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ON LINEAR MAPS LEAVING INVARIANT THE COPOSITIVE/COMPLETELY POSITIVE CONES

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Abstract. The objective of this manuscript is to investigate the structure of linear maps on the space of real symmetric matrices $\mathcal{S}^n$ that leave invariant the closed convex cones of copositive and completely positive matrices (COP_n and CP_n). A description of an invertible linear map on $\mathcal{S}^n$ such that $L(\mathrm{CP}_n) \subset CP_n$ is obtained in terms of semipositive maps over the positive semidefinite cone $\mathcal{S}^n_+$ and the cone of symmetric nonnegative matrices $\mathcal{N}^n_+$ for $n \leqslant 4$, with specific calculations for $n = 2$. Preserver properties of the Lyapunov map $X \mapsto AX + XA^t$, the generalized Lyapunov map $X \mapsto AXB + B^tXA^t$, and the structure of the dual of the cone $\pi(\mathrm{CP}_n)$ (for $n \leqslant 4$) are brought out. We also highlight a different way to determine the structure of an invertible linear map on $\mathcal{S}^2$ that leaves invariant the closed convex cone $\mathcal{S}^2_+$.

Keywords: completely positive/copositive matrix; proper cone; semipositive matrix; positive semidefinite matrix; linear preserver problem

MSC 2020: 15A86, 15B48

1. Introduction

We work throughout over the field $\mathbb{R}$ of real numbers. Let $M_{m,n}(\mathbb{R})$ denote the set of all $m \times n$ matrices over $\mathbb{R}$. When $m = n$, this set is denoted by $M_m(\mathbb{R})$ or $M_n(\mathbb{R})$. The subspace of real symmetric matrices is denoted by $\mathcal{S}^n$. A subset K of finite dimensional real Hilbert space H is called a *convex cone* if $K + K \subseteq K$ and $\alpha K \subseteq K$ for every $\alpha \geqslant 0$. The *conic hull* of a subset S of H is defined to be

$$\mathrm{cone}\,(S) = \left\{ \sum_{i=1}^{m} \alpha_i x_i \colon\ x_i \in S,\ \alpha_i \geqslant 0,\ m \in \mathbb{N} \right\}.$$

It is obvious that $\mathrm{cone}\,(S)$ is a convex cone in H. If K is a subset of H, the dual cone of K is defined as $K^* := \{y \in H \colon\ \langle y, x\rangle \geqslant 0 \text{ for all } x \in K\}$. A convex cone K

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is said to be proper if it is topologically closed, pointed ($K \cap -K = \{0\}$), and has nonempty interior, denoted by K°. The dual K^* is a closed convex cone for any subset K of H. If K is a convex cone, then $(K^*)^* = \text{closure}\,(K)$. A subset K is a closed convex cone in H if and only if $(K^*)^* = K$. A closed convex cone K in H is said to be self-dual if $K = K^*$. If K_1, K_2 are closed convex cones in H, then $(K_1 + K_2)^* = K_1^* \cap K_2^*$ and $\text{closure}\,(K_1^* + K_2^*) = K_1^* \cap K_2^*$. These definitions may be found in [14]. Further $\mathbb{R}^n_+$ and $\mathcal{L}^n_+$ denote the nonnegative orthant and the Lorentz or the ice cream cone in $\mathbb{R}^n$, respectively. The set $\mathcal{S}^n_+$ denotes the closed convex cone of all symmetric positive semidefinite matrices in the space $\mathcal{S}^n$ of real symmetric matrices. A cone K is said to be polyhedral if $K = X(\mathbb{R}^m_+)$ for some $X \in M_{m,n}(\mathbb{R})$ and simplicial if X is an invertible matrix. Our focus in this manuscript is on the copositive and complete positive cones, as defined below.

Definition 1.1. An $n \times n$ real symmetric matrix A is called *copositive* if its quadratic form is nonnegative on the nonnegative orthant of $\mathbb{R}^n$. A matrix A is said to be completely positive if $A = BB^t$ for some nonnegative (not necessarily square) matrix B.

The set of copositive matrices forms a closed convex cone COP_n in $\mathcal{S}^n$ with nonempty interior and its dual CP_n consists of the closed convex cone of completely positive matrices. Moreover, $\text{CP}_n \subseteq \mathcal{S}^n_+ \subseteq \text{COP}_n$. If $\mathcal{N}^n_+$ denotes the cone of nonnegative matrices in $\mathcal{S}^n$, then $\mathcal{S}^n_+ + \mathcal{N}^n_+ \subseteq \text{COP}_n$, with equality when $n \leqslant 4$. Note that in such a case, $\text{CP}_n = \mathcal{S}^n_+ \cap \mathcal{N}^n_+$. It can also be easily proved that $\mathcal{N}^n_+$ is a self-dual cone in $\mathcal{S}^n$. In fact, $\mathcal{N}^n_+$ is isomorphic to $\mathbb{R}^{n(n+1)/2}_+$. For a comprehensive treatment of these two cones, we refer to the recent monograph by Berman and Shaked-Monderer, see [14].

Definition 1.2. For proper cones K_1 and K_2 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, we have the following notions. $A \in M_{m,n}(\mathbb{R})$ is:
(1) (K_1, K_2)-nonnegative if $A(K_1) \subseteq K_2$.
(2) (K_1, K_2)-positive if $A(K_1 \setminus \{0\}) \subseteq K_2^\circ$.
(3) (K_1, K_2)-semipositive if there exists $x \in K_1^\circ$ such that $Ax \in K_2^\circ$.

The above definition holds with obvious modifications when K_1 and K_2 are proper cones in finite dimensional real Hilbert spaces H_1 and H_2 and $A\colon\ H_1 \to H_2$ is a linear map. It is clear that if A is (K_1, K_2)-positive, then A is (K_1, K_2)-semipositive. We denote the set of all matrices that are (K_1, K_2)-nonnegative by $\pi(K_1, K_2)$. When $K_1 = K_2 = K$, this is denoted by $\pi(K)$. The automorphism group of a proper cone K (the set of all invertible linear maps on H such that $L(K) = K$) is denoted by $\text{Aut}\,(K)$. It is a well-known fact that when K is a proper cone, $\pi(K)^\circ$ is precisely the set of all K-positive elements. A well known result due to Tam (see [18])

says that for proper cones K_1 and K_2 in a finite dimensional Hilbert space H, $\pi(K_1, K_2)^* = \text{cone}\,(K_2^* \otimes K_1)$, where $K_2^* \otimes K_1$ is the tensor product of K_2^* and K_1. The tensor product of convex cones has been studied by several mathematicians in different contexts, see for instance the recent articles [12], [15]. The particular case when $K_1 = K_2 = \text{CP}_n$ will be discussed in Section 3.4. Let us also denote the set of all matrices that are (K_1, K_2)-semipositive by $\text{Sem}\,(K_1, K_2)$. When $K_1 = K_2 = K$, this is denoted by $\text{Sem}(K)$. Semipositive matrices occur very naturally in certain optimization problems, for example in linear complementarity problems. For a complete description of this class, one may refer to [2]. The following result is easily proved and we skip the proof.

Lemma 1.3. *Let K_1 and K_2 be proper cones in finite dimensional real Hilbert spaces H_1 and H_2 that are isomorphic through a map T, that is, $T\colon\ H_1 \to H_2$ is an invertible linear map such that $T(K_1) = K_2$. Then:*
▷ $\pi(K_1) = T^{-1}\pi(K_2)T$.
▷ $\text{Sem}\,(K_1) = T^{-1}\text{Sem}\,(K_2)T$.

2. Description of the problem

Describing/determining $\pi(K)$ is a hard problem. When K happens to be the nonnegative orthant $\mathbb{R}^n_+$, this set is just the collection of all entrywise nonnegative matrices. The automorphism group of the cone $\mathbb{R}^n_+$ is the collection of all nonnegative monomial matrices (a matrix that has exactly one entry in each row and column). A complete description of both $\pi(K)$ as well as $\text{Aut}\,(K)$ when K is the Lorentz cone was given by Loewy and Schneider, see [11]. In a recent work, Sznajder provides a nice representation of the automorphism of the Lorentz cone, see [17] for details. When K happens to be a cone of symmetric positive semidefinite matrices $\mathcal{S}^n_+$, a complete description of $\pi(K)$ is still not known for all values of n. A classical result of Choi says that if L is a linear map on $\mathcal{S}^2$ that leaves invariant $\mathcal{S}^2_+$, then there exist 2×2 matrices A_i such that $L(X) = \sum\limits_i A_i^t X A_i$ for all symmetric matrices X. There are also two other ways to describe $\pi(K)$ in this case:

(1) when $n = 2$, one can identify the cone $\mathcal{S}^2_+$ with the three dimensional Lorentz cone $\mathcal{L}^3_+$ and use Loewy-Schneider's result to describe this set, and
(2) an alternate way is to use the characterization of nonnegativity, see Theorem 2.1 in the next section.

However, automorphisms of the positive semidefinite cone are well known. A well known result due to Schneider says that any automorphism of the positive semidefinite cone $\mathcal{S}^n_+$ is of the form $X \mapsto AXA^t$ for some invertible matrix A.

This work began in trying to find possible applications of an interesting characterization of nonnegativity obtained recently. We state this below.

Theorem 2.1 ([3], Theorems 2.2 and 2.3)**.** *Let $A \in M_{m,n}(\mathbb{R})$ and let K_1, K_2 be proper cones in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. Then $A + B \in \mathrm{Sem}\,(K_1, K_2)$ for every $B \in \mathrm{Sem}\,(K_1, K_2)$ if and only if $A \in \pi(K_1, K_2)$.*

A restatement of this to a linear map defined on a finite dimensional real Hilbert space H leaving invariant a proper cone K in H holds as well. Determining the structure of the collection of all linear maps with the above property has been studied for a long time and considering the characterization of nonnegativity stated above, we were naturally led to the following preserver problem.

Question 2.2. Determine the structure of an invertible linear map L on $\mathcal{S}^n$ such that $L(K) \subset K$, where K is either COP_n or CP_n.

We infer from the discussion in the previous paragraph that $L \in \pi(K) \Leftrightarrow L + L_1 \in \mathrm{Sem}\,(K)$ for every $L_1 \in \mathrm{Sem}\,(K)$. A recent result from [2] says that $\mathrm{Sem}\,(K) = \{T_1 T_2^{-1}\colon \text{ where } T_1, T_2 \in \pi(K)^\circ\}$. We thus have the following theorem.

Theorem 2.3. *Let K be a proper cone in a finite dimensional real Hilbert space H. Then $\pi(K) \subseteq \mathrm{Sem}\,(K) - \mathrm{Sem}\,(K) = \{T_1 T_2^{-1} - T_3 T_4^{-1}\colon\ T_i \in \pi(K)^\circ\}$.*

However, the reverse inclusion in the above theorem is not true in general as the following example illustrates.

Example 2.4. Consider the following two linear maps on $\mathcal{S}^2$ defined as follows, where $K = \mathcal{N}_+^2$:

$$L_2(X) = \begin{bmatrix} 2x_1 - x_3 & -x_1 + 2x_2 \\ -x_1 + 2x_2 & -x_1 + 2x_3 \end{bmatrix} \quad \text{and} \quad L_1(X) = \begin{bmatrix} x_1 - 2x_2 & x_1 - x_2 \\ x_1 - x_2 & 2x_3 \end{bmatrix}.$$

By considering the matrices

$$X_0 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad Y_0 = \begin{bmatrix} 3 & 1 \\ 1 & 1 \end{bmatrix},$$

both of which are in the interior of $\mathcal{N}_+^2$, we see that $L_2(X_0)$ and $L_1(Y_0)$ are also in the interior of $\mathcal{N}_+^2$, proving that the maps L_2 and L_1 are semipositive relative to $\mathcal{N}_+^2$. However,

$$(L_2 - L_1)(X) = \begin{bmatrix} x_1 + 2x_2 - x_3 & -2x_1 + 3x_2 \\ -2x_1 + 3x_2 & -x_1 \end{bmatrix}$$

certainly does not belong to $\pi(\mathcal{N}_+^2)$.

Nevertheless, determining $\mathrm{Sem}\,(K)$ will reveal the structure of $\pi(K)$. Since $\mathrm{CP}_n = \mathcal{S}_+^n \cap \mathcal{N}_+^n$ for $n \leqslant 4$, we have $\pi(\mathcal{N}_+^n)^\circ \cap \pi(\mathcal{S}_+^n)^\circ \subseteq \pi(\mathrm{CP}_n)^\circ$ for $n \leqslant 4$. While $\pi(\mathcal{N}_+^n)^\circ$ is not hard to determine, $\pi(\mathcal{S}_+^n)$ is not easy to determine. This makes it difficult to determine the interior of $\pi(\mathrm{CP}_n)$ even for small values of n. However, when $n \leqslant 4$, it turns out that $\mathrm{Sem}\,(\mathrm{CP}_n) \subsetneq \mathrm{Sem}\,(\mathcal{N}_+^n) \cap \mathrm{Sem}\,(\mathcal{S}_+^n)$, see Lemma 3.9 and Example 3.10. Thus, when $n \leqslant 4$, $\pi(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathrm{CP}_n) - \mathrm{Sem}\,(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathcal{N}_+^n) \cap \mathrm{Sem}\,(\mathcal{S}_+^n) - \mathrm{Sem}\,(\mathcal{N}_+^n) \cap \mathrm{Sem}\,(\mathcal{S}_+^n)$. We determine $\mathrm{Sem}\,(\mathcal{N}_+^2)$ and $\mathrm{Sem}\,(\mathcal{S}_+^2)$ in Lemmas 3.13 and 3.14 in order to get information about $\pi(\mathrm{CP}_2)$. This manuscript is organized as follows. Section 1 is introductory. The main results are presented in Section 3, which is subdivided into subsections for ease of reading. Each of these subsections are self-explanatory.

3. Main results

The main results are presented in this section.

3.1. A brief detour into preservers of $\mathrm{COP}_n/\mathrm{CP}_n$. We begin with a brief description of the question that we are interested in. Any linear map on $\mathcal{S}^n$ can be expressed as $L(X) = \sum_{i=1}^{n(n+1)/2} A_i X B_i$ for appropriate square matrices A_i and B_i in $M_n(\mathbb{R})$. A *standard map* L on $M_{m,n}(\mathbb{R})$ is a map of the form $X \mapsto AXB$ for matrices A and B of appropriate sizes. Such a map L is invertible if and only if A and B are invertible. A *linear preserver* is a linear map on a space of matrices that preserves a subset $\mathcal{A}$ or a relation $\mathcal{R}$. There is rich history on this topic within linear algebra/matrix theory as can be evidenced from MathSciNet. A good reference to linear preservers is [13]. The monograph [10] collects several interesting resources on preserver problems. There are two types of preserver problems. The first one is to determine the structure of a map L defined on a matrix space such that $L(\mathcal{A}) = \mathcal{A}$ ($\mathcal{A}$ is a subset of the matrix space). These are called *onto/strong* preservers. The other one is to determine the structure of L such that $L(\mathcal{A}) \subset \mathcal{A}$, and they are called *into* preservers. Strong preservers are many times tractable, especially when $\mathcal{A}$ contains a basis for the underlying space. In this case, a strong preserver is an *into* preserver that is invertible with L^{-1} being an *into* preserver. Many linear preservers arise through standard maps, although there are exceptions. For instance, *into* preservers of copositive matrices that are not in standard form can be seen as follows: If A is a strictly copositive matrix (such a matrix is in the interior of the copositive cone) and B is a rank one completely positive matrix, both of the same size, then the map L defined by $X \mapsto \langle A, X\rangle B$ preserves completely positive matrices, but is not in standard form. For recent results on linear preservers of copositive matrices, the reader may look into

the paper [5]. As the authors point out in [5], the structure of an *into* linear preserver of copositivity is subtle and remains unsolved till date. Choi proved that any linear map on $\mathcal{S}^2$ that leaves invariant the cone of positive semidefinite matrices is precisely in the form $\sum_i A_i^t X A_i$ for some $A_i \in M_2(\mathbb{R})$ for all $X \in \mathcal{S}^2$, see Theorem 7 of [4] for details. As Choi himself points out this is no longer true when $n \geqslant 3$. It is therefore reasonable to believe that any *into* preserver of copositive matrices is necessarily of the form $\sum_i A_i X A_i^t$ for nonnegative matrices A_i. This turns out to be false even in the 2×2 case, as pointed out in [5]. It is not hard to prove that a standard map L on $\mathcal{S}^n$ preserves copositive matrices if and only if $L(X) = RXR^t$ for some nonnegative matrix R, see Theorem 2.2 of [5]. Notice that both into and onto preservers of copositive matrices that are in standard form are preservers of $\mathcal{S}_+^n$. However, there are maps that preserve $\mathcal{S}_+^n$ but do not preserve COP_n. Several other interesting results on linear preservers of copositive matrices were obtained in [5]. Strong preservers of copositive matrices arise only through a nonnegative monomial congruence, see Example 2 and Corollary 12 of [9] as well as [16]. We record this result below.

Theorem 3.1 ([9, 16])**.** *A linear mapping $L\colon \mathcal{S}^n \to \mathcal{S}^n$ is a strong preserver of COP_n if and only if L is a monomial congruence; that is, $L(X) = MXM^t$ for all $X \in \mathcal{S}^n$ and some nonnegative monomial matrix M.*

3.2. Standard maps. This section is devoted to the structure of standard maps that preserve copositive/completely positive matrices. Since our linear map L is defined on $\mathcal{S}^n$, the map L has a standard form if and only if it is of the form $X \mapsto RXR^t$ for some $R \in M_n(\mathbb{R})$. Recall that a standard map L on $\mathcal{S}^n$ preserves copositive matrices if and only if $L(X) = RXR^t$ for some nonnegative matrix R, see Theorem 2.2 of [5]. The general structure of *into* preservers of COP_n and CP_n when $n \leqslant 4$ is taken up in the next section.

The discussion in the previous section motivates us to ask the following question.

Question 3.2. Given a finite collection $\{A_1, \ldots, A_N\}$ of $n \times n$ real matrices, consider the linear map L on $\mathcal{S}^n$ defined by $L(X) = \sum_{i=1}^{N} A_i^t X A_i$. Under what conditions on the matrices A_i is the map L a preserver CP_n or COP_n?

As pointed out earlier, if the family $\{A_1, \ldots, A_N\}$ consists of only nonnegative matrices, then it is obvious that L will leave invariant both the cones CP_n and COP_n. It is therefore pertinent to seek other reasonable assumptions on the family $\{A_1, \ldots, A_N\}$ that will ensure that the map L leaves invariant either of the above two cones. A complete answer to this question is nontrivial and unless the map reduces to a familiar form, this is quite hard to tackle. Nevertheless, we identify certain family

of matrices that will ensure that the map L preserves these two cones. In the next section, we take up the question of determining semipositivity of L over these cones.

(1) If the family $\{A_1, \ldots, A_N\}$ consists of commuting normal matrices (or more generally, the matrices $A_1, \ldots, A_N$ are simultaneously unitarily diagonalizable), then the map L reduces to a standard form. If U is a unitary matrix that simultaneously diagonalizes $A_1, \ldots, A_N$, then

$$L(X) = U \operatorname{diag}\Big(\sum_{i=1}^{N} \lambda_1^i, \ldots, \sum_{i=1}^{N} \lambda_n^i\Big)(U^t X U) \operatorname{diag}\Big(\sum_{i=1}^{N} \lambda_1^i, \ldots, \sum_{i=1}^{N} \lambda_n^i\Big) U^t = BXB^t.$$

The map L is then a preserver of $\mathrm{CP}_n/\mathrm{COP}_n$ if B is a nonnegative matrix. This is equivalent to saying that for each j, $\sum_{i=1}^{N} \lambda_j^i \geqslant 0$, where the λ_j^i's are the eigenvalues of the matrices A_i's.

(2) If the family $\{A_1, \ldots, A_N\}$ consists of diagonalizable matrices such that A_1 anti-commutes with $A_2, \ldots, A_N$, then the family $\{A_1, A_2^2, \ldots, A_N^2\}$ is simultaneously diagonalizable. In particular, if A_1 is a normal matrix and $A_2, \ldots, A_N$ are positive semidefinite matrices such that A_1 anti-commutes with $\sqrt{A_2}, \ldots, \sqrt{A_N}$, then the family $\{A_1, A_2^2, \ldots, A_N^2\}$ is simultaneously unitarily diagonalizable, thereby reducing the map L to standard form. One can then argue as in the previous case for the map to be a preserver.

Given a proper cone K in a finite dimensional real Hilbert space $(H, \langle ,. \rangle)$, a linear map L on H is said to have the $\mathcal{Z}$-property relative to K (written $L \in \mathcal{Z}(K)$) if

$$x \in K, \quad y \in K^*, \quad \langle x, y\rangle = 0 \Rightarrow \langle L(x), y\rangle \leqslant 0.$$

The above notion is a generalization of the notion of $\mathcal{Z}$-matrices. Recall that a closed convex cone K is *reducible* if there exist closed convex cones K_1 and K_2 such that $K = K_1 + K_2$ and $\operatorname{span}(K_1) \cap \operatorname{span}(K_2) = \{0\}$. The cone K is *irreducible* if it is not reducible. In [8], Gowda and Sznajder prove that the completely positive cone is irreducible, see Theorem 4.1 of [8]. In [7], Gowda et al. also proves that when K is an irreducible cone, the only linear transformations in $\mathcal{Z}(K) \cap \pi(K)$ are nonnegative multiples of I. We thus conclude the following: For $K = \mathrm{CP}_n$, if $L \in \mathcal{Z}(K) \cap \pi(K)$, then $L = \alpha I$ for some $\alpha \geqslant 0$. In what follows below, we consider a well known map that preserves completely positive matrices, which actually reduces to the standard form.

Observe that both COP_n and CP_n contain a basis for $\mathcal{S}^n$, as these are generating cones. For instance, when $n = 2$, the collection of matrices $\{E_{11}, E_{22}, J_2\}$, where E_{ii} are the matrices with 1 at the iith position and 0 elsewhere and J_2 is the 2×2 matrix of all 1s, is a basis for CP_2. Consider the *generalized Lyapunov map* $\mathcal{L}_{A,B}$ on $\mathcal{S}^n$ defined by $X \mapsto AXB + B^t X A^t$, $A, B \in M_n(\mathbb{R})$ with B invertible. Notice that

when $B = I$, this reduces to the Lyapunov operator $\mathcal{L}_A$ on $\mathcal{S}^n$ defined by $X \mapsto AX + XA^t$. Before proceeding further, we wish to point out an interesting result from [6], namely, the Lyapunov operator $\mathcal{L}_A$ has the $\mathcal{Z}$-property relative to CP_n if and only if A is a Z-matrix, see Corollary 3.6 of [6]. In the theorem that follows, we prove a stronger statement when $n \leqslant 4$, namely, the Lyapunov operator $\mathcal{L}_A$ not only has the $\mathcal{Z}$-property relative to CP_n, but the matrix A is a constant multiple of the identity (assuming that $\mathcal{L}_A$ preserves CP_n). However, we do not know if this is true when $n \geqslant 5$, as the proof relies on the fact that when $n \leqslant 4$, the cone $\mathrm{CP}_n = \mathcal{S}_+^n \cap \mathcal{N}_+^n$. It is also easy to check that $\mathcal{L}_{A,B}(X) = B^t \mathcal{L}_C(X) B$, where $C = (B^t)^{-1}A$. These observations lead us to the following theorem.

Theorem 3.3. *Let L be an invertible linear map on $\mathcal{S}^n$ such that $L(\mathrm{CP}_n) \subseteq \mathrm{CP}_n$. If L is a standard map, then $L = \mathcal{L}_{A,B}$ for suitable choices of nonnegative matrices A and B. For $n \leqslant 4$, if $X \mapsto (B^t)^{-1}\mathcal{L}_{A,B}(X)B^{-1}$ preserves CP_n, then B is a nonnegative matrix. If $(B^t)^{-1}A$ is a Z-matrix and if $\mathcal{L}_{A,B}$ preserves CP_n, then $\mathcal{L}_{A,B}$ is a standard map, $\mathcal{L}_C$ is a standard map and B is a nonnegative matrix.*

P r o o f. Suppose L is an invertible standard map on $\mathcal{S}^n$ such that $L(\mathrm{CP}_n) \subseteq \mathrm{CP}_n$. Then, Theorem 2.2 of [5] implies that there exists a nonnegative invertible matrix A such that $L(X) = AXA^t$ for all $X \in \mathcal{S}^n$. Take $A_1 = A$ and $B_1 = \frac{1}{2}A^t$. It then follows that $L = \mathcal{L}_{A_1,B_1}$. To prove the second part of the theorem, we first observe that if $\mathcal{L}_A(\mathrm{CP}_2) \subseteq \mathrm{CP}_2$, then $A = \alpha I$ for some $\alpha > 0$ and hence, $\mathcal{L}_A$ is a constant multiple of the identity map. We indicate the proof for $n = 2$. This can be easily modified for $n = 3, 4$.

Step 1: If $\mathcal{L}_A(\mathrm{CP}_2) \subseteq \mathrm{CP}_2$, then $A = \alpha I$ for some $\alpha > 0$ and hence, $\mathcal{L}_A$ is a constant multiple of the identity map.

P r o o f of Step 1. Let $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. Since $\mathrm{CP}_2 = \mathcal{S}_+^2 \cap \mathcal{N}_+^2$, it is easy to show that $\mathcal{L}_A(E_{11}) \in \mathrm{CP}_2$ and $\mathcal{L}_A(E_{22}) \in \mathrm{CP}_2$. These two together imply that $a \geqslant 0$, $d \geqslant 0$, $b = c = 0$. Now, we also have $\mathcal{L}(J_2) \in \mathrm{CP}_2$, which implies that $a = d$. Therefore, the matrix A equals αI for some $\alpha > 0$. □

If $X \mapsto (B^t)^{-1}\mathcal{L}_{A,B}(X)B^{-1}$ preserves CP_n, then from Step 1 above we infer that $C = (B^t)^{-1}A = \alpha I$ for some $\alpha > 0$, thereby yielding that $\mathcal{L}_C(X) = 2\alpha X$. Since $\mathcal{L}_C(X) = (B^t)^{-1}\mathcal{L}_{A,B}(X)B^{-1}$, we see that $\mathcal{L}_{A,B}(X) = AXB + B^tXA^t = (2\alpha)B^tXB$ for all $X \in \mathcal{S}^n$. It then follows from Theorem 2.2 of [5] that B is a nonnegative matrix.

If $C := (B^t)^{-1}A$ is a Z-matrix, then the Lyapunov map $\mathcal{L}_C$ has the $\mathcal{Z}$-property relative to the cone CP_n and vice versa, see Corollary 3.6 of [6]. It then easily follows that the map $B^t\mathcal{L}_C B$ has the $\mathcal{Z}$-property relative to CP_n. Therefore, if $\mathcal{L}_{A,B}$ preserves CP_n, then it is a constant multiple of the identity and therefore a standard map. It now easily follows that the Lyapunov map $\mathcal{L}_C$ is a standard map. Since C is

a Z-matrix, the map $\mathcal{L}_C$ has the $\mathcal{Z}$-property, which in turn implies that the inverse of the map $X \mapsto (B^{-1})^t X B^{-1}$ is nonnegative relative to CP_n. It now follows that B is a nonnegative matrix. □

We end this section by determining invertible elements of the automorphism group of a proper cone K. Given any proper cone K in a finite dimensional real Hilbert space H, it is obvious that $\overline{\mathrm{Aut}\,(K)} \subseteq \pi(K)$, where the closure is taken in the space of all linear maps on H. An interesting result due to Gowda and Sznajder says that $\mathrm{Sem}\,(K) \cap \overline{\mathrm{Aut}\,(K)} = \mathrm{Aut}\,(K)$, see Lemma 2.7 of [8]. It is then not very hard to determine invertible elements of $\overline{\mathrm{Aut}\,(K)}$, when K is a proper cone in a finite dimensional real Hilbert space. We state this below.

Theorem 3.4. *Let K be a proper cone in a finite dimensional real Hilbert space H. If L is an invertible linear transformation on H with $L \in \overline{\mathrm{Aut}\,(K)}$, then $L \in \mathrm{Aut}\,(K)$.*

P r o o f. The proof follows from the fact that in any finite dimensional Hilbert space H, if $L_k \to L$ with all L_k and L invertible, then $L_k^{-1} \to L^{-1}$. If $L_k \to L$ with $L_k \in \mathrm{Aut}(K)$ and L invertible, then from $L_k(K) \subseteq K$ and $L_k^{-1}(K) \subseteq K$, we deduce that $L(K) \subseteq K$ and $L^{-1}(K) \subseteq K$, thereby proving that $L \in \mathrm{Aut}\,(K)$. □

We thus have the following equivalence for any proper cone K in a finite dimensional real Hilbert space H.

Observation 3.5. $L \in \mathrm{Sem}\,(K) \cap \overline{\mathrm{Aut}\,(K)} \Leftrightarrow L \in \mathrm{Aut}\,(K) \Leftrightarrow L$ is invertible and $L \in \overline{\mathrm{Aut}\,(K)}$.

The following corollary follows from Theorem 3.4 and Observation 3.5.

Corollary 3.6. *Let K be either of the cones CP_n or COP_n and L an invertible element on the boundary of the automorphism group of K. Then L arises only through monomial congruence.*

P r o o f. The proof follows from Theorem 3.1 as L is an automorphism of K. □

3.3. General into preservers of $\mathrm{CP}_n/\mathrm{COP}_n, n \leqslant 4$. This section is devoted to determining the structure of *into* linear preservers of CP_n for $n \leqslant 4$. The structure of *into* linear preservers of COP_n can be obtained by taking the transpose of the into preserver of CP_n. We infer from Theorems 2.1 and 2.3 that the problem reduces to determining $\mathrm{Sem}\,(\mathrm{CP}_n)$. We shall make use of two results, which we state in the order in which they are used. The first one is the following theorem of the alternative, one that can be suitably modified in the context of this manuscript. We use it in Example 3.10.

Theorem 3.7 ([1], Theorem 2.8)**.** *For proper cones K_1 and K_2 in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and an $m \times n$ matrix A, one and only one of the following alternatives holds.*

(1) *There exists $x \in K_1$ such that $Ax \in K_2^\circ$.*

(2) *There exists $0 \neq y \in K_2^*$ such that $-A^t y \in K_1^*$.*

The second one is from [2] and will be used in determining the structure of $\mathrm{Sem}\,(\mathcal{S}_+^2)$.

Theorem 3.8 ([2], Theorem 2.4)**.** *For proper cones K_1, K_2 in $\mathbb{R}^n$, let $S \in \pi(K_1, K_2)$ be an invertible linear map on $\mathbb{R}^n$. If a matrix A is K_1-semipositive, then the matrix $B = SAS^{-1}$ is K_2-semipositive. Conversely, if the cones are self-dual and if C is K_2-semipositive, then there exists a K_1-semipositive matrix A such that $C = (S^t)^{-1}AS^t$.*

We now state the first result of this section.

Lemma 3.9. *For $n \leqslant 4$, the following holds,* $\mathrm{Sem}\,(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathcal{S}_+^n) \cap \mathrm{Sem}\,(\mathcal{N}_+^n)$.

P r o o f. Suppose $n \leqslant 4$ and $L \in \mathrm{Sem}\,(\mathrm{CP}_n)$. Then, there exists $A \in (\mathrm{CP}_n)^\circ$ such that $L(A) \in (\mathrm{CP}_n)^\circ$. Since $\mathrm{CP}_n = \mathcal{S}_+^n \cap \mathcal{N}_+^n$, we see that $(\mathrm{CP}_n)^\circ = (\mathcal{S}_+^n)^\circ \cap (\mathcal{N}_+^n)^\circ$. Thus, the matrix $A \in (\mathcal{S}_+^n)^\circ$ such that $L(A) \in (\mathcal{S}_+^n)^\circ$ as well as $A \in (\mathcal{N}_+^n)^\circ$ with $L(A) \in (\mathcal{N}_+^n)^\circ$. This implies that $\mathrm{Sem}\,(\mathrm{CP}_n) \subseteq \mathrm{Sem}(\mathcal{S}_+^n) \cap \mathrm{Sem}(\mathcal{N}_+^n)$. □

The reverse inclusion in the above lemma is not true in general. The following example illustrates this.

Example 3.10. Let L be defined by

$$X \mapsto \begin{bmatrix} x_1 + x_2 + x_3 & -x_1 + x_2 - x_3 \\ -x_1 + x_2 - x_3 & x_1 + x_2 + x_3 \end{bmatrix}, \quad X = \begin{bmatrix} x_1 & x_2 \\ x_2 & x_3 \end{bmatrix}.$$

Further, L maps the matrix $X_0 = \left[\begin{smallmatrix} 1 & 3 \\ 3 & 1 \end{smallmatrix}\right]$ to $X_1 = \left[\begin{smallmatrix} 5 & 1 \\ 1 & 5 \end{smallmatrix}\right]$, thereby proving that L is in $\mathrm{Sem}\,(\mathcal{N}_+^2)$. The map L is also $\mathcal{S}_+^2$-semipositive as it maps the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right]$ to the positive definite matrix $\left[\begin{smallmatrix} 3 & -1 \\ -1 & 3 \end{smallmatrix}\right]$. The transpose of the map L is given by

$$L^t(X) = \begin{bmatrix} x_1 - x_2 + x_3 & x_1 + x_2 + x_3 \\ x_1 + x_2 + x_3 & x_1 - x_2 + x_3 \end{bmatrix}.$$

If $Y_0 = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right] \in \mathrm{COP}_2$, then $-L^t(Y_0) = \left[\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right]$, again an element in COP_2. We then conclude from Theorem 3.7 that $L \notin \mathrm{Sem}\,(\mathrm{CP}_2)$.

Remark 3.11. It turns out that these two sets $\mathrm{Sem}\,(\mathcal{N}_+^n)$ and $\mathrm{Sem}\,(\mathcal{S}_+^n)$ are not equal even when $n = 2$. This can be seen as follows: Consider the Lyapunov operator $\mathcal{L}_C$, where C is the matrix $\left[\begin{smallmatrix} 1 & 2 \\ 2 & 1 \end{smallmatrix}\right]$. It is easy to check that $\mathcal{L}_C \in \mathrm{Sem}\,(\mathcal{N}_+^2)$. However, it is not in $\mathrm{Sem}\,(\mathcal{S}_+^2)$, as the matrix C is not positive stable.

The preceding discussion yields the following structure theorem.

Theorem 3.12. *Let $n \leqslant 4$ and let L be an invertible linear map on $\mathcal{S}^n$. Then,* $\pi(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathrm{CP}_n) - \mathrm{Sem}\,(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathcal{N}_+^n) \cap \mathrm{Sem}\,(\mathcal{S}_+^n) - \mathrm{Sem}\,(\mathcal{N}_+^n) \cap \mathrm{Sem}\,(\mathcal{S}_+^n)$.

P r o o f. If L is an invertible linear map on $\mathcal{S}^n$ such that $L\pi(\mathrm{CP}_n)$, then we know from Theorem 2.3 that $L \in \mathrm{Sem}\,(\mathrm{CP}_n) - \mathrm{Sem}\,(\mathrm{CP}_n)$. The rest of the proof follows from Lemma 3.9. □

We now proceed to determining $\mathrm{Sem}\,(\mathcal{N}_+^n)$. Notice that $\mathcal{N}_+^n$ is a polyhedral cone as it is isomorphic to $\mathbb{R}_+^{n(n+1)/2}$. It follows from Lemma 1.3 that it now suffices to determine $\mathrm{Sem}(\mathbb{R}_+^{n(n+1)/2})$. Elements of the set $\mathrm{Sem}(\mathbb{R}_+^{n(n+1)/2})$ can be characterized as YX^{-1} for some (entrywise) positive matrices X and Y with X invertible, see [2], Theorem 2.3. We thus have the lemma.

Lemma 3.13. *The set $\mathrm{Sem}\,(\mathcal{N}_+^n)$ can be determined as the set of all maps on $\mathcal{S}^n$ of the form $T^{-1}(YX^{-1})T$, where X and Y are entrywise positive matrices in $\mathbb{R}^{n(n+1)/2}$ with X invertible and T is the isomorphism between $\mathcal{N}_+^n$ and $\mathbb{R}_+^{n(n+1)/2}$.*

P r o o f. There is a natural isomorphism T between $\mathcal{N}_+^n$ and $\mathbb{R}^{n(n+1)/2}$. The proof now follows from the second statement of Lemma 1.3 and the fact that semipositive maps over the nonnegative orthant of $\mathbb{R}^k$ are of the form YX^{-1} for entrywise positive matrices with X invertible. □

Determining $\mathrm{Sem}\,(\mathcal{S}_+^n)$ is not that simple. Although Theorem 2.3 of [2] gives a characterization of semipositive maps over all proper cones, determining the structure of positive maps over $\mathcal{S}_+^n$ (and possibly many other proper cones) is extremely challenging. In fact, the answer is not known for $\mathcal{S}_+^n$. We therefore restrict to $n = 2$. We will use Theorem 3.8 here. Recall that the Lorentz cone $\mathcal{L}_+^3$ is isomorphic to $\mathcal{S}_+^2$ via the map

$$T_1 \colon\ (x_1, x_2, x_3)^t \mapsto \begin{bmatrix} x_3 - x_1 & x_2 \\ x_2 & x_3 + x_1 \end{bmatrix}.$$

We therefore have $\mathrm{Sem}(\mathcal{S}_+^2) = T_1^{-1} S(\mathcal{L}_+^3) T_1$. Thus, the problem now reduces to determining semipositive maps over the Lorentz cone.

Theorem 3.8 says that if A is $\mathcal{L}_+^3$-semipositive, then there exists a semipositive matrix B (with respect to the usual nonnegative orthant) such that $A = (S^t)^{-1} B (S^t)$, where $S \in \pi(\mathbb{R}_+^3, \mathcal{L}_+^3)$ is an invertible map. For instance, one can take S to be the map $(x_1, x_2, x_3)^t \mapsto (x_1, x_2, x_1 + x_2 + x_3)^t$. We thus have the lemma.

Lemma 3.14. *The set* $\mathrm{Sem}\,(\mathcal{S}^2_+)$ *can be determined as the set of all maps on* $\mathcal{S}^2$ *of the form* $(S^tT_1^{-1})^{-1}(YX^{-1})(S^tT_1^{-1})$*, where* X *and* Y *are* 3×3 *entrywise positive matrices with* X *invertible,* T_1 *is the isomorphism between* $\mathcal{S}^2_+$ *and* $\mathcal{L}^3_+$ *and* S *is the invertible map described above.*

P r o o f. The preceding discussion says that

$$\mathrm{Sem}\,(\mathcal{S}^2_+) = \{T_1AT_1^{-1}\colon\ A \in \mathrm{Sem}\,(\mathcal{L}^3_+)\},$$

which is equal to $\{(T_1^{-1})^{-1}(S^t)^{-1}BS^tT_1^{-1}\colon\ B \in \mathrm{Sem}\,(\mathbb{R}^3_+)\}$. The statement follows since $B = YX^{-1}$ for entrywise positive matrices X and Y. □

Recall that $\mathrm{CP}_n \subseteq \mathcal{S}^n_+ \subseteq \mathrm{COP}_n$. It easily follows from this that $\mathrm{Sem}\,(\mathrm{CP}_n) \subseteq \mathrm{Sem}\,(\mathcal{S}^n_+) \subseteq \mathrm{Sem}\,(\mathrm{COP}_n)$. Example 3.10 illustrates that the first containment is proper. We also infer from Remark 3.11 that the second containment is also proper.

We end this section with the following result. Theorem 3.8 tells us that any map L that is semipositive relative to a proper cone K is necessarily of the form $L = L_1L_2^{-1}$ for some K-positive maps L_1 and L_2. Since $L^t = (L_2^{-1})^tL_1^t$ and the fact that if a map T is K-positive, then T^t is positive relative to the dual cone K^*, we have the following theorem. The proof is skipped.

Theorem 3.15. *Suppose* K *is a proper cone in a finite dimensional real Hilbert space* H *and let* $L \in \mathrm{Sem}\,(K)$*. Suppose, for any representation* $L_1L_2^{-1}$ *of* L *with* L_1 *and* L_2 *being* K*-positive, that the map* $L_2 \in \mathrm{Aut}\,(K_1)$*. Then* L^t *is* K^**-semipositive. The converse is not true.*

3.4. The dual of $\pi(K)$**.** If K is a proper cone, then so are its dual and $\pi(K)$. Therefore, $\pi(K)^*$ is a proper cone as well. This suggests that if one can determine $\pi(K)^*$, then one can determine its dual, namely $\pi(K)$, as well. Besides, the dual of $\pi(K)$ involves the boundary/facial structure of the cone K. For a proper cone K, $K^* \otimes K$ is defined to be the set $\{k \otimes l\colon\ \ k \in K^*, l \in K\}$, where $k \otimes l$ is the rank one operator $y \mapsto \langle l, y\rangle k$. As pointed earlier, it is well known that for a proper cone K, $\pi(K)^* = \mathrm{cone}\,\{K^* \otimes K\}$. In what follows, we describe a possible way to determine $\pi(K)^*$, when K is the cone $\mathcal{CP}_2$. Since $\mathcal{CP}_2 = \mathcal{N}^2_+ \cap \mathcal{S}^2_+$, we see that $\pi(\mathcal{N}^2_+) \cap \pi(\mathcal{S}^2_+) \subseteq \pi(\mathcal{CP}_2)$. Therefore, $\pi(\mathcal{CP}_2)^* \subseteq \{\pi(\mathcal{N}^2_+) \cap \pi(\mathcal{S}^2_+)\}^*$. Recall from the introduction, that the right hand side of the above containment equals closure $\{\pi(\mathcal{N}^2_+)^* + \pi(\mathcal{S}^2_+)^*\}$. We determine $\pi(\mathcal{N}^2_+)^*$ and $\pi(\mathcal{S}^2_+)^*$ below.

Theorem 3.16. *Let* $\pi(\mathcal{N}^n_+)^*$ *be the self-dual cone of all* $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ *nonnegative matrices (with respect to the trace inner product).*

P r o o f. Since $\mathcal{N}_+^n$ is isomorphic to the nonnegative orthant of $\mathbb{R}^{n(n+1)/2}$, one can use the representation of $\pi(\mathcal{N}_+^n)^*$ to deduce that $\pi(\mathcal{N}_+^n)^*$ is the self-dual cone of all $\frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1)$ nonnegative matrices. (w.r.t. the trace inner product). $\square$

In particular, $\pi(\mathcal{N}_+^2)^*$ is the self-dual cone of all 3×3 nonnegative matrices (with respect to the trace inner product).

Theorem 3.17. $\pi(\mathcal{S}_+^n)^* = \Big\{ \sum_{i=1}^{m} \operatorname{trace}(A_i Y) B_i,\ A_i, B_i \in \mathcal{S}_+^n,\ Y \in \mathcal{S}^n,\ m \in \mathbb{N} \Big\}$.

P r o o f. Since $\mathcal{S}_+^n$ is a self-dual cone in $\mathcal{S}^n$, we see that $\pi(\mathcal{S}_+^n)^* = \operatorname{cone}\{\mathcal{S}_+^n \otimes \mathcal{S}_+^n\}$. Any element of $\mathcal{S}_+^n \otimes \mathcal{S}_+^n$ is an operator on $\mathcal{S}^n$ of the form $Y \mapsto \operatorname{trace}(AY)B$, where $A, B \in \mathcal{S}_+^n$. The result follows from this. $\square$

We thus conclude that

$$\pi(\mathcal{CP}_2)^* \subseteq \operatorname{closure} \Big\{ M_3(\mathbb{R})^+ + \sum_{k=1}^{m} \operatorname{trace}(A_i Y) B_i,\ A_i, B_i \in \mathcal{S}_+^2,\ Y \in \mathcal{S}^2,\ m \in \mathbb{N} \Big\}.$$

Notice that this calculation carries over for $n \leqslant 4$ with appropriate modifications, as $\mathcal{CP}_n = \mathcal{N}_+^n \cap \mathcal{S}_+^n$ for such n. Since $\pi(K)$ is a proper cone when K is a proper cone, we infer that $\pi(\mathrm{CP}_2) \subseteq \pi(\mathrm{CP}_2)^*$ and therefore,

$$\pi(\mathrm{CP}_2) \subseteq \operatorname{closure} \Big\{ M_3(\mathbb{R})^+ + \sum_{k=1}^{m} \operatorname{trace}(A_i Y) B_i,\ A_i, B_i \in \mathcal{S}_+^2,\ Y \in \mathcal{S}^2,\ m \in \mathbb{N} \Big\}.$$

The above containment is also proper since $\pi(\mathrm{CP}_2) \subsetneq \pi(\mathrm{CP}_2)^*$ as CP_2 cannot be written as the image of the nonnegative orthant $\mathbb{R}_+^3$ under an orthogonal transformation. Once again, the above calculations also hold for $n = 3, 4$.

There is, however, a different way to look at $\pi(\mathcal{S}_+^2)^*$. As before, we shall make use of the fact that $\mathcal{S}_+^2$ is isomorphic to the Lorentz cone $\mathcal{L}_+^3$ in $\mathbb{R}^3$ through the map T_1 described in the previous section. Let us also denote by $\Theta(K)$ the set of all strong linear preservers (automorphisms) of K for an arbitrary cone K. We use this notation only to point out a theorem due to Loewy and Schneider. Lemma 1.3 says that $\pi(\mathcal{S}_+^2) = T_1^{-1} \pi(\mathcal{L}_+^3) T_1$. Therefore, $\pi(\mathcal{S}_+^2)^* = (T_1)^t \pi(\mathcal{L}_+^3)^* (T_1^{-1})^t$. The problem thus reduces to determining $\pi(\mathcal{L}_+^3)^*$. Once again, it is possible to write a representation of an element of L in $\pi(\mathcal{L}_+^3)^*$ as $\sum_{k=1}^{m} ab^t,\ a, b \in \mathcal{L}_+^3,\ m \in \mathbb{N}$ (recall that the Lorentz cone is a self-dual cone). However, an alternate description can be derived. The following well known result is due to Loewy and Schneider, see [11], Theorem 4.7.

Theorem 3.18. $\pi(\mathcal{L}_+^n) = \text{cone}\,(\text{closure}\,(\Theta(\mathcal{L}_+^n)))$.

Notice also that $\Theta(\mathcal{L}_+^3) = T_1\Theta(\mathcal{S}_+^2)T_1^{-1}$. It is well known that any automorphism of the cone $\mathcal{S}_+^n$ is of the form PXP^t for some fixed invertible matrix P. We therefore have

$$\Theta(\mathcal{L}_+^3) = \{(T_1P)X(P^tT_1^{-1})\colon\ X \in \mathcal{S}^2,\ P \text{ invertible}\}.$$

Thus, a typical element of $\pi(\mathcal{L}_+^3)$ is a linear map of the form $\sum_{i=1}^{k} \alpha_i(T_1P_i)X(P_i^tT_1^{-1})$ for some $\alpha_i \geqslant 0$, $X \in \mathcal{S}^2$, and P_i (not necessarily invertible), and k varies over the set of natural numbers (notice that we are taking the closure of $\Theta(\mathcal{L}_+^3)$ before taking the conic hull). Notice that the above representation of $\pi(\mathcal{S}_+^2)$ confirms with the one given by Choi in Theorem 7 of [4]. Although this representation is not new, this is a nice way to derive it and to the best of our knowledge, this derivation seems new, although using it to compute the dual of $\pi(\mathcal{S}_+^2)$ can be difficult. We end by pointing out that for any n, $\pi(\mathcal{S}_+^n) \subsetneq \text{Sem}(\mathcal{S}_+^n) - \text{Sem}(\mathcal{S}_+^n)$.

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