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SOME HÖLDER-LOGARITHMIC ESTIMATES ON HARDY-SOBOLEV SPACES

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Abstract. We prove some optimal estimates of Hölder-logarithmic type in the Hardy-Sobolev spaces $H^{k,p}(G)$, where $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and G is either the open unit disk $\mathbb{D}$ or the annular domain G_s, $0 < s < 1$ of the complex space $\mathbb{C}$. More precisely, we study the behavior on the interior of G of any function f belonging to the unit ball of the Hardy-Sobolev spaces $H^{k,p}(G)$ from its behavior on any open connected subset I of the boundary ∂G of G with respect to the L^1-norm. Our results can be viewed as an improvement and generalization of those established in S. Chaabane, I. Feki (2009), I. Feki, H. Nfata, F. Wielonsky (2012), I. Feki (2013), I. Feki, H. Nfata (2014). As an application, we establish a logarithmic stability results for the Cauchy problem of the identification of Robin's coefficient by boundary measurements.

Keywords: Hardy-Sobolev space; annular domain; Kernel function

MSC 2020: 30H10, 30C40, 30H05, 35R30

1. Notations, definitions and main result

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ with boundary the unit circle $\mathbb{T}$ and denote for $0 < s < 1$ by $G_s = \{z \in \mathbb{C}\colon\ s < |z| < 1\}$ the annulus of inner boundary $s\mathbb{T}$ and outer boundary $\mathbb{T}$.

In the sequel, we denote by G the open unit disk $\mathbb{D}$ or the annulus G_s, $0 < s < 1$ and by I, an open connected subset of the boundary of G of length $2\pi\lambda$, $0 < \lambda < 1$. We recall that the boundary ∂G_s is equipped with the usual Lebesgue measure μ normalized so that both the circles $\mathbb{T}$ and $s\mathbb{T}$ have unit measure.

For any measurable function $f\colon\ I \mapsto \mathbb{C}$ we denote

$$\|f\|_{L^1(I)} = \frac{1}{2\pi\lambda} \int_I |f(r\mathrm{e}^{\mathrm{i}\theta})| \,\mathrm{d}\theta$$

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for the L^1-norm of f on I, where $r = s$ if $I \subset s\mathbb{T}$ and $r = 1$ if $I \subset \mathbb{T}$. For $1 < p \leqslant \infty$ and for an analytic function f on G, the integral means

$$M_p(f,r) = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|^p \,\mathrm{d}\theta\right)^{1/p} \quad \text{and} \quad M_\infty(f,r) = \sup_{0\leqslant\theta\leqslant 2\pi} |f(r\mathrm{e}^{\mathrm{i}\theta})|$$

provide one measure of growth and lead a simple definition of Hardy spaces: The Hardy space $H^p(G)$ is defined as the set of all analytic functions f in G such that $M_p(f,r)$ remains bounded as $r \to 1$ if $G = \mathbb{D}$ and $M_p(f,r)$ remains bounded as $r \to 1$ and $r \to s$ in the annulus case. We have also the equivalent definition

$$H^p(G_s) = H^p(\mathbb{D}) \oplus H_0^p(\mathbb{C} \setminus s\overline{\mathbb{D}}),$$

where the Hardy space $H_0^p(\mathbb{C} \setminus s\overline{\mathbb{D}})$ is defined as the set of analytic functions in $\mathbb{C} \setminus s\overline{\mathbb{D}}$ with a zero limit at infinity.

For $p = \infty$, the Hardy space $H^\infty(G_s)$ can also be viewed as the space of bounded analytic functions on G_s and according to [6], Theorem 7.1, it can be identified to a closed subspace $H^\infty(\partial G_s)$ of $L^\infty(\partial G_s)$. We refer the reader also to [2] for more details.

For $1 < p < \infty$, the space $H^p(\partial G_s)$ is defined to be the closure in the complex Banach space $L^p(\partial G_s)$ of the set $R(\partial G_s)$ consisting of rationals whose poles lie in the complement of $\overline{G_s}$. We have a natural one-to-one isomorphic correspondence between the spaces $H^p(G_s)$ and $H^p(\partial G_s)$ and this induces a Banach space structure on $H^p(G_s)$. We refer the reader to [5], [7], [11], [16], [17], [18] for more details on the theory of Hardy spaces.

The Hardy-Sobolev space $H^{1,p}(G)$ is defined as the set of all functions f in $H^p(G)$ such that the complex derivative f' belongs to $H^p(G)$. We endow the Banach space $H^{1,p}(G)$ with the usual Sobolev norm

$$\|f\|^p_{H^{1,p}(G)} = \|f\|^p_{L^p(\partial G)} + \|f'\|^p_{L^p(\partial G)},$$

where

$$\|f\|^p_{L^p(\partial G)} = \|f\|^p_{L^p(\mathbb{T})} + \|f\|^p_{L^p(s\mathbb{T})} \quad \text{if } G = G_s$$

and

$$\|f\|_{L^p(\partial G)} = \|f\|_{L^p(\mathbb{T})} \quad \text{if } G = \mathbb{D}.$$

In the sequel, we denote by $\mathbf{B}_{1,p}(G)$ the closed unit ball of $H^{1,p}(G)$

$$\mathbf{B}_{1,p(G)} = \{f \in H^{1,p}(G)\colon\ \|f\|_{H^{1,p}(G)} \leqslant 1\}.$$

For every $\theta \in [-\pi, \pi]$, we introduce the radial primitive of a function $f \in H^\infty(G)$ as

$$F_\theta(r) = \int_{x_0}^{r} f(x\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}x \quad \forall\, r \in]x_0, 1[, \tag{1.1}$$

where $x_0 = 0$ if $G = \mathbb{D}$ and $x_0 = s$ if $G = G_s$.

Control problems in Hardy spaces were considered by Baratchart and Zerner. They study in [3] an interpolation scheme for analytic functions in $\mathbb{D}$ from boundary values on the unit circle $\mathbb{T}$. In particular, they proved an estimate in the Hardy-Sobolev space $H^{1,2}$ of the unit disk, which shows that an upper bound on the L^2-norm on $\mathbb{T}$ can be derived from the L^2-norm on a subarc of $\mathbb{T}$ and that the relation between the two norms is of $(\log - \log)/\log$-type.

In 2003, Alessandrini, Del Piero and Rondi proved in [1], with quite different method, a similar estimate of $1/\log^\alpha$-type, $0 < \alpha < 1$ and $p = 2$. They applied their result to an inverse Robin boundary value problem in corrosion detection by electrostatic boundary measurements. Simultaneously and with quite different method Chaabane and Feki proved in [4] a $(\log - \log)/\log$-type inequality for the same inverse problem.

For the annular case $G = G_s$ and for $p = 2$, the particular case $I = \mathbb{T}$ has been considered by Leblond, Mahjoub and Partington in [12]. In the same context, Meftahi and Wielonsky in [13] gave a similar estimate and made explicit the dependance of the constant C upon the magnitude of the radius s of the inner boundary. Their methods strongly depend on the Hilbertian structure of the Hardy-Sobolev space $H^{1,2}(G_s)$. These estimates are interestingly used to prove logarithmic stability results for an inverse Robin's problem. In [8], the first author of the paper proved the logarithmic-type estimate with the uniform norm $p = \infty$ for any open connected subset $I \subset \partial G_s$. The proof is based on the estimates established on the Poisson kernel of the annular domain together with the notion of the radial primitive of any analytic function.

For bounded analytic functions in the open unit disk $G = \mathbb{D}$, a similar logarithmic estimate has been established in [4]. The authors proved an $1/\log^k$ inequality in the Hardy-Sobolev spaces $H^{k,\infty}(\mathbb{D})$ for any integer k, and have shown that their estimate is of optimal type. Their results are interestingly used to establish logarithmic stability estimates for the Cauchy problem of identifying Robin coefficient with a Laplace operator, and to prove an error estimate for the inverse problem of the identification of a Robin coefficient. Similar estimates of optimal type for the Hilbertian spaces $H^{k,2}(\mathbb{D})$ have been proved in [10], where the authors derive also logarithmic stability results with respect to the L^2-norm for the same inverse problem of identifying Robin coefficients and for a recovering interpolation scheme with Hardy-Sobolev space $H^{1,2}(\mathbb{D})$ with interpolation points located on the boundary $\mathbb{T}$ of the unit disk.

2. Estimates in Hardy-Sobolev spaces of the disk

Let $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and $\mathbb{D}$ the unit disk of the complex plane $\mathbb{C}$. We establish in this section a new estimates of Hölder-logarithmic type in the Hardy-Sobolev spaces $H^{k,p}(\mathbb{D})$ of the unit disk. These results extend those established by Chaabane and Feki in the case $p = \infty$ (cf. [4]) and by Feki et al. when $p = 2$, cf. [10].

We start by recording a variant of the Hardy-Landau-Littlewood inequality which will crucially be used in the proof of Theorems 2.4 and 2.8, see [15], Theorem 1.

Lemma 2.1. *Let $1 \leqslant p \leqslant \infty$, $\mathcal{I}$ be a bounded interval and let $j \in \mathbb{N}$ such that $j \geqslant 2$. Then there exists a nonnegative constant C_p such that*

$$\|g'\|_{L^p(\mathcal{I})} \leqslant C_p \|g\|^{1/j}_{W^{j,p}(\mathcal{I})} \|g\|^{1-1/j}_{L^p(\mathcal{I})} \quad \forall\, g \in W^{j,p}(\mathcal{I}). \tag{2.1}$$

Referring to [4], Lemma 2.2 we get:

Lemma 2.2. *Let I be a subarc of $\mathbb{T}$ of length $2\pi\lambda$, $0 < \lambda < 1$ and let $f \in H^\infty(\mathbb{D})$. For every constant $m \geqslant \|f\|_{L^\infty(r\mathbb{T})}$ and $z \in \overline{\mathbb{D}}(0,r)$ we have*

$$|f(z)| \leqslant m \Big\| \frac{f}{m} \Big\|^{\lambda(1-|z|)/(1+|z|)}_{L^1(I)}. \tag{2.2}$$

Next, we establish the following control lemma:

Lemma 2.3. *Let $(\theta, r) \in [-\pi, \pi] \times [0,1]$, F_θ be the radial primitive of f defined by* (1.1). *For every constant $m \geqslant \|f\|_{L^\infty(r\mathbb{T})}$ we have*

$$|F_\theta(r)| \leqslant m \frac{\|f/m\|^{\lambda(1-r)/(1+r)}_{L^1(I)}}{|\frac{1}{2}\lambda \log \|f/m\|_{L^1(I)}|}. \tag{2.3}$$

P r o o f. Let $f \in H^{1,\infty}(\mathbb{D})$ and let θ be a real number such that $|\theta| \leqslant \pi$. We designate by F_θ the radial primitive of f defined by

$$F_\theta(r) = \int_0^r f(t\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}t \quad \forall\, r \in \mathcal{I} =]0,1[.$$

According to Lemma 2.2, we have

$$|F_\theta(r)| \leqslant m \int_0^r \Big\| \frac{f}{m} \Big\|^{\lambda(1-t)/(1+t)}_{L^1(I)} \mathrm{d}t \leqslant m \int_0^r \Big\| \frac{f}{m} \Big\|^{\lambda(1-t)/(1+r)}_{L^1(I)} \mathrm{d}t. \tag{2.4}$$

Integration by parts gives

$$|F_\theta(r)| \leqslant \frac{m\|f/m\|^{\lambda/(1+r)}_{L^1(I)}}{-(\lambda/(1+r)) \log \|f/m\|_{L^1(I)}} (\mathrm{e}^{-(\lambda r/(1+r)) \log \|f/m\|_{L^1(I)}} - 1).$$

Since the second term of the right-hand side is negative, we deduce the desired inequality:

$$|F_\theta(r)| \leqslant m\,\frac{\|f/m\|_{L^1(I)}^{\lambda(1-r)/(1+r)}}{|\frac{1}{2}\lambda\log\|f/m\|_{L^1(I)}|}.$$

□

We now state our main result of this chapter.

Theorem 2.4. *Let $1 \leqslant p \leqslant \infty$ and $f \in \mathbf{B}_{1,p}(\mathbb{D})$. Then for every $r \in]0,1[$ we have*

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\delta}{|\lambda\log\|f\|_{L^1(I)}|}\|\|f\|_{L^1(I)}^{\eta(r)(1-r)}, \tag{2.5}$$

where $\delta > 0$ and $\eta(r) = \frac{1}{2}\lambda(1-1/\mathrm{e})(1+r)$.

P r o o f. For every $|\theta| \leqslant \pi$, let F_θ be the radial primitive of f defined by equation (1.1). Let $r \in]0,1[$ and $m \geqslant \|f\|_{L^\infty(r\mathbb{T})}$. According to Lemma 2.3, we have

$$|F_\theta(r)| \leqslant 2m\frac{\|f/m\|_{L^1(I)}^{\lambda(1-r)/(1+r)}}{|\lambda\log\|f/m\|_{L^1(I)}|}. \tag{2.6}$$

Applying Lemma 2.1 to the radial primitive F_θ, there exists a constant $\beta > 0$ such that

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \beta\|F_\theta\|_{L^p(r\mathbb{T})}^{1/2}. \tag{2.7}$$

By using the estimates established in [4], Theorem 2.4 we have that there exists $C > 0$ such that

$$\|f\|_{L^p(\mathbb{T})} \leqslant \frac{C}{|\lambda\log\|f\|_{L^1(I)}|}.$$

Now by applying (2.6) and (2.7) with $m = C/|\lambda\log\|f\|_{L^1(I)}|$, we obtain

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\beta(2C)^{1/2}}{|\lambda\log\|f\|_{L^1(I)}|^{1/2}}\frac{\|f/m\|_{L^1(I)}^{\lambda(1-r)/2(1+r)}}{|\lambda\log\|f/m\|_{L^1(I)}|^{1/2}}. \tag{2.8}$$

In the sequel, let $\mu(x) = x|\log x|$ and $\alpha = 1-\mathrm{e}^{-1}$. Since $C \geqslant 1$, $\lambda \leqslant 1$ and $\mu(x) \leqslant x^\alpha$ in $]0,1]$, we get

$$\Big\|\frac{f}{m}\Big\|_{L^1(I)} \leqslant \frac{\lambda}{C}\mu(\|f\|_{L^1(I)}) \leqslant \|f\|_{L^1(I)}^\alpha. \tag{2.9}$$

From (2.8) and the monotonicity of the mapping $\varepsilon(x) = |\log x|^{-1}$, we conclude the proof of the theorem:

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \beta\frac{(2C/\alpha)^{1/2}\|f\|_{L^1(I)}^{\alpha\lambda(1-r)/2(1+r)}}{|\lambda\log\|f\|_{L^1(I)}|}.$$

□

Actually, Theorem 2.4 is of optimal type as shown by the following proposition.

Proposition 2.5. *For $I = \mathbb{T}$ and $0 < r < 1$, we consider the sequence of normalized functions in $\mathbf{B}_{1,p}(\mathbb{D})$*

$$f_n = \frac{u_n}{\|u_n\|_{H^{1,p}(\mathbb{D})}}, \quad u_n(z) = \frac{1}{z^n}, \quad n \in \mathbb{N}^*.$$

Then for every $r \in]0,1[$ we get

$$\lim_{n\to\infty} \frac{\|f_n\|_{L^p(r\mathbb{T})} |\log \|f_n\|_{L^1(\mathbb{T})}|}{\|f_n\|_{L^1(\mathbb{T})}^{\eta(r)(1-r)}} = \infty, \tag{2.10}$$

where $\eta(r) = \lambda(1 - \mathrm{e}^{-1})/(1 + r)$.

P r o o f. By making use of the asymptotic estimate (cf. [7]), we get

$$\|f_n\|_{L^p(r\mathbb{T})}^p = \frac{1}{r^{np}(1+n^p)} \quad \text{and} \quad \|f_n\|_{L^\infty(\mathbb{T})}^p = \frac{1}{1+n^p}. \tag{2.11}$$

Let

$$\Phi_n(r) = \frac{\|f_n\|_{L^p(r\mathbb{T})} |\log \|f_n\|_{L^1(\mathbb{T})}|}{\|f_n\|_{L^1(\mathbb{T})}^{\eta(r)(1-r)}}.$$

Then, from the monotonicity of the function $x \to x^{\eta(r)(1-r)}|\log x|$, we get

$$\Phi_n(r) \geqslant \mathrm{e}^{np(-\log r + (\eta(r)(1-r)-1)\log n/n)} \Big(1 + \frac{1}{n^p}\Big)^{\eta(r)(1-r)-1} (\log(1+n^p))^p.$$

Hence, $\lim\limits_{n\to\infty} \Phi_n(r) = \infty$ and this completes the proof. □

Remark 2.6. It is clear that the constant $\eta(r)$ of estimate (2.5) cannot exceed the value 1. Note also that the constant δ cannot be replaced by $\varepsilon(\|f\|_{L^1(I)})$ with ε, a function which tends to zero at zero.

In the case, where r tends to 1, we found the results given by Chaabane and Feki in [4] and those given by Nfata, Wielonsky and Feki in [10].

The following corollary is a direct consequence of Theorem 2.4.

Corollary 2.7. *Let $k \in \mathbb{N}^*$ and $1 \leqslant p \leqslant \infty$. Then for any $f \in H^{1,p}(\mathbb{D})$ satisfying $\|f\|_{L^1(I)} < 1$ and $\|f\|_{H^{1,p}(\mathbb{D})} \leqslant K$, we get*

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\delta \max(1,K)}{|\lambda \log \|f\|_{L^1(I)}|} \|f\|_{L^1(I)}^{\eta(r)(1-r)},$$

where $\delta > 0$ and $\eta(r) = \frac{1}{2}(1 - \mathrm{e}^{-1})\lambda/(1 + r)$.

P r o o f. For $K > 1$, we consider the function $g = fK^{-1}$, which satisfies $\|g\|_{H^{1,p}(\mathbb{D})} \leqslant 1$. According to Theorem 2.4, we have

$$\|g\|_{L^p(r\mathbb{T})} \leqslant \frac{\delta}{|\lambda \log \|g\|_{L^1(I)}|} \|g\|_{L^1(I)}^{\eta(r)(1-r)},$$

where $\delta > 0$ and $\eta(r) = \frac{1}{2}(1 - \mathrm{e}^{-1})\lambda/(1 + r)$. This gives us the inequality

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\delta\ K}{|\lambda \log \|f\|_{L^1(I)}|} \|f\|_{L^1(I)}^{\eta(r)(1-r)}.$$

If $K \leqslant 1$, then we have

$$\|f\|_{H^{1,p}(\mathbb{D})} \leqslant K \leqslant 1,$$

and Theorem 2.4 guaranties the result. □

If we suppose that f is sufficiently regular, we can improve inequality (2.5) in the same way as in the proof of Theorem 2.4.

Theorem 2.8. *Let $k \in \mathbb{N}^*$ and $1 \leqslant p \leqslant \infty$. Then for every $f \in \mathbf{B}_{k,p}(\mathbb{D})$ we have*

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{C_k}{|\lambda \log \|f\|_{L^1(I)}|^k} \|f\|_{L^1(I)}^{\eta_k(r)(1-r)},$$

where $C_k > 0$ and $\eta_k(r) = k\lambda(1 - \mathrm{e}^{-1})/((k + 1)(1 + r))$.

P r o o f. For every $r \in]0, 1]$ we have according to the previous Theorem 2.4 that the radial primitive F_θ of f satisfies inequality (2.5). Since $f \in \mathbf{B}_{k,p}(\mathbb{D})$, using inequality (2.1) applied to $j = k + 1$ we prove that there exist a nonnegative constant $\beta > 0$ such that

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \beta \Big(\frac{2m\|f/m\|_{L^1(I)}^{\lambda(1-r)/(1+r)}}{|\lambda \log \|f/m\|_{L^1(I)}|} \Big)^{k/(k+1)}.$$

By using the estimates established by [4], Theorem 2.6 there exists $C > 0$ such that

$$\|f\|_{L^p(r\mathbb{T})} \leqslant m = \frac{C}{|\lambda \log \|f\|_{L^1(I)}|^k},$$

then we obtain

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\beta(2m)^{k/(k+1)}}{|\lambda \log \|f\|_{L^1(I)}|^{k^2/(k+1)}} \frac{\|f/m\|_{L^1(I)}^{k\lambda(1-r)/((k+1)(1+r))}}{|\lambda \log \|f\|_{L^1(I)}|^{k/(k+1)}}.$$

Similarly to the proof of Theorem 2.4, we consider $\varrho = k/(k + 1)$, $g_\varrho(x) = x|\log x|^\varrho$ and $\alpha = 1 - \varrho/\mathrm{e}$. Since $g(x) \leqslant x^\alpha$ in $]0, 1]$, we obtain the inequality

$$\|f\|_{L^p(r\mathbb{T})} \leqslant \frac{\beta(2C/\alpha)^{k/(k+1)}}{|\lambda \log \|f\|_{L^1(I)}|^k} \|f\|_{L^1(I)}^{\alpha k\lambda(1-r)/((k+1)(1+r))}.$$

□

Corollary 2.9. *Let $1 \leqslant p \leqslant \infty$, $K > 0$ and k, m be some integers with $0 \leqslant m < k$. Let $f \in H^{k,p}(\mathbb{D})$ such that $\|f\|_{H^{m,p}(I)} \leqslant K$. Then there exist nonnegative constants γ, ε and $\eta(r)$ depending only on k, m and K such that*

$$\|f\|_{H^{m,p}(r\mathbb{T})} \leqslant \frac{\gamma}{|\log \|f\|_{L^1(I)}|^{k-m}} \|f\|_{L^1(I)}^{(1-m/k)\eta_{k-m}(r)(1-r)}$$

provided that $\|f\|_{L^1(I)} < \varepsilon$.

P r o o f. Let $K_1 = \max(K, 1)$ and let $g = f/K_1$. Then the derivative $g^{(j)}$ of order $j \in \{0, \dots, m\}$ belongs to $\mathbf{B}_{k-j,p}(\mathbb{D})$ and satisfies the assumptions of Theorem 2.8. Hence, there exists a nonnegative constant C_1 depending only on K, k, j and λ such that

$$\|g^{(j)}\|_{L^p(r\mathbb{T})} \leqslant \frac{C_1}{|\log \|g^{(j)}\|_{L^1(I)}|^{k-j}} \|g^{(j)}\|_{L^1(I)}^{\eta_{k-j}(r)(1-r)}. \tag{2.12}$$

According to [15], Theorem 1 and the assumption that $g \in \mathbf{B}_{k,p}(\mathbb{D})$, there exist a nonnegative constant C_2 such that

$$\|g^{(j)}\|_{L^1(I)} \leqslant C_2 \|g\|_{L^1(I)}^{1-j/k},$$

implying that

$$\|g^{(j)}\|_{L^1(I)}^{\eta_{k-j}(r)(1-r)} \leqslant C_2 \|g\|_{L^1(I)}^{(1-j/k)\eta_{k-j}(r)(1-r)}. \tag{2.13}$$

We derive from (2.12), (2.13) and the monotony of the map $\eta_j(x) = (-\log x)^{(j-k)}$ that

$$\|g^{(j)}\|_{L^p(r\mathbb{T})} \leqslant C_1 C_2 \|g\|_{L^1(I)}^{(1-j/k)\eta_{k-j}(r)(1-r)} \eta_j(C_2 \|g\|_{L^1(I)}^{1-j/k}). \tag{2.14}$$

Let us choose $\varepsilon > 0$ small enough such that

$$\eta_j(C_2 \|g\|_{L^1(I)}^{1-j/k}) \leqslant 2\eta_j(\|g\|_{L^1(I)}). \tag{2.15}$$

Then from (2.14) and (2.15) we obtain

$$\|g^{(j)}\|_{L^p(r\mathbb{T})} \leqslant \frac{2C_1 C_2}{|\log \|g\|_{L^1(I)}|^{k-j}} \|g\|_{L^1(I)}^{(1-j/k)\eta_{k-j}(r)(1-r)}.$$

Taking the maximum over all $j = 0, \dots, m$ we achieved the proof of the corollary. □

3. Estimates in Hardy-Sobolev spaces of the annulus

We establish in this section some estimates of Hölder-logarithmic type in the Hardy-Sobolev spaces $H^{k,p}(G_s)$, $k \in \mathbb{N}^*$ and $1 \leqslant p \leqslant \infty$, that can be viewed as a generalization of the results establised in [8].

We first recall a pointwise estimate based on a lower bound for the Poisson kernel of the annulus G_s.

Lemma 3.1. *Let I be a subarc of ∂G_s of length $2\pi\lambda$, $0 < \lambda < 1$ and let $f \in H^\infty(G_s)$. For every constant $m \geqslant \|f\|_{L^\infty(r\partial G_s)}$ there exists a nonnegative constant C_s depending only on s such that for every $z \in \overline{G_s}$ we have*

$$|f(z)| \leqslant m\Big\|\frac{f}{m}\Big\|_{L^1(I)}^{(2\lambda C_s/\log s)(\log s - \log|z|)} \quad \text{if } s < |z| \leqslant \sqrt{s},$$
$$|f(z)| \leqslant m\Big\|\frac{f}{m}\Big\|_{L^1(I)}^{(2\lambda C_s/\log s)\log|z|} \quad \text{if } \sqrt{s} \leqslant |z| < 1.$$

Lemma 3.2. *Let $f \in H^\infty(G_s)$, $r \in]s,1[$ and $m \geqslant \|f\|_{L^\infty(r\partial G_s)}$. We suppose that f is not identically zero and that $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$. Then for all $\theta \in [-\pi,\pi[$ we get*

$$|F_\theta(r)| \leqslant m\frac{\|f/m\|_{L^1(I)}^{-2\lambda C_s q_0^{-1}\log r}}{|\lambda_0 \log\|f/m\|_{L^1(I)}|}, \tag{3.1}$$

where $q_0 = -\log s$, $C_s > 0$ and $\lambda_0 = \max(1, \lambda C_s/\widetilde{C}q_0 t)$.

Proof. Let $r \in]s,1[$ and $m \geqslant \|f\|_{L^\infty(r\partial G_s)}$. For every $\theta \in [-\pi,\pi[$ we designate by F_θ the radial primitive of f defined by equation (1.1). Then we have

$$|F_\theta(r)| \leqslant \int_s^{\sqrt{s}} |f(t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t + \int_{\sqrt{s}}^r |f(t\mathrm{e}^{\mathrm{i}\theta})|\,\mathrm{d}t$$

and according to Lemma 3.1, we get

$$\begin{aligned}|F_\theta(r)| &\leqslant m\int_s^{\sqrt{s}} \Big\|\frac{f}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log s^{-1}(\log s - \log t)}\,\mathrm{d}t + m\int_{\sqrt{s}}^r \Big\|\frac{f}{m}\Big\|_{L^1(I)}^{2\lambda C_s \log s^{-1}\log t}\,\mathrm{d}t\\ &\leqslant m\Big\|\frac{f}{m}\Big\|_{L^1(I)}^{-2\lambda C_s q_0^{-1}\log r}\Big(\frac{s^{1+(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}}}{|1+(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}|}\\ &\quad + \frac{1}{|1-(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}|}\Big).\end{aligned}$$

From the assumption that $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$ we have

$$\frac{1}{|1+(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}|} \leqslant \frac{2}{|(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}|},$$

and therefore, we conclude the desired inequality

$$|F_\theta(r)| \leqslant \frac{mq_0\widetilde{C}\|f/m\|_{L^1(I)}^{-2\lambda C_s q_0^{-1}\log r}}{|\lambda C_s \log\|f/m\|_{L^1(I)}|},$$

where $\widetilde{C} = (2s^{1+(2\lambda C_s/q_0)\log\|f/m\|_{L^1(I)}}+1)$. □

We are now in a position to establish the main control theorem in the Hardy-Sobolev space $H^{1,p}(G_s)$, $1 \leqslant p \leqslant \infty$.

Theorem 3.3. *Let $1 \leqslant p \leqslant \infty$ and $f \in \mathbf{B}_{1,p}(G_s)$. We suppose that f is not identically zero and that $\|f\|_{L^1(I)} \leqslant \mathrm{e}^{-q_0/\lambda C_s}$. Then there exist a constant $\delta > 0$ such that for every $r \in]s,1[$ we have*

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{\delta}{|\lambda_0 \log\|f\|_{L^1(I)}|}\|f\|_{L^1(I)}^{\eta\log r}, \tag{3.2}$$

where $q_0 = -\log s$, $C_s > 0$, $\lambda_0 = \min(1, \lambda C_s/\widetilde{C}q_0)$ and $\eta = -(1-\mathrm{e}^{-1})\lambda C_s q_0^{-1}$.

P r o o f. For every $r \in]s,1[$ and $\theta \in [-\pi,\pi[$, let F_θ be the radial primitive of f defined by equation (1.1) and let $m \geqslant \|f\|_{L^\infty(r\partial G_s)}$.

According to Lemma 2.3, there exists a constant $\lambda_0 > 0$ such that

$$|F_\theta(r)| \leqslant \frac{m\|f/m\|_{L^1(I)}^{-2\lambda C_s q_0^{-1}\log r}}{|\lambda_0 \log\|f/m\|_{L^1(I)}|}. \tag{3.3}$$

Applying Lemma 2.1 to the radial primitive F_θ, there exists a constant $\beta > 0$ such that

$$\|f\|_{L^p(r\partial G_s)} \leqslant \beta\|F_\theta\|_{L^p(r\partial G_s)}^{1/2},$$

consequently,

$$\|f\|_{L^p(r\partial G_s)} \leqslant \beta\left(\frac{m\|f/m\|_{L^1(I)}^{-2\lambda C_s q_0^{-1}\log r}}{|\lambda_0 \log\|f/m\|_{L^1(I)}|}\right)^{1/2}. \tag{3.4}$$

By using the estimate established by [8], Theorem 3.5 there exists $C > 0$ such that

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{C}{|\lambda_0 \log\|f\|_{L^1(I)}|}.$$

Then, if we replace in (3.3) and (3.4) the constant m by the new estimate $C/|\log\|f\|_{L^1(I)}|$, we obtain

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{\beta C^{1/2}}{|\lambda_0 \log\|f\|_{L^1(I)}|^{1/2}} \frac{\|f/m\|_{L^1(I)}^{-\lambda C_s q_0^{-1}\log r}}{|\lambda_0 \log\|f/m\|_{L^1(I)}|^{1/2}}. \tag{3.5}$$

Let $\mu(x) = x|\log x|$ and $\alpha = 1 - \mathrm{e}^{-1}$. Since $C \geqslant 1$, $m \geqslant 1$ and $\mu(x) \leqslant x^\alpha$ in $]0,1]$, we get

$$\Big\|\frac{f}{m}\Big\|_{L^1(I)} = \frac{\lambda_0}{C}\mu(\|f\|_{L^1(I)}) \leqslant \|f\|_{L^1(I)}^\alpha. \tag{3.6}$$

From (3.6) and the monotonicity of the mapping $\varepsilon(x) = |\log x|^{-1}$, we obtain

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{\beta(C/\alpha)^{1/2}}{|\lambda_0 \log\|f\|_{L^1(I)}|}\|f\|_{L^1(I)}^{-\alpha\lambda C_s q_0^{-1}\log r}.$$

□

Actually, Theorem 3.3 is of optimal type as shown by the following proposition.

Proposition 3.4. *Let $I = \mathbb{T}$ and $0 < r < 1$. Then there exists a sequence of functions $f_n \in \mathbf{B}_{1,p}(G_s)$ such that*

$$\lim_{n\to\infty} \frac{\|f_n\|_{L^p(r\partial G_s)}|\log\|f_n\|_{L^1(\mathbb{T})}|}{\|f_n\|_{L^1(\mathbb{T})}^{\eta\log r}} = \infty, \tag{3.7}$$

where $\eta = -(1-\mathrm{e}^{-1})\lambda C_s q_0^{-1}$.

P r o o f. To prove (3.2), we consider the sequence of functions $u_n(z) = z^{-n}$, $n \in \mathbb{N}^*$. Let $I = \mathbb{T}$ and let $f_n = u_n/\|u_n\|_{H^{1,p}(G_s)}$ be the $H^{1,p}(G_s)$ normalized function of u_n. Then by making use of the asymptotic estimate [14], we get

$$\|u_n\|_{L^p(\mathbb{T})}^p = 1, \quad \|u_n\|_{L^p(r\mathbb{T})}^p = \frac{1}{r^{np}} \quad \text{and} \quad \|u_n\|_{L^p(s\mathbb{T})}^p = \frac{1}{s^{np}}.$$

Furthermore, we have

$$\|u_n\|_{H^{1,p}(G_s)}^p = n^p\Big(1 + \frac{1}{s^{p(n+1)}}\Big)(1+o(1)).$$

Hence,

$$\|f_n\|_{L^p(r\partial G_s)}^p = \frac{s^p((s/r)^{np}+1)}{n^p\big(s^{p(n+1)}+1\big)}(1+o(1)), \quad \|f_n\|_{L^\infty(\mathbb{T})}^p = \frac{s^{p(n+1)}}{n^p\big(s^{p(n+1)}+1\big)}(1+o(1)).$$

Let

$$A_n = \frac{\|f_n\|_{L^p(r\partial G_s)}^p|\log\|f_n\|_{L^\infty(\mathbb{T})}^p|^p}{\|f_n\|_{L^\infty(\mathbb{T})}^{p\eta\log r}},$$

where $\eta = -\frac{1}{2}(1-\mathrm{e}^{-1})\lambda C_s q_0^{-1}$. Then, from the monotonicity of $x \mapsto |\log x|/(x^{\eta \log r})$ we get

$$A_n \geqslant \Big(\frac{n(s^{p(n+1)}+1)}{s^{p(n+1)}}\Big)^{\eta \log r} \frac{s^p((s/r)^{pn}+1)}{(1+s^{p(n+1)})} \times \Big(\frac{1}{n}\Big(\log n + \frac{1}{p}\log(1+s^{n+1}) - (n+1)\log s\Big)\Big)^p (1+o(1)).$$

Finally, by letting n to ∞, we conclude the proof of (3.2). □

The following corollary is a direct consequence of Theorem 3.3.

Corollary 3.5. *Let $K \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and $f \in H^{1,p}(G_s)$ such that $\|f\|_{H^{1,p}(G_s)} \leqslant K$ and $\|f\|_{L^1(I)} \leqslant \mathrm{e}^{-q_0/\lambda C_s}$. Then we have*

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{\delta \max(1,K)}{|\lambda_0 \log \|f\|_{L^1(I)}|}\|f\|_{L^1(I)}^{\eta \log r},$$

where $q_0 = -\log s$, $C_s > 0$, $\lambda_0 = \max(1, \lambda C_s/\widetilde{C}q_0)$ and $\eta = -(1-1/\mathrm{e})\lambda C_s q_0^{-1}$.

If we suppose that f is more regular, then we can improve inequality (3.2) in the same way as in the proof of Theorem 3.3.

Theorem 3.6. *Let $k \in \mathbb{N}^*$, $1 \leqslant p \leqslant \infty$ and $r \in]s,1[$. Then there exists a constant $C_k > 0$ depending only on k, s and λ such that for every $f \in \mathbf{B}_{k,p}(G_s)$ satisfying $\|f\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$ also satisfies*

$$\|f\|_{L^p(r\partial G_s)} \leqslant \frac{C_k}{|\log \|f\|_{L^1(I)}|^k}\|f\|_{L^1(I)}^{\eta_k \log r},$$

where $q_0 = -\log s$, $\eta_k = -(1-k/((k+1)\mathrm{e}))k\lambda C_s q_0^{-1}/(k+1)$.

Corollary 3.7. *Let $K > 0$, $1 \leqslant p \leqslant \infty$ and m, k be integers with $0 \leqslant m < k$. Let $f \in H^{k,p}(G_s)$ be such that $\|f\|_{H^{k,p}(G_s)} \leqslant K$ and $\|f\|_{H^{m,p}(I)} < \mathrm{e}^{-q_o/\lambda C_s}$. Then there exists positive constants γ, ε and η depending only on K, k, m, s and λ such that*

$$\|f\|_{H^{m,p}(r\partial G_s)} \leqslant \frac{\gamma}{|\log \|f\|_{L^1(I)}|^{k-m}}\|f_1 - f_2\|_{L^1(I)}^{\eta_{k-m} \log r},$$

provided that $\|f\|_{L^1(I)} < \varepsilon$.

4. Applications

In this section, we consider two applications of the previous estimate. Let us consider first the Cauchy problem with Laplace operator in the following case of the unit disk:

$$(\mathrm{CP})\begin{cases} \triangle u = 0 & \text{in } \mathbb{D},\\ \dfrac{\partial u}{\partial n} = \Phi & \text{on } I,\\ u = f & \text{on } I, \end{cases}$$

where ∂n stands for the partial derivative with respect to the outer normal derivative of u, Φ denotes the imposed current flux and f the potential measurement. Then we establish the following theorem.

Theorem 4.1. *Let $\Phi \in \mathcal{C}^0(\overline{I})$, $c > 0$ and $1 \leqslant p \leqslant \infty$, k and m some integers with $0 \leqslant m < k$. We denote by $\mathcal{H}_c^k$ the set*

$$\mathcal{H}_c^k = \{v \in \mathcal{C}^k(\overline{\mathbb{D}})\colon\ \|v\|_{W^{k,p}(\overline{\mathbb{D}})} \leqslant c\}.$$

Let $u_i \in \mathcal{H}_c^k$ the solution of (CP) *when $f = f_i$, $i = 1, 2$. Then there exists a constant $\delta > 0$ depending only on Φ, I, m and c such that if $\|f_1 - f_2\|_{L^1(I)} < 1$, we get*

$$\|u_1 - u_2\|_{L^p(r\mathbb{T})} \leqslant \frac{\delta}{|\log \|f_1 - f_2\|_{L^1(I)}|^{k-m}} \|f_1 - f_2\|_{L^1(I)}^{\eta_{k-m}(r)(1-r)}, \tag{4.1}$$

where

$$\eta_{k-m}(r) = \frac{\lambda(k-m)(1-1/\mathrm{e})}{(k-m+1)(1+r)}.$$

P r o o f. The proof follows the one of Theorem 3.1 in [4], except that we use our Corollary 2.7 instead of Corollary 2.7 in [4]. □

Next we consider the following stability result:

Theorem 4.2. *Let $\Phi \in \mathcal{C}^0(\overline{I})$, $K > 0$ and $1 \leqslant p \leqslant \infty$, k and m some integers with $0 \leqslant m < k$. We denote by W_K the set*

$$W_K = \{u \in W^{k,p}(G_s)\colon\ \|u\|_{W^{k,p}(G_s)} \leqslant K\}.$$

Let $u_i \in W_K$ be the solution of (CP) *when $f = f_i$, $i = 1, 2$. Then there exist two positive constants δ, η depending only on k and m such that if $\|f_1 - f_2\|_{L^1(I)} < \mathrm{e}^{-q_0/\lambda C_s}$, we get*

$$\|u_1 - u_2\|_{W^{k,p}(r\partial G_s)} \leqslant \frac{\delta}{|\log \|f_1 - f_2\|_{L^1(I)}|^{k-m}} \|f_1 - f_2\|_{L^1(I)}^{\eta_{k-m}\log r}.$$

P r o o f. The proof of this theorem follows immediately from Corollary 3.7. □

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