly covariantly finite* if for any object $C \in \mathscr{C}$, there exists an $(n+2)$-angle

$$C \xrightarrow{f} X_1 \to X_2 \to \ldots \to X_n \to B \to \Sigma C,$$

where f is a left $\mathscr{X}$-approximation of C and $X_1, \ldots, X_n \in \mathscr{X}$. Dually, we can define a *strongly contravariantly finite* subcategory.

A strongly covariantly finite and strongly contravariantly finite subcategory is called *strongly functorially finite*.

Definition 2.17 ([16], Proposition 3.3)**.** Let $(\mathscr{C}, \Sigma, \Phi)$ be an $(n+2)$-angulated category and $\mathscr{X}$ be a strongly covariantly finite subcategory of $\mathscr{C}$. Then there exists an additive endofunctor $\mathbb{G}\colon \mathscr{C}/\mathscr{X} \to \mathscr{C}/\mathscr{X}$, defined as follows:

For any object $A \in \mathscr{C}$ there exists an $(n+2)$-angle

$$A \xrightarrow{a_0} X_1 \xrightarrow{a_1} X_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} X_n \xrightarrow{a_n} B \xrightarrow{a_{n+1}} \Sigma A\ ,$$

where $X_1, \dots, X_n \in \mathscr{X}$, a_0 is a left $\mathscr{X}$-approximation of A. For any morphism $f\colon A \to A'$, since a_0 is a left $\mathscr{X}$-approximation of A, we have the following commutative diagram:

$$\begin{array}{ccccccccccccc}
A & \xrightarrow{a_0} & X_1 & \xrightarrow{a_1} & X_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & X_n & \xrightarrow{a_n} & B & \xrightarrow{a_{n+1}} & \Sigma A \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle \Sigma f} \\
A' & \xrightarrow{a'_0} & X'_1 & \xrightarrow{a'_1} & X'_2 & \xrightarrow{a'_2} & \cdots & \xrightarrow{a'_{n-1}} & X'_n & \xrightarrow{a'_n} & B' & \xrightarrow{a'_{n+1}} & \Sigma A'
\end{array}$$

Then we can define an endofunctor $\mathbb{G}\colon \mathscr{C}/\mathscr{X} \to \mathscr{C}/\mathscr{X}$ so that $\mathbb{G}A = B$ and $\mathbb{G}\overline{f} = \overline{g}$.

Definition 2.18 ([16], Definition 3.4)**.** Let $(\mathscr{C}, \Sigma, \Phi)$ be an $(n+2)$-angulated category and $\mathscr{X}$ be a strongly covariantly finite subcategory of $\mathscr{C}$. Assume that

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma A_0$$

is an $(n+2)$-angle in $\mathscr{C}$, where a_0 is $\mathscr{X}$-monic. Then there exists a commutative diagram

$$\begin{array}{ccccccccccccc}
A_0 & \xrightarrow{a_0} & A_1 & \xrightarrow{a_1} & A_2 & \xrightarrow{a_2} & \cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma A_0 \\
\| & & \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n+1}} & & \| \\
A_0 & \xrightarrow{b_0} & X_1 & \xrightarrow{b_1} & X_2 & \xrightarrow{b_2} & \cdots & \xrightarrow{b_{n-1}} & X_n & \xrightarrow{b_n} & \mathbb{G}A_0 & \xrightarrow{b_{n+1}} & \Sigma A_0
\end{array}$$

of $(n+2)$-angles. Then we have a complex

$$A_0 \xrightarrow{\overline{a_0}} A_1 \xrightarrow{\overline{a_1}} A_2 \xrightarrow{\overline{a_2}} \cdots \xrightarrow{\overline{a_n}} A_{n+1} \xrightarrow{(-1)^n\overline{f_{n+1}}} \mathbb{G}A_0.$$

We define *right* $(n+2)$*-angles* in $\mathscr{C}/\mathscr{X}$ as the complexes which are isomorphic to the complexes obtained in this way. We denote by Θ the class of right $(n+2)$-angles.

Theorem 2.19 ([17], Remark 3.8)**.** *Let $(\mathscr{C}, \Sigma, \Phi)$ be an $(n+2)$-angulated category and $\mathscr{X}$ be a strongly covariantly finite subcategory of $\mathscr{C}$. Then the quotient category $\mathscr{C}/\mathscr{X}$ is a right $(n+2)$-angulated category in the sense of Lin (see* [17]*, Definition* 2.1*), where the endofunctor $\mathbb{G}$ is defined in Definition* 2.17 *and right $(n+2)$-angles are defined in Definition* 2.18*.*

Lemma 2.20 ([23], Lemma 2.7)**.** *Let $(\mathscr{C}, \Sigma, \Theta)$ be a right $(n+2)$-angulated category and the sequence*

$$A_0 \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma A_0$$

be a right $(n+2)$-angle. Then a_i is a weak cokernel of a_{i-1} for $i = 1, 2, \dots, n+1$.

3. Main result

The notion of a cluster tilting subcategory in $(n+2)$-angulated categories was introduced by [23].

Definition 3.1 ([23], Definition 1.1)**.** Let $\mathscr{C}$ be an $(n+2)$-angulated category and $\mathscr{X}$ be an additive subcategory of $\mathscr{C}$, $\mathscr{X}$ is called *cluster tilting* if

(1) $\mathscr{C}(\mathscr{X}, \Sigma\mathscr{X}) = 0$,

(2) for any object $C \in \mathscr{C}$ there exists an $(n+2)$-angle

$$X_0 \to X_1 to \ldots \to X_{n-1} \to X_n \xrightarrow{f} C \to \Sigma X_0, \tag{3.1}$$

where $X_0, X_1, \ldots, X_n \in \mathscr{X}$.

Remark 3.2. In Definition 3.1, it is easy to see that $\mathscr{X}$ is functorially finite. In fact, for any object $C \in \mathscr{C}$ there exists an $(n+2)$-angle as (3.1). Applying the functor $\mathscr{C}(\mathscr{X}, -)$ to the $(n+2)$-angle (3.1), we have the following exact sequence:

$$\mathscr{C}(\mathscr{X}, X_n) \xrightarrow{\mathscr{C}(\mathscr{X}, f)} \mathscr{C}(\mathscr{X}, C) \longrightarrow \mathscr{C}(\mathscr{X}, \Sigma X_0).$$

By Definition 3.1, we have $\mathscr{C}(\mathscr{X}, \Sigma X_0) = 0$. This shows that $\mathscr{C}(\mathscr{X}, f)$ is an epimorphism. Hence, f is a right $\mathscr{X}$-approximation of C. It follows that $\mathscr{X}$ is contravariantly finite. Dually, we can show that $\mathscr{X}$ is covariantly finite.

Theorem 3.3 ([23], Theorem 3.4)**.** *Let $\mathscr{C}$ be an $(n+2)$-angulated category and $\mathscr{X}$ be a cluster tilting subcategory of $\mathscr{C}$. Then the quotient category $\mathscr{C}/\mathscr{X}$ is an n-abelian category.*

Now we establish a connection between Auslander-Reiten $(n+2)$-angles in $(n+2)$-angulated categories and Auslander-Reiten n-exact sequences in n-abelian categories.

Theorem 3.4. *Let $(\mathscr{C}, \Sigma, \Phi)$ be an $(n+2)$-angulated category and $\mathscr{X}$ be a cluster tilting subcategory of $\mathscr{C}$. If $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles, then $\mathscr{C}/\mathscr{X}$ has Auslander-Reiten n-exact sequences.*

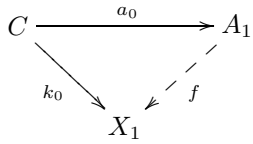
Proof. By Theorem 3.3, we know that $\mathscr{C}/\mathscr{X}$ is an n-abelian category.

For any non-injective indecomposable object $C \in \mathscr{C}/\mathscr{X}$ we have $C \oplus X \in \mathscr{C}$, where $X \in \mathscr{X}$, and then C is indecomposable in $\mathscr{C}$. Since $\mathscr{C}$ has Auslander-Reiten $(n+2)$-angles, we have the following Auslander-Reiten $(n+2)$-angle:

$$C \xrightarrow{a_0} A_1 \xrightarrow{a_1} A_2 \xrightarrow{a_2} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma C. \tag{3.2}$$

Since $\mathscr{X}$ is functorially finite, take $k_0 \colon C \to X_1$ a left $\mathscr{X}$-approximation of C; note that it is not a split monomorphism (otherwise $C \in \mathscr{X}$ and then $C = 0$ in $\mathscr{C}/\mathscr{X}$, this

is a contradiction). Since a_0 is left almost split, there exists a morphism $f\colon A_1 \to X_1$ such that $fa_0 = k_0$; it is reflected in the following commutative diagram:



For object $X_1 \in \mathscr{C}$ there exists a trivial $(n+2)$-angle $0 \to X_1 \xrightarrow{1} X_1 \to 0 \to \ldots \to 0$.

Since Φ is closed under direct sums, considering the direct sum of the $(n+2)$-angle (3.2) and the trivial $(n+2)$-angle of X_1, we obtain the following $(n+2)$-angle:

$$C \xrightarrow{\binom{a_0}{0}} A_1 \oplus X_1 \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} A_2 \oplus X_1 \xrightarrow{(a_2,\ 0)} A_3 \xrightarrow{a_3} \cdots \xrightarrow{a_{n-1}} A_n \xrightarrow{a_n} A_{n+1} \xrightarrow{a_{n+1}} \Sigma C.$$

The commutative diagram

$$\begin{array}{ccccccccccc}
C & \xrightarrow{\binom{a_0}{0}} & A_1 \oplus X_1 & \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ 0 & 1 \end{smallmatrix}\right)} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & A_3 & \xrightarrow{a_3} & \cdots \\
\| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & 0 \\ f & 1 \end{smallmatrix}\right)} & & \| & & \| & & \\
C & \xrightarrow{\binom{a_0}{k_0}} & A_1 \oplus X_1 & \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ -f & 1 \end{smallmatrix}\right)} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & A_3 & \xrightarrow{a_3} & \cdots
\end{array}$$

$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma C \\
 & & \| & & \| & & \| \\
\cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma C
\end{array} \tag{3.3}$$

shows that the second row of (3.3) is an $(n+2)$-angle. It is easy to know that $\binom{a_0}{k_0}$ is $\mathscr{X}$-monic since k_0 is $\mathscr{X}$-monic.

Note that we have the following commutative diagram:

$$\begin{array}{ccccccccccc}
C & \xrightarrow{\binom{a_0}{k_0}} & A_1 \oplus X_1 & \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ -f & 1 \end{smallmatrix}\right)} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & A_3 & \xrightarrow{a_3} & \cdots \\
\| & & \downarrow{\scriptstyle (0,\ 1)} & & \downarrow{\scriptstyle \alpha_2} & & \downarrow{\scriptstyle \alpha_3} & & \\
C & \xrightarrow{k_0} & X_1 & \xrightarrow{k_1} & X_2 & \xrightarrow{k_2} & X_3 & \xrightarrow{k_3} & \cdots
\end{array}$$

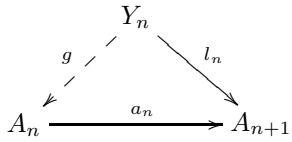
$$\begin{array}{ccccccccc}
\cdots & \xrightarrow{a_{n-1}} & A_n & \xrightarrow{a_n} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma C \\
 & & \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha_{n+1}} & & \| \\
\cdots & \xrightarrow{k_{n-1}} & X_n & \xrightarrow{k_n} & X_{n+1} & \xrightarrow{k_{n+1}} & \Sigma C
\end{array} \tag{3.4}$$

By Theorem 2.19, we obtain a right $(n+2)$-angle

$$C \xrightarrow{\overline{a_0}} A_1 \xrightarrow{\overline{a_1}} A_2 \xrightarrow{\overline{a_2}} \cdots \xrightarrow{\overline{a_{n-1}}} A_n \xrightarrow{\overline{a_n}} A_{n+1} \longrightarrow 0\ . \tag{3.5}$$

By Lemma 2.20, we know that $\overline{a_i}$ is a weak cokernel of $\overline{a_{i-1}}$, note that $\overline{a_n}$ is an epimorphism, thus $\overline{a_n}$ is a cokernel of $\overline{a_{n-1}}$. That is to say, (3.5) is a right n-exact sequence.

Since $\mathscr{X}$ is functorially finite, take $l_n\colon Y_n \to A_{n+1}$ a right $\mathscr{X}$-approximation of A_{n+1}; note that it is not a split epimorphism. Since a_n is right almost split, there exists a morphism $g\colon Y_n \to A_n$ such that $a_n g = l_n$; it is reflected in the following commutative diagram:



Since any $(n+2)$-angulated category is closed under direct sums, considering the direct sum of the second row of (3.3) and the trivial $(n+2)$-angle of Y_n

$$0 \to 0 \to \ldots \to 0 \to Y_n \xrightarrow{1} Y_n \to 0,$$

we obtain the following $(n+2)$-angle:

$$\begin{aligned} C \xrightarrow{\binom{a_0}{b_0}} A_1 \oplus X_1 \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ -f & 1 \end{smallmatrix}\right)} A_2 \oplus X_1 \xrightarrow{(a_2,\ 0)} A_3 \xrightarrow{a_3} \cdots \\ \cdots \xrightarrow{\binom{a_{n-2}}{0}} A_{n-1} \oplus Y_n \xrightarrow{\left(\begin{smallmatrix} a_{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)} A_n \oplus Y_n \xrightarrow{(a_n,\ 0)} A_{n+1} \xrightarrow{a_{n+1}} \Sigma C. \end{aligned}$$

The commutative diagram

$$\begin{array}{ccccccccc} C & \xrightarrow{\binom{a_0}{k_0}} & A_1 \oplus X_1 & \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ -f & 1 \end{smallmatrix}\right)} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & A_3 & \xrightarrow{a_3} & \cdots \\ \| & & \| & & \| & & \| & & \\ C & \xrightarrow{\binom{a_0}{k_0}} & A_1 \oplus X_1 & \xrightarrow{\left(\begin{smallmatrix} a_1 & 0 \\ -f & 1 \end{smallmatrix}\right)} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & A_3 & \xrightarrow{a_3} & \cdots \end{array} \tag{3.6}$$

$$\begin{array}{ccccccccc} \cdots & \xrightarrow{\binom{a_{n-2}}{0}} & A_{n-1} \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} a_{n-1} & 0 \\ 0 & 1 \end{smallmatrix}\right)} & A_n \oplus Y_n & \xrightarrow{(a_n,\ 0)} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma C \\ & & \| & & \downarrow{\scriptstyle \left(\begin{smallmatrix} 1 & -g \\ 0 & 1 \end{smallmatrix}\right)} & & \| & & \| \\ \cdots & \xrightarrow{\binom{a_{n-2}}{0}} & A_{n-1} \oplus Y_n & \xrightarrow{\left(\begin{smallmatrix} a_{n-1} & -g \\ 0 & 1 \end{smallmatrix}\right)} & A_n \oplus Y_n & \xrightarrow{(a_n,\ l_n)} & A_{n+1} & \xrightarrow{a_{n+1}} & \Sigma C \end{array}$$

shows that the second row of (3.6) is an $(n+2)$-angle. It is easy to see that (a_n, l_n) is $\mathscr{X}$-epic since l_n is $\mathscr{X}$-epic. Note that we have the following commutative diagram

$$
\begin{array}{ccccccccccc}
\Sigma^{-1}A_{n+1} & \xrightarrow{\Sigma^{-1}l_{n+1}} & Y_0 & \xrightarrow{l_0} & Y_1 & \xrightarrow{l_1} & Y_2 & \xrightarrow{l_2} & \cdots & & \\
\| & & \downarrow{\scriptstyle \beta_0} & & \downarrow{\scriptstyle \beta_1} & & \downarrow{\scriptstyle \beta_2} & & & & \\
\Sigma^{-1}A_{n+1} & \xrightarrow{\Sigma^{-1}a_{n+1}} & C & \xrightarrow{\binom{a_0}{k_0}} & A_1 \oplus X_1 & \xrightarrow{\binom{a_1\ 0}{-f\ 1}} & A_2 \oplus X_1 & \xrightarrow{(a_2,\ 0)} & \cdots & & \\
\end{array}
$$

$$
\begin{array}{ccccccccc}
\cdots & \xrightarrow{l_{n-2}} & Y_{n-1} & \xrightarrow{l_{n-1}} & Y_n & \xrightarrow{l_n} & A_{n+1} \\
 & & \downarrow{\scriptstyle \beta_{n-1}} & & \downarrow{\scriptstyle \binom{0}{1}} & & \| \\
\cdots & \xrightarrow{\binom{a_{n-2}}{0}} & A_{n-1} \oplus Y_n & \xrightarrow{\binom{a_{n-1}\ -g}{0\ \ 1}} & A_n \oplus Y_n & \xrightarrow{(a_n,\ l_n)} & A_{n+1}
\end{array}
\tag{3.7}
$$

By the duality of Theorem 2.19, we obtain a left $(n+2)$-angle

$$
0 \longrightarrow C \xrightarrow{\overline{a_0}} A_1 \xrightarrow{\overline{a_1}} A_2 \xrightarrow{\overline{a_2}} \cdots \xrightarrow{\overline{a_{n-1}}} A_n \xrightarrow{\overline{a_n}} A_{n+1}\ . \tag{3.8}
$$

From the duality of Lemma 2.20, we can see that (3.8) is a left n-exact sequence. Thus,

$$
0 \longrightarrow C \xrightarrow{\overline{a_0}} A_1 \xrightarrow{\overline{a_1}} A_2 \xrightarrow{\overline{a_2}} \cdots \xrightarrow{\overline{a_{n-1}}} A_n \xrightarrow{\overline{a_n}} A_{n+1} \longrightarrow 0 \tag{3.9}
$$

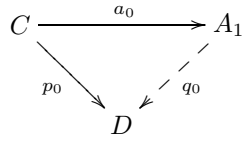
is an n-exact sequence.

Next, we prove that the above n-exact sequence (3.9) is Auslander-Reiten n-exact sequence. Since $a_0\colon\ C \to A_1$ is left almost split, a_0 is not a split monomorphism, that is, C is not a direct summand of A_1. Assume that $\overline{a_0}\colon\ C \to A_1$ is a split monomorphism, then there exists a morphism $\overline{u_0}\colon\ A_1 \to C$ such that $\overline{u_0 a_0} = 1_C$, thus, there exists an object $X \in \mathscr{X}$ such that $1_C - u_0 a_0 = ts$, as the commutative diagram below:

$$
\begin{array}{ccc}
C & \xrightarrow{1_{C'} - u_0 a_0} & C \\
 & {\scriptstyle s}\searrow \quad \nearrow{\scriptstyle t} & \\
 & X &
\end{array}
$$

Then $u_0 a_0 + ts = 1_C \Rightarrow (u_0, t)\binom{a_0}{s} = 1_C$, that is to say $\binom{a_0}{s}\colon\ C \to A_1 \oplus X$ is a split monomorphism. Thus, C is a direct summand of $A_1 \oplus X$. Since $a_0\colon\ C \to A_1$ is left almost split, then C is indecomposable in $\mathscr{C}$, in particular, C is not a direct summand of X. Thus, C is a direct summand of A_1, which is a contradiction. Therefore, $\overline{a_0}$ is not a split monomorphism. For every object $D \in \mathscr{C}/\mathscr{X}$, every

morphism $\overline{p_0}\colon C \to D$ which is not a split monomorphism, and $p_0\colon C \to D$ is not a split monomorphism, there exists a morphism $q_0\colon A_1 \to D$ such that $q_0a_0 = p_0$, as the commutative diagram below:



Then we have $\overline{q_0a_0} = \overline{p_0}$. Therefore, $\overline{a_0}$ is left almost split. Similarly, we can get that $\overline{a_n}$ is right almost split.

Since (3.2) is an Auslander-Reiten $(n+2)$-angle, we have $a_i\colon A_i \to A_{i+1} \in \mathrm{rad}_{\mathscr{C}}$ for $i = 1, 2, \ldots, n-1$. Thus, for every morphism $\overline{h_i}\colon A_{i+1} \to A_i$ we have $h_i\colon A_{i+1} \to A_i \in \mathscr{C}$ such that $1_{A_i} - h_ia_i$ is invertible. Then there exists a morphism $e_i\colon A_i \to A_i$ such that we have the following equalities:

$$e_i(1_{A_i} - h_ia_i) = 1_{A_i}, \quad (1_{A_i} - h_ia_i)e_i = 1_{A_i}.$$

Then we have

$$\overline{e_i}(\overline{1}_{A_i} - \overline{h_ia_i}) = \overline{1}_{A_i}, \quad (\overline{1}_{A_i} - \overline{h_ia_i})\overline{e_i} = \overline{1}_{A_i}.$$

Therefore, $\overline{a_i} \in \mathrm{rad}_{\mathscr{C}/\mathscr{X}}$ for $i = 1, 2, \ldots, n-1$. This shows that the n-exact sequence (3.9) is Auslander-Reiten n-exact sequence.

Dually, for any non-projective indecomposable object $C \in \mathscr{C}/\mathscr{X}$, we can get an Auslander-Reiten n-exact sequence

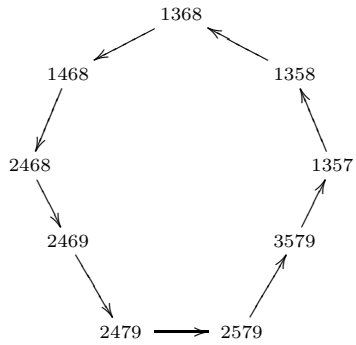
$$0 \longrightarrow B_0 \xrightarrow{\overline{b_0}} B_1 \xrightarrow{\overline{b_1}} B_2 \xrightarrow{\overline{b_2}} \cdots \xrightarrow{\overline{b_{n-1}}} B_n \xrightarrow{\overline{b_n}} C \longrightarrow 0.$$

Therefore, $\mathscr{C}/\mathscr{X}$ has Auslander-Reiten n-exact sequences. This completes the proof. □

Example 3.5. Let Λ be an n-representation finite algebra. In [19], Section 5, it was constructed a so-called $(n+2)$-angulated cluster category $\mathscr{O}_\Lambda$. In particular, we let $n = 3$ and $\mathscr{C} = \mathscr{O}_{A_2^3}$. This is the 5-angulated cluster category of type A_2, see [19], Definition 5.2, Sections 6, and 8. The indecomposable objects can be identified with the elements of the set

$$\mathrm{ind}\mathscr{C} = \{1357, 1358, 1368, 1468, 2468, 2469, 2479, 2579, 3579\}.$$

The Auslander-Reiten quiver of $\mathscr{C}$ is the following:



We denote by $\mathrm{add}(M)$ the full subcategory of $\mathscr{C}$ consisting of direct summands of direct sums of finitely many copies of M. It is straightforward to verify that the subcategory

$$\mathrm{add}(T) := \mathrm{add}(1357 \oplus 1358 \oplus 1368 \oplus 1468)$$

is a cluster tilting subcategory of $\mathscr{C}$.

Example 3.6. Let Λ be an n-representation finite algebra. In [19], Section 5, it was constructed a so-called $(n+2)$-angulated cluster category $\mathscr{O}_\Lambda$. Let $\mathrm{add}(T)$ be a cluster tilting subcategory of $\mathscr{O}_\Lambda$. According to [19], Theorem 5.2, we know that $\mathscr{O}_\Lambda$ has a Serre functor. Furthermore, by [22], Theorem 4.5, $\mathscr{O}_\Lambda$ has Auslander-Reiten $(n+2)$-angles if and only if $\mathscr{O}_\Lambda$ has a Serre functor. Therefore, $\mathscr{O}_\Lambda$ has Auslander-Reiten $(n+2)$-angles. By Theorem 3.4, we deduce that $\mathscr{O}_\Lambda/\mathrm{add}(T)$ has Auslander-Reiten n-exact sequences.

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