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AREA DIFFERENCES UNDER ANALYTIC MAPS AND OPERATORS

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Abstract. Motivated by the relationship between the area of the image of the unit disk under a holomorphic mapping h and that of zh, we study various L^2 norms for $T_\varphi(h)$, where T_φ is the Toeplitz operator with symbol φ. In Theorem 2.1, given polynomials p and q we find a symbol φ such that $T_\varphi(p) = q$. We extend some of our results to the polydisc.

Keywords: unit disk; polydisc; polynomial; Toeplitz operator; Bergman projection

MSC 2020: 30J99, 30H05, 47B35, 32A36

1. Introduction

For the paper, the domain Ω is the unit disk $\mathbb{D}$ in the complex plane, $\mathbb{D} \subset \mathbb{C}$, or the polydisc $\mathbb{D}^n = \{z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n \colon |z_j| < 1\}$. Let $L^2(\Omega)$ denote the set of square integrable functions on Ω, $L^2(\Omega)$ is an inner-product space with inner product given by

$$\langle f, g\rangle_{L^2(\Omega)} = \int_\Omega f\overline{g}\,\mathrm{d}A, \quad \text{inducing} \quad \|f\|^2_{L^2(\Omega)} = \int_\Omega |f|^2\,\mathrm{d}A.$$

Let $\mathcal{O}(\Omega)$ denote the set of holomorphic functions on Ω, $A^2(\Omega) = L^2(\Omega) \cap \mathcal{O}(\Omega)$ be the space of square integrable holomorphic functions on Ω, known as Bergman space on Ω, and (for $k \in \mathbb{N}$) $A^{k,2}(\Omega)$ be the space of $h \in A^2(\Omega)$ such that $\partial^j h/\partial z^j \in A^2(\Omega)$ for all $j \leqslant k$. The inner product of $A^{k,2}(\Omega)$ is

$$\langle f, g\rangle_{A^{k,2}(\Omega)} = \int_\Omega \sum_{l=0}^{k} \frac{\partial^l f}{\partial z^l}\overline{\frac{\partial^l g}{\partial z^l}}\,\mathrm{d}A(z), \quad \text{inducing} \quad \|f\|^2_{A^{k,2}(\Omega)} = \int_\Omega \sum_{l=0}^{k} \Big|\frac{\partial^l f}{\partial z^l}\Big|^2\,\mathrm{d}A(z).$$

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The subspaces $A^2(\Omega)$ and $A^{k,2}(\Omega)$ are closed in $L^2(\Omega)$ and $A^2(\Omega)$, respectively, see [8], pages 70–71. For information on Bergman spaces, see [13], Chapter 14 or [14]. Regarding $A^{k,2}(\Omega)$ spaces, refer to [9], Chapter 3 or [11], Chapter 6.

The set of monomials $\{z^\alpha\colon\ \alpha \in \mathbb{N}_0^n\}$ forms an orthogonal basis for $A^2(\Omega)$. Let $c_\alpha^2 := \|z^\alpha\|_{L^2(\Omega)}^2 = \int_\Omega |z^\alpha|^2 \,\mathrm{d}V(z)$ (for $n = 1$, $c_\alpha^2 = \pi/(\alpha+1)$). Then the set $\{z^\alpha/c_\alpha\}$ forms an orthonormal basis for the Bergman space. Thus, $A^2(\Omega)$ is a reproducing kernel Hilbert space, and we have the following representation for the Bergman kernel by using the orthonormal basis $K_\Omega(z,w) = \sum\limits_{\alpha\in\mathbb{N}_0^n} z^\alpha \overline{w}^\alpha/c_\alpha^2$ for $n = 1$:

$$K_\Omega(z,w) = \frac{1}{\pi}\sum_{\alpha\in\mathbb{N}_0}(\alpha+1)z^\alpha\overline{w}^\alpha = \frac{1}{\pi}\frac{1}{(1-z\overline{w})^2}. \tag{1.1}$$

The Bergman projection, which is the orthogonal projection from $L^2(\Omega)$ onto the subspace $A^2(\Omega)$, is the integral operator associated to the kernel $K_\Omega(z,w)$ and is defined as

$$(P^\Omega h)(z) = \int_\Omega K_\Omega(z,w)h(w)\,\mathrm{d}V(w) \quad \text{for any } h \in L^2(\Omega).$$

We refer to [16] for more on the basic of the Bergman projection.

Let $M_\varphi\colon\ L^2(\Omega) \to L^2(\Omega)$ be the multiplication operator by $\varphi \in L^\infty(\Omega)$. The Toeplitz operator $T_\varphi^\Omega\colon\ A^2(\Omega) \to A^2(\Omega)$ is defined as the composition of the multiplication operator M_φ with a symbol function $\varphi \in L^\infty(\Omega)$ and the Bergman projection. That is, for $h \in A^2(\Omega)$, $T_\varphi^\Omega h = P^\Omega M_\varphi h$.

Let $D = \partial/\partial z$ with domain $A^{1,2}(\Omega)$. Then D is an unbounded linear operator. For $\Omega = \mathbb{D}$, the commutator $[T_{\overline{z}}^{\mathbb{D}} M_z, DM_z] = 0$ on the monomials z^m. This means $T_{\overline{z}}^{\mathbb{D}} M_z$ and DM_z have z^m as their eigenfunctions. In other words, when $T_{\overline{z}}^{\mathbb{D}} M_z$ or DM_z operates on z^m, the outcome is a constant times z^m. In this paper, we attempt to mimic the derivative operator with Toeplitz operators, Theorem 2.1, and observe its consequences. We also explore extending the technique developed for Theorem 2.1 to higher dimensions, as presented in Theorem 2.2. Furthermore, we offer generalizations of a relationship between areas of images of the unit disk under holomorphic mappings h and zh, demonstrated in Corollary 4.1 and Proposition 4.1. Here is an outline of the paper's structure.

In Section 2 of the paper, we present our main findings – Theorem 2.1 and Corollary 2.1. Additionally, we extend our results to the polydisc in Theorem 2.2. In Section 3, we provide the proof for Theorems 2.1 and 2.2. In Section 4, we present generalizations of an equation related to areas of images of holomorphic mappings. Finally, we conclude the paper with some remarks in Section 5 and an appendix in Section 6. In the appendix, we offer a discussion on the regularity of Toeplitz operators on the unit disk used in Proposition 4.1.

2. Main results with some observations

This section presents the main findings of the paper. Theorem 2.1, Corollary 2.1, and Proposition 2.1 provide a symbol construction of a Toeplitz operator to modify its output. Theorem 2.2 offers an extension of the idea to the polydisk.

Lemma 2.1. *For all $n, m \in \mathbb{N}_0$,*

$$T^{\mathbb{D}}_{a\overline{z}^n}(bz^m) = \begin{cases} \dfrac{ab(m-n+1)}{m+1} z^{m-n} & \text{if } m \geqslant n; \\ 0 & \text{otherwise.} \end{cases} \tag{2.1}$$

P r o o f of Lemma 2.1. Let n and m be nonnegative integers. Then

$$\begin{aligned} T_{\overline{z}^n}(z^m) = P(\overline{z}^n z^m) &= \int_{\mathbb{D}} K(z,w)\overline{w}^n w^m \,\mathrm{d}A(w) = \int_{\mathbb{D}} \sum_{k=0}^{\infty} \frac{z^k}{c_k^2} \overline{w}^{n+k} w^m \,\mathrm{d}A(w) \\ &= \sum_{k=0}^{\infty} \frac{z^k}{c_k^2} \int_{\mathbb{D}} \overline{w}^{n+k} w^m \,\mathrm{d}A(w) \\ &= \begin{cases} \dfrac{c_m^2}{c_{m-n}^2} z^{m-n} = \dfrac{(m-n+1)}{m+1} z^{m-n} & \text{if } m-n \geqslant 0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Note that $m - n \neq k$ implies $\int_{\mathbb{D}} \overline{w}^{k+n} w^m \,\mathrm{d}A(w) = 0$. In the second equality, we use the series representation of the Bergman kernel stated in (1.1).

The form of identity (2.1) as stated in Lemma 2.1 follows directly from bilinearity of the Toeplitz operator in its symbol and argument. □

Theorem 2.1. *Let p, q be (holomorphic) polynomials and suppose that $p(z) \not\equiv 0$. There exists a polynomial symbol $\varphi(z, \overline{z}) = \phi_1(z) + \overline{\phi_2(z)}$ such that $T^{\mathbb{D}}_{\varphi}(p) = q$, where ϕ_j is holomorphic for $j = 1, 2$.*

To establish the existence of a symbol φ of a Toeplitz operator in the proof of Theorem 2.1, given any two holomorphic polynomials p and q such that the operator with that symbol acts on p and produces q, i.e., $T^{\mathbb{D}}_{\varphi}(p) = q$, an explicit algorithm is developed to assemble the symbol φ. The formed algorithm takes into account the L^2-norms of the monomials on the domain (the unit disc).

Remark 2.1. It follows from Lemma 2.1 that, given the monomials z^m and z^{n+m} and a nonnegative integer $r > (n + m)$, there is a constant c_{mn} such that

$$T_{\overline{z}^n + \overline{z}^r}(c_{mn} z^{n+m}) = P(c_{mn}\overline{z}^n z^{n+m} + c_{mn}\overline{z}^r z^{n+m}) = z^m.$$

Therefore, the symbol φ in Theorem 2.1 is not unique in a trivial manner. (Just vary r for example.)

The following corollary enables us to find a Toeplitz operator that maps a given polynomial to one of its derivatives. This is a direct consequence of Theorem 2.1.

Corollary 2.1. *Let p be a holomorphic polynomial. For each $k \geqslant 0$, there is an antiholomorphic polynomial symbol φ such that $T^{\mathbb{D}}_{\varphi}(p) = p^{(k)}$, where $p^{(k)}$ denotes the kth derivative of p.*

P r o o f of Corollary 2.1. The existence of some φ follows immediately from Theorem 2.1. The proof of Theorem 2.1 shows that φ can be chosen to be antiholomorphic. □

For any holomorphic polynomial p we choose, there exists a symbol φ_p such that $T^{\mathbb{D}}_{\varphi_p}(p) = \partial p/\partial z$.

Remark 2.2. We can also use the procedure that is established in the proof of Theorem 2.1 (in Section 3) to form a symbol that takes a sufficiently nice rational function to any given polynomial. If $f = p/q$ is a rational function with poles outside of $\overline{\mathbb{D}}$, and r is a polynomial, let $\varphi_{(p,r)}$ be a symbol that takes the polynomial p to the polynomial r. The desired symbol is then $\psi = q\varphi_{(p,r)}$

$$\begin{aligned}(T^{\mathbb{D}}_{\psi}(p/q))(z) = P^{\mathbb{D}}(\psi p/q) &= (P^{\mathbb{D}}(q\varphi_{(p,r)}p/q))(z)\\ &= (P^{\mathbb{D}}(\varphi_{(p,r)}p))(z) = (T^{\mathbb{D}}_{\varphi_{(p,r)}}(p))(z) = r(z).\end{aligned}$$

Remark 2.3. In the case of holomorphic polynomials, particularly if they have zeros in the unit disc, with the developed procedure in the proof of Theorem 2.1, we observe that the symbol that takes a holomorphic polynomial to its derivative should be antiholomorphic. However, we may also use a similar approach for holomorphic functions that are not polynomials but not vanishing in the unit disc and assemble symbols that are holomorphic, e.g. $\exp(z^2)$ is taken to $2z\exp(z^2)$ by the symbol $2z$. This leads us to the following proposition.

Proposition 2.1 (Exponential version)**.** *Let Ω be a simply connected domain that contains $\overline{\mathbb{D}}$ and let $f \in \mathcal{O}(\Omega)$ be nonvanishing on $\overline{\mathbb{D}}$. Then there is a holomorphic symbol φ such that $T^{\mathbb{D}}_{\varphi} f = f^{(k)}$, where $f^{(k)}$ denotes the kth derivative of f.*

P r o o f of Proposition 2.1. Given that Ω is simply connected and f is holomorphic and nonzero, it follows from [13], Lemma 6.6.4, that there exists a function $g \in \mathcal{O}(\Omega)$ such that $f = \exp(g)$ on Ω. Therefore, $f^{(k)}(z) = h(z)\exp(g(z))$, where h combines different order derivatives and degrees of g and so is a holomorphic function on Ω. We complete the proof by letting $\varphi(z) = h(z)$. □

Remark 2.4. In Proposition 2.1, we can draw the same conclusion under different assumptions. If we assume that f is holomorphic, nonzero on $\mathbb{D}$, and that $f^{(k)}$ is bounded for a natural number k, then there exists a holomorphic symbol φ such that $T_\varphi^{\mathbb{D}} f = f^{(j)}$, where $1 \leqslant j \leqslant k$. The reason, why in Proposition 2.1 we assume that f is holomorphic on a larger simply connected domain, is to guarantee that f (or, more appropriately, g) has bounded derivatives. These bounded derivatives of g can then serve as the symbol for a Toeplitz operator on the disc.

Next, we extend the construction procedure from Theorem 2.1 to higher dimensions in Theorem 2.2 by examining a restricted class of polynomials under a Toeplitz operator on the polydisc $\mathbb{D}^n$. The symbol construction will resemble the one-dimensional case.

Definition 2.1. Consider a complex polynomial p in variables $z_1, \ldots, z_n$. We say p is *scissile* if there exists an index j such that $1 \leqslant j \leqslant n$, and we can write p in the form

$$p = \sum_{l=0}^{k} p_l(z_1, \ldots, \widehat{z}_j, \ldots, z_n) z_j^l, \tag{2.2}$$

where the p_l are monomials in variables $z_1, \ldots, z_{j-1}, z_{j+1}, \ldots, z_n$.

In simpler terms, p is scissile if we can express it as a polynomial in one variable (z_j) with monomial coefficients in the other variables.

Example 2.1. $p(z) = z_1 + z_1^2 z_2 + z_1^3 z_2^2 z_3 + z_1^4 z_2^3 z_3^2 z_4$ is a scissile polynomial, whereas $p(z) = z_1 + z_2 + z_3 + z_1^2 z_2 + z_2^2 z_3 + z_3^2 z_1$ is not.

Theorem 2.2 (Multivariable)**.** *Let p, q be (holomorphic) polynomials in variables $z_1, \ldots, z_n$ and suppose that p is scissile and $p(z) \not\equiv 0$. Then there is a polynomial $\varphi(z, \overline{z})$ such that $T_\varphi^{\mathbb{D}^n}(p) = q$, where $T_\varphi^{\mathbb{D}^n}$ is a Toeplitz operator on $\mathbb{D}^n$.*

The reason for choosing this particular class of polynomials in Theorem 2.2 is twofold: one, it allows us to use the same techniques as in the proof of Theorem 2.1, and two, the fact that the coefficients of z_j^a, $1 \leqslant a \leqslant k$ are monomials greatly simplifies the process of finding a suitable symbol.

In the proof of Theorem 2.2, we only rely on the fact that $p_k(z)$ is a monomial in the variables $z_1, \ldots, \widehat{z}_j, \ldots, z_n$ (where $p = \sum_{l=0}^{k} p_l(z_1, \ldots, \widehat{z}_j, \ldots, z_n) z_j^l$). This allows us to establish a more general result, as demonstrated in Corollary 3.1 of Section 3.

3. Proof of Theorems 2.1 and 2.2

Within this section, we prove our main results and some of their consequences and generalizations.

The algorithm in Theorem 2.1 generates a symbol $\phi_{p,q}$ for holomorphic polynomials p and q, such that the Toeplitz operator with symbol $\phi_{p,q}$ applied to p yields q.

P r o o f of Theorem 2.1. Let $p, q \in \mathbb{C}[z]$ with $p \neq 0$ and set $\deg(p) = n$, $\deg(q) = m$. If $m \geqslant n$, write $q = sp+r$, where $\deg(r) < n$. If $r = 0$, then we are done, since $\varphi = s$ gives $T_\varphi(p) = P^{\mathbb{D}}(sp) = q$. Otherwise, we must find an antiholomorphic ϕ such that $T_\phi(p) = r$, so that $T_{s+\phi}(p) = P^{\mathbb{D}}(sp+\phi p) = sp+T_\phi(p) = sp+r = q$. By linearity, it is sufficient to find such a ϕ for which $r = z^{n-k}$ for each $1 \leqslant k \leqslant n$. According to Lemma 2.1, it suffices to consider ϕ of degree $k \leqslant \deg(\phi) \leqslant n$, so write

$$\phi = \sum_{j=k}^{n} a_j \overline{z}^j$$

so that

$$\phi p = \sum_{l=1}^{n} \sum_{j=k}^{n} p_l a_j z^l \overline{z}^j.$$

Now again by Lemma 2.1 we have

$$P^{\mathbb{D}}(z^l \overline{z}^j) = \begin{cases} 0 & \text{if } l < j; \\ c_{l,j} z^{l-j} & \text{if } l \geqslant j, \end{cases}$$

where $c_{l,j} = (l-j+1)/(l+1)$, so setting $P^{\mathbb{D}}(\phi p)$ equal to z^{n-k} yields a triangular linear system for $a_1, \ldots, a_k$:

$$\sum_{l=k}^{n} \sum_{j=k}^{n} p_l a_j c_{l,j} z^{l-j} = z^{n-k},$$

or equivalently:

$$\begin{pmatrix} p_n a_k c_{n,k} & p_{n-1} a_k c_{n-1,k} & \ldots & p_k a_k c_{k,k} \\ 0 & p_n a_{k+1} c_{n,k+1} & & p_{k+1} a_{k+1} c_{k+1,k+1} \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & p_n a_n c_{n,n} \end{pmatrix} \begin{pmatrix} z^{n-k} \\ z^{n-k-1} \\ \vdots \\ z^0 \end{pmatrix} = \begin{pmatrix} z^{n-k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

The $(n-k) \times (n-k)$ coefficient matrix is nonsingular as all of the $c_{l,j}$ are nonzero and p_n is nonzero by assumption. The system may be solved recursively by first solving it for a_k, then using this value to solve it for a_{k-1} and so on. This gives us ϕ such that $T_\phi(p) = z^{n-k}$; it follows that we may find $\varphi = s + \phi_1 + \ldots + \phi_k$ such that $T_\varphi(p) = q$ by linearity of the Toeplitz operator in its symbol and argument. Now if $m < n$, we simply reduce to the case above, where $r = q$. □

To prove Theorem 2.2, we recall Definition 2.1: a polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ is called *scissile* if there exists an index $1 \leqslant j \leqslant n$ such that we can express p in the form $p = \sum_{l=0}^{k} p_l(z_1, \ldots, \widehat{z_j}, \ldots, z_n) z_j^l$, where the coefficients $p_0, \ldots, p_k$ are monomials, and $\widehat{z_j}$ denotes the omission of the variable z_j.

We will now introduce a notation that will be used throughout the proof of Theorem 2.2. For a monomial p, we define γ as the tuple $(\gamma_1, \ldots, \gamma_n)$, where γ_j, $1 \leqslant j \leqslant n$, is the degree of p as a polynomial in z_j. We will use superscripts $+$ and $-$ to denote the positive and negative parts, respectively, of a point in $\mathbb{Z}^n$. Additionally, if $m \in \mathbb{Z}$, we will use m_j to represent the n-tuple consisting of all zeros except for m in the jth position.

Before we start the proof, we need to extend Lemma 2.1 from $\mathbb{D}$ to $\mathbb{D}^n$.

Lemma 3.1 (Generalizing Lemma 2.1)**.** *For the Toeplitz operator on the polydisc $\mathbb{D}^n$,*

$$T_{a\overline{z}^\gamma}^{\mathbb{D}^n}(bz^\beta) = \begin{cases} abz^{\beta-\gamma} \prod_{k=1}^{n} \dfrac{\beta_k - \gamma_k + 1}{\beta_k + 1} & \text{if } \beta_1 \geqslant \gamma_1, \ldots, \beta_n \geqslant \gamma_n; \\ 0 & \text{otherwise,} \end{cases} \tag{3.1}$$

where γ, β are multi-indices.

P r o o f. We show the result directly.

$$\begin{aligned} T_{\overline{z}^\gamma}^{\mathbb{D}^n}(z^\beta) &= \int_{\mathbb{D}^n} K_{\mathbb{D}^n}(z, w) \overline{w}^\gamma w^\beta \, \mathrm{d}V(w) = \int_{\mathbb{D}^n} \prod_{j=1}^{n} K_{\mathbb{D}}(z_j, w_j) \overline{w_j}^{\gamma_j} w_j^{\beta_j} \, \mathrm{d}V(w) \\ &= \prod_{j=1}^{n} \int_{\mathbb{D}} K_{\mathbb{D}}(z_j, w_j) \overline{w_j}^{\gamma_j} w_j^{\beta_j} \, \mathrm{d}A(w_j) = \int_{\mathbb{D}} K(z_n, w_n) \overline{w_n}^{\gamma_n} w_n^{\beta_n} \, \mathrm{d}A(w_n) \\ &= \begin{cases} z^{\beta-\gamma} \prod_{k=1}^{n} \dfrac{\beta_k - \gamma_k + 1}{\beta_k + 1} & \text{if } \beta_1 \geqslant \gamma_1, \ldots, \beta_n \geqslant \gamma_n; \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

where the last step is an application of Lemma 2.1, and the form as in equation (3.1) follows from the linearity of the integral. □

P r o o f of Theorem 2.2. Since p is scissile, write

$$\begin{aligned} p &= p_0(z_1, \ldots, \widehat{z_j}, \ldots, z_n) + p_1(z_1, \ldots, \widehat{z_j}, \ldots, z_n) z_j + \ldots + p_k(z_1, \ldots, \widehat{z_j}, \ldots, z_n) z_j^k \\ &= a_0 z^{m_0} + a_1 z^{m_1} z_j + \ldots + a_k z^{m_k} z_j^k \end{aligned}$$

for some $1 \leqslant j \leqslant n$, where $p_0, \ldots, p_k$ are monomials in the other variables and $m_0, \ldots, m_k$ are their respective degrees. Write q in this manner as well, so that

$$q = q_0(z_1, \ldots, \widehat{z_j}, \ldots, z_n) + q_1(z_1, \ldots, \widehat{z_j}, \ldots, z_n) z_j + \ldots + q_l(z_1, \ldots, \widehat{z_j}, \ldots, z_n) z_j^l,$$

where $q_0, \ldots, q_l$ are *polynomials* in $z_1, \ldots, \widehat{z_j}, \ldots, z_n$. Write q_l in terms of its constituent monomials:

$$q_l(z) = b_{0,1} z^{\gamma_{0,1}} + b_{0,2} z^{\gamma_{0,2}} + \ldots + b_{0,i_0} z^{\gamma_{0,i_0}}.$$

We now want to find a polynomial f_0 of the form

$$f_0 = (c_{0,1} z^{(\gamma_{0,1}-m_k)^+} \overline{z}^{(\gamma_{0,1}-m_k)^-} + \ldots + c_{0,i_0} z^{(\gamma_{0,i_0}-m_k)^+} \overline{z}^{(\gamma_{0,i_0}-m_k)^-}) z_j^{(l-k)^+} \overline{z}_j^{(l-k)^-}$$

such that $T_{f_0}^{\mathbb{D}^n}(p_k(z) z_j^k) = q_l(z) z_j^l$. We have

$$\begin{aligned} T_{f_0}^{\mathbb{D}^n}(p_k(z) z_j^k) &= \sum_{l=1}^{i_0} T^{\mathbb{D}^n}_{c_{0,l} z^{(\gamma_{0,l}-m_k)^+} \overline{z}^{(\gamma_{0,l}-m_k)^-} z_j^{(l-k)^+} \overline{z}_j^{(l-k)^-}} (p_k(z) z_j^k) \\ &= \sum_{l=1}^{i_0} c_{0,l} a_k T^{\mathbb{D}^n}_{z^{(\gamma_{0,l}-m_k)^+} \overline{z}^{(\gamma_{0,l}-m_k)^-} z_j^{(l-k)^+} \overline{z}_j^{(l-k)^-}} (p_k(z) z_j^k) \\ &= \sum_{l=1}^{i_0} c_{0,l} a_k z^{\gamma_{0,l}} z_j^l \prod_{\varepsilon=1}^{n} \frac{(\gamma_{0,l})_\varepsilon + l_j + 1}{[\gamma_{0,l} + (\gamma_{0,l} - m_k)^-]_\varepsilon + [l + (l-k)^-]_j + 1}. \end{aligned}$$

There are i_0 monomials of distinct degrees above, each corresponding to a monomial of the same degree in $q_l(z) z_j^l$, so in order for the above to equal $q_l(z) z_j^l$, we must have

$$c_{0,\eta} = \frac{b_{0,\eta}}{a_k} \prod_{\varepsilon=1}^{n} \frac{[\gamma_{0,\eta} + (\gamma_{0,\eta} - m_k)^-]_\varepsilon + [l + (l-k)^-]_j + 1}{(\gamma_{0,\eta})_\varepsilon + l_j + 1}, \quad 1 \leqslant \eta \leqslant i_0.$$

We then have

$$T_{f_0}^{\mathbb{D}^n}(p) = q_l(z) z_j^l + T_{f_0}^{\mathbb{D}^n}(p_0(z) + p_1(z) z_j + \ldots + p_{k-1}(z) z_j^{k-1}).$$

We see that $T_{f_0}^{\mathbb{D}^n}(p_{k-\omega}(z) z_j^{k-\omega})$ is a monomial (in z_j) of degree $(l-k)^+ - (l-k)^- + (k-\omega) = l - \omega$ for $1 \leqslant \omega \leqslant k$ as long as $l - \omega > 0$; when $l - \omega = 0$, it is constant (as a function of z_j), and for $l - \omega < 0$, it is 0. We obtain possibly one additional term of degree $l-1$ in z_j, $T_{f_0}^{\mathbb{D}^n}(p_{k-1}(z) z_j^{k-1})$, and hence wish to find f_1 such that

$$T_{f_1}^{\mathbb{D}^n}(p_k(z) z_j^k) = q_{l-1}(z) z_j^{l-1} - T_{f_0}^{\mathbb{D}^n}(p_{k-1}(z) z_j^{k-1}).$$

Writing the right-hand side above in terms of its constituent monomials we get

$$q_{l-1}(z) z_j^{l-1} - T_{f_0}^{\mathbb{D}^n}(p_{k-1}(z) z_j^{k-1}) = (b_{1,1} z^{\gamma_{1,1}} + b_{1,2} z^{\gamma_{1,2}} + \ldots + b_{1,i_1} z^{\gamma_{1,i_1}}) z_j^{l-1}$$

and we see that we should look for f_1 of the form

$$f_1 = (c_{1,1} z^{(\gamma_{1,1}-m_k)^+} \overline{z}^{(\gamma_{1,1}-m_k)^-} + \ldots + c_{1,i_1} z^{(\gamma_{1,i_1}-m_k)^+} \overline{z}^{(\gamma_{1,i_1}-m_k)^-}) z_j^{(l-k)^+} \overline{z}_j^{(l-k)^-+1}.$$

Again, we come up with i_1 monomials of distinct degree, each corresponding to a monomial of the same degree in $q_l(z) z_j^l - T_{f_0}^{\mathbb{D}^n}(p_{k-1}(z) z_j^{k-1})$, so we must set

$$c_{1,\eta} = \frac{b_{1,\eta}}{a_k} \prod_{\varepsilon=1}^{n} \frac{[\gamma_{1,\eta} + (\gamma_{1,\eta} - m_k)^-]_\varepsilon + [l + (l-k)^-]_j + 1}{(\gamma_{1,\eta})_\varepsilon + (l-1)_j + 1}, \quad 1 \leqslant \eta \leqslant i_1.$$

We then have

$$\begin{aligned} T_{f_0+f_1}^{\mathbb{D}^n}(p) = q_l(z) z_j^l + q_{l-1}(z) z_j^{l-1} + T_{f_1}^{\mathbb{D}^n}(p_0(z) + \ldots + p_{k-1}(z) z_j^{k-1}) \\ + T_{f_0}^{\mathbb{D}^n}(p_0(z) + \ldots + p_{k-2}(z) z_j^{k-2}), \end{aligned}$$

and again $T_{f_1}^{\mathbb{D}^n}(p_{k-\omega}(z) z_j^{k-\omega})$ is a monomial in z_j of degree $l - \omega - 1$, $1 \leqslant \omega \leqslant k$ for $l - \omega > 1$, constant for $l - \omega = 1$, and 0 for $l - \omega < 1$. Continuing this process, we see that on step s, we wish to find f_s such that

$$T_{f_s}^{\mathbb{D}^n}(p_k(z) z_j^k) = q_{l-s}(z) z_j^{l-s} - \sum_{\alpha=0}^{s-1} T_{f_\alpha}^{\mathbb{D}^n}(p_{k-s+\alpha}(z) z_j^{k-s+\alpha}).$$

We split the right-hand side into its constituent monomials as

$$q_{l-s}(z) z_j^{l-s} - \sum_{\alpha=0}^{s-1} T_{f_\alpha}^{\mathbb{D}^n}(p_{k-s+\alpha}(z) z_j^{k-s+\alpha}) = (b_{s,1} z^{\gamma_{s,1}} + b_{s,2} z^{\gamma_{s,2}} + \ldots + b_{s,i_s} z^{\gamma_{s,i_s}}) z_j^{l-s}$$

and look for f_s in the form

$$f_s = (c_{s,1} z^{(\gamma_{s,1}-m_k)^+} \overline{z}^{(\gamma_{s,1}-m_k)^-} + \ldots + c_{s,i_s} z^{(\gamma_{s,i_s}-m_k)^+} \overline{z}^{(\gamma_{s,i_s}-m_k)^-}) z_j^{(l-k)^+} \overline{z}_j^{(l-k)^-+s}.$$

In this case, we see that the coefficients $c_{s,\eta}$ should be

$$c_{s,\eta} = \frac{b_{s,\eta}}{a_k} \prod_{\varepsilon=1}^{n} \frac{[\gamma_{s,\eta} + (\gamma_{s,\eta} - m_k)^-]_\varepsilon + [l + (l-k)^-]_j + 1}{(\gamma_{s,\eta})_\varepsilon + (l-s)_j + 1}, \quad 1 \leqslant \eta \leqslant i_s.$$

This process must end on the lth step, since for $s \geqslant l+1$, the degree of $\overline{z}_j$ in f_s, $(l-k)^- + l$, will exceed the sum of the powers of z_j in f_s and $p_k(z) z_j^k$, $(l-k)^+ + k$,

and thus annihilate p. We get

$$\begin{aligned}
T^{\mathbb{D}^n}_{f_0+\ldots+f_l}(p) = q_l(z)z_j^l &+ \sum_{\beta=0}^{k-1} T^{\mathbb{D}^n}_{f_0}(p_\beta(z)z_j^\beta) \\
&+ q_{l-1}(z)z_j^{l-1} - T^{\mathbb{D}^n}_{f_0}(p_{k-1}(z)z_j^{k-1}) + \sum_{\beta=0}^{k-1} T^{\mathbb{D}^n}_{f_1}(p_\beta(z)z_j^\beta) \\
&\vdots \\
&+ q_{l-s}(z)z_j^{l-s} - \sum_{\alpha=0}^{s-1} T^{\mathbb{D}^n}_{f_\alpha}(p_{k-s+\alpha}(z)z_j^{k-s+\alpha}) + \sum_{\beta=0}^{k-1} T^{\mathbb{D}^n}_{f_s}(p_\beta(z)z_j^\beta) \\
&\vdots \\
&+ q_0(z) - \sum_{\alpha=0}^{l-1} T^{\mathbb{D}^n}_{f_\alpha}(p_{k-l+\alpha}(z)z_j^{k-l+\alpha}) \\
= q_l(z)z_j^l &+ q_{l-1}z_j^{l-1} + \ldots + q_1(z)z_j + q_0(z) = q(z),
\end{aligned}$$

and thus the desired symbol is $\varphi(z,\overline{z}) = f_0 + \ldots + f_l$. It is important to note that the sums in the equation above containing the "error" terms start containing zero terms once a step s is reached such that $l-k-s<0$, and so the algorithm will eventually stop generating error terms, allowing it to end. This can be seen as a consequence of Lemma 2.1, and is not represented in the above equation for the ease of notation. □

In the proof of Theorem 2.2, we only utilize the fact that $p_k(z)$ is a monomial in $z_1,\ldots,\widehat{z}_j,\ldots,z_n$, which allows us to prove a more general case:

Corollary 3.1 (Generalization to polynomials with monomial coefficient)**.** *Let p,q be (holomorphic) polynomials in variables $z_1,\ldots,z_n$ and suppose that $p(z)\not\equiv 0$, and there is some j such that p may be written as*

$$p(z) = p_0(z_1,\ldots,\widehat{z}_j,\ldots,z_n) + p_1(z_1,\ldots,\widehat{z}_j,\ldots,z_n)z_j + \ldots + p_k(z_1,\ldots,\widehat{z}_j,\ldots,z_n)z_j^k,$$

where only $p_k(z)$ is a monomial in $z_1,\ldots,\widehat{z}_j,\ldots,z_n$. Then there is a polynomial $\varphi(z,\overline{z})$ such that $T^{\mathbb{D}^n}_\varphi(p) = q$, where $T^{\mathbb{D}^n}_\varphi$ is a Toeplitz operator on the polydisc $\mathbb{D}^n$.

P r o o f of Corollary 3.1. The algorithm used in proving Theorem 2.2 can also be applied here. □

Remark 3.1. The algorithm behaves differently when applied on $\mathbb{D}$ as compared to $\mathbb{D}^n$. When applied on $\mathbb{D}$, the algorithm generates only one symbol, which is non-unique in a trivial way (refer to Remark 2.1). However, in the multivariable case, such as in Example 3.1 on $\mathbb{D}^2$, the algorithm produces two symbols. This highlights the significance of selecting the correct variable to guide the algorithm.

We provide an example utilizing the algorithm presented in the proof of Theorem 2.2. The polynomial class considered in Corollary 3.1 is used.

Example 3.1. (Application of Corollary 3.1 for two complex variables). We will apply Corollary 3.1 to the polynomials $p(z) = z_1^2 z_2 + z_1 z_2^2$ and $q(z) = -z_1 + z_1^2 z_2 - z_2^3$. We will perform this computation in two ways to show that the generated symbol is not unique. First, we establish notation similar to that of Corollary 3.1:

$$q(z) = -z_1 + z_1^2 z_2 - z_2^3, \quad q_0(z) = -z_1, \quad q_1(z) = z_1^2, \quad q_2(z) = 0, \quad q_3(z) = -1,$$
$$p(z) = z_1^2 z_2 + z_1 z_2^2, \quad p_0(z) = 0, \quad p_1(z) = z_1^2, \quad p_2(z) = z_1.$$

We first wish to find f_0 such that $T_{f_0}^{\mathbb{D}^n}(z_1 z_2^2) = -z_2^2$, and so look for $f_0 = c_0 z_2 \overline{z}_1$. We have

$$T_{f_0}^{\mathbb{D}^n}(z_1 z_2^2) = c_0 T_{\overline{z}_1}^{\mathbb{D}^n}(z_1 z_2^3) = \frac{1}{2} c_0 z_2^3 = -z_2^3, \quad \text{which implies} \quad c_0 = -2.$$

Calculating our error term, we have

$$T_{f_0}^{\mathbb{D}^n}(z_1^2 z_2) = -2 T_{\overline{z}_1}^{\mathbb{D}^n}(z_1^2 z_2^2) = -\frac{4}{3} z_1 z_2^2.$$

We now wish to find f_1 such that $T_{f_1}^{\mathbb{D}^n}(z_1 z_2^2) = q_2(z) z_2^2 - T_{f_0}^{\mathbb{D}^n}(z_1^2 z_2) = \frac{4}{3} z_1 z_2^2$, and thus look for $f_1 = c_1$. We have $T_{f_1}^{\mathbb{D}^n}(z_1 z_2^2) = c_1 z_1 z_2^2$, so $c_1 = \frac{4}{3}$. Again, calculating our error term, we get $T_{f_1}^{\mathbb{D}^n}(z_1^2 z_2) = \frac{4}{3} z_1^2 z_2$. For the second step, we want f_2 such that $T_{f_2}^{\mathbb{D}^n}(z_1 z_2^2) = q_1(z) z_2 - T_{f_1}^{\mathbb{D}^n}(z_1^2 z_2) = -\frac{1}{3} z_1^2 z_2$, and hence look for $f_2 = c_2 z_1 \overline{z}_2$. So,

$$T_{f_2}^{\mathbb{D}^n}(z_1 z_2^2) = c_2 T_{\overline{z}_2}^{\mathbb{D}^n}(z_1^2 z_2^2) = \frac{2}{3} c_2 z_1^2 z_2 = -\frac{1}{3} z_1^2 z_2, \quad \text{which implies } c_2 = -\frac{1}{2}.$$

Calculating our final error term, we have

$$T_{f_2}^{\mathbb{D}^n}(z_1^2 z_2) = -\frac{1}{2} T_{\overline{z}_2}^{\mathbb{D}^n}(z_1^3 z_2) = -\frac{1}{4} z_1^3.$$

Now, we want to find f_3 such that $T_{f_3}^{\mathbb{D}^n}(z_1 z_2^2) = q_0(z) - T_{f_2}^{\mathbb{D}^n}(z_1^2 z_2) = \frac{1}{4} z_1^3 - z_1$, so we look for $f_3 = (c_{3,1} z_1^2 + c_{3,2}) \overline{z}_2^2$. We have

$$T_{f_3}^{\mathbb{D}^n}(z_1 z_2^2) = c_{3,1} T_{\overline{z}_2^2}^{\mathbb{D}^n}(z_1^3 z_2^2) + c_{3,1} T_{\overline{z}_2^2}^{\mathbb{D}^n}(z_1 z_2^2) = \frac{1}{3} c_{3,1} z_1^3 + \frac{1}{3} c_{3,2} z_1 = \frac{1}{4} z_1^3 - z_1,$$

which implies $c_{3,1} = \frac{3}{4}$ and $c_{3,2} = -3$. Hence, the desired symbol is

$$\varphi(z, \overline{z}) = f_0 + f_1 + f_2 + f_3 = -2 z_2 \overline{z}_1 + \frac{4}{3} - \frac{1}{2} z_1 \overline{z}_2 + \Big(\frac{3}{4} z_1^2 - 3 \Big) \overline{z}_2^2.$$

Plugging φ into the Toeplitz operator and applying it to p, we see that it yields the desired result:

$$\begin{aligned} T_{\varphi}^{\mathbb{D}^n}(p) = & -z_2^3 - \frac{4}{3} z_1 z_2^2 + \frac{4}{3} z_1 z_2^2 + \frac{4}{3} z_1^2 z_2 - \frac{1}{3} z_1^2 z_2 - \frac{1}{4} z_1^3 + \frac{1}{4} z_1^3 - z_1 \\ = & -z_2^3 + z_1^2 z_2 - z_1 = q(z). \end{aligned}$$

Now, we begin calculations for a different symbol that yields the same result:

$$\begin{aligned} q(z) &= -z_2^3 - z_1 + z_2 z_1^2, & p(z) &= z_2^2 z_1 + z_2 z_1^2, \\ q_0(z) &= -z_1, & p_0(z) &= 0, \\ q_1(z) &= z_1^2, & p_1(z) &= z_1^2, \\ q_2(z) &= 0, & p_2(z) &= z_1, \\ q_3(z) &= -1. \end{aligned}$$

Since the leading terms in p and q are equal, it is easy to see that $f_0 = 1$, and thus $T_{f_0}^{\mathbb{D}^n}(z_2^2 z_1) = z_2^2 z_1$. We now wish to find f_1 such that $T_{f_1}^{\mathbb{D}^n}(z_2 z_1^2) = -z_1 - z_2^2 z_1$, and we look for f_1 of the form $f_1 = (c_{1,1}\overline{z}_2 + c_{1,2} z_2)\overline{z}_1$. Then

$$T_{f_1}^{\mathbb{D}^n}(z_2 z_1^2) = \frac{1}{3} c_{1,1} z_1 + \frac{2}{3} c_{1,2} z_2^2 z_1, \quad \text{which implies} \quad c_{1,1} = -3,\ c_{1,2} = -\frac{3}{2},$$

and we have

$$T_{f_1}^{\mathbb{D}^n}(z_2^2 z_1) = -z_2 - \frac{3}{4} z_2^3.$$

Finally, we wish to find f_2 such that $T_{f_2}^{\mathbb{D}^n}(z_2 z_1^2) = z_2 - \frac{1}{4} z_2^3$, and look for $f_2 = (c_{2,1} + c_{2,2} z_2^2)\overline{z}_1^2$. We have

$$T_{f_2}^{\mathbb{D}^n}(z_2 z_1^2) = \frac{1}{3} c_{2,1} z_2 + \frac{1}{3} c_{2,2} z_2^3 = z_2 - \frac{1}{4} z_2^3, \quad \text{which implies} \quad c_{2,1} = 3,\ c_{2,2} = -\frac{3}{4}.$$

Hence,

$$\varphi(z, \overline{z}) = 1 + \Big(-3\overline{z}_2 - \frac{3}{2} z_2\Big)\overline{z}_1 + \Big(3 - \frac{3}{4} z_2^2\Big)\overline{z}_1^2,$$

which differs from the one found previously (e.g. the constant terms are not equal).

The version presented in Corollary 3.1 is more robust than the case in Theorem 2.2. However, it is not as general as the case in a single variable Theorem 2.1. It should be noted that even in $\mathbb{C}[z_1, z_2]$, some polynomials cannot be placed in this form, such as $p(z) = z_1^2(z_2^2+1) + z_2^2(z_1^2+1)$. Nevertheless, it is worth mentioning that any nonzero polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$ can be represented as the sum of a polynomial of the described form in Corollary 3.1 and monomials in z that have a degree k in z_j. That is, for any polynomial $p \in \mathbb{C}[z_1, \ldots, z_n]$, one can select an index $1 \leqslant j \leqslant n$ (appearing to maximum power k) and split p up as a polynomial in z_j, where the coefficient of z_j^k is a monomial in the other indices and the coefficients of lower powers are allowed to be polynomials, plus other terms z_j^k whose coefficients are monomials in the other indices. For example: $p(z) = z_1 z_2 + z_2 z_3 + z_1^2 z_2 z_3 + z_1 z_3^2 + z_2^2 z_3^2 + z_1 z_2 z_3^3 + z_1 z_2^2 z_3^3$, selecting index $j = 3$. Then $p(z) = z_1 z_2 + (z_2 + z_1^2 z_2) z_3 + (z_1 + z_2^2) z_3^2 + z_1 z_2 z_3^3 + z_1 z_2^2 z_3^3$. The key point is that one can split p into the form described in Corollary 3.1, plus a few extra terms which are monomials with z_j^k.

4. On generalization of the excess area

Let Ω be an open subset of the complex plane $\mathbb{C}$. Consider a function $h\colon \Omega \to \mathbb{C}$ that is complex analytic on Ω. We define the area of the image of h, denoted by $\mathrm{Area}(h)$, as the L^2 norm of the partial derivative of h with respect to z over Ω. The definition incorporates the concept of counting multiplicity, and it coincides with the expected value when h is a complex analytic and injective function. In such cases, the area of $h(\Omega)$ is equal to $\|\partial h/\partial z\|_{L^2(\Omega)}$, as noted in [9], Lemma 4.1. Suppose that h is complex analytic near z_0. If the function $h(z)-h(z_0)$ has a zero of order m, then there is a (deleted) neighborhood of z_0, where the equation $f(z)=w$ has precisely m solutions, for w sufficiently close to $h(z_0)$, see [1]. By dividing Ω into sets on which h has constant multiplicity, we can justify the definition of $\mathrm{Area}(h)$.

A simple observation is made in [9] by relating L^2-norm of h to the l^2-norm of the Taylor coefficients of $h \in A^{1,2}(\mathbb{D})$, $(h(z) = \sum\limits_{k=0}^{\infty} h_k z^k$ on $\mathbb{D})$:

$$\mathrm{Area}(zh) - \mathrm{Area}(h) = \Big\|\frac{\partial(zh)}{\partial z}\Big\|_{L^2(\mathbb{D})} - \Big\|\frac{\partial h}{\partial z}\Big\|^2_{L^2(\mathbb{D})} = \pi \sum_{k=0}^{\infty} |h_k|^2. \tag{4.1}$$

When h extends continuously to the circle, let Sh represent the restriction of h to the unit circle. Then for $h \in A^{1,2}(\mathbb{D})$ we have $\pi\|Sh\|^2_{L^2(\partial\mathbb{D})} = \frac{1}{2}\int_0^{2\pi} |h(\mathrm{e}^{\mathrm{i}\theta})|^2\,\mathrm{d}\theta = \frac{1}{2}\sum\limits_{k=0}^{\infty} |h_k|^2 \int_0^{2\pi} |\mathrm{e}^{\mathrm{i}k\theta}|^2\,\mathrm{d}\theta = \pi \sum\limits_{k=0}^{\infty} |h_k|^2$. The measure used on the unit circle is normalized: $\mathrm{d}\theta/2\pi$. The area difference between the images of the unit disc under holomorphic function h and its modification with M_z has a simple geometric interpretation.

Theorem 4.1 ([9], Corollary 4.2)**.** *Let Sh be the restriction of h to the unit circle for $h \in A^{1,2}(\mathbb{D})$. Then Sh is square integrable on the unit circle and*

$$\|DMh\|^2_{L^2(\mathbb{D})} - \|Dh\|^2_{L^2(\mathbb{D})} = \pi\|Sh\|^2_{L^2(\partial\mathbb{D})} \quad \text{for } h \in A^{1,2}(\mathbb{D}). \tag{4.2}$$

In a recent paper (see [2]), some modifications to the excess area growth identity (4.2) were explored. Corollary 4.1 and Proposition 4.1 are built as variations of identity (4.1). Corollary 4.1 is an application of Theorem 2.1, Corollary 2.1, and Proposition 2.1.

Corollary 4.1. *Let h be a holomorphic function in a region that is simply connected and contains the closed unit disc $\overline{\mathbb{D}}$. Assume that h has finitely many zeros within $\mathbb{D}$. Then, h can be expressed as a product of a polynomial p and a holomorphic function f, where f has no zeros within $\overline{\mathbb{D}}$. Moreover, there exist symbols φ_{zp}, φ_p (both in $\mathbb{C}[z,\overline{z}]$), and a holomorphic symbol φ_f such that*

$$\|T^{\mathbb{D}}_f T^{\mathbb{D}}_{\varphi_{zp}+\varphi_f}(zp)\|^2_{L^2(\mathbb{D})} - \|T^{\mathbb{D}}_f T^{\mathbb{D}}_{\varphi_p+\varphi_f}(p)\|^2_{L^2(\mathbb{D})} = \pi\|Sh\|^2_{L^2(\partial\mathbb{D})}.$$

P r o o f of Corollary 4.1. Suppose h is holomorphic on $\mathbb{D}$ and has the zeros $\{z_l\}_{l=1}^n$ with different orders. From the power series expansion of h, it follows that $p(z) := \prod_{l=1}^{n}(z-z_l)^{m_l}$ is a factor of h, and $h = pf$, where f does not vanish on $\overline{\mathbb{D}}$. Then, by Corollary 2.1 there are $\varphi_{zp}, \varphi_p \in \mathbb{C}[z,\overline{z}]$ such that $T^{\mathbb{D}}_{\varphi_{zp}}(zp) = \partial(zp)/\partial z$ and $T^{\mathbb{D}}_{\varphi_p}(p) = \partial p/\partial z$; by Proposition 2.1 there is a holomorphic symbol φ_f such that $T^{\mathbb{D}}_{\varphi_f}(f) = \partial f/\partial z$. Then

(4.3)
$$\begin{aligned}\operatorname{Area}(zh) - \operatorname{Area}(h) &= \Big\|\frac{\partial(zh)}{\partial z}\Big\|^2_{L^2(\mathbb{D})} - \Big\|\frac{\partial h}{\partial z}\Big\|^2_{L^2(\mathbb{D})} = \Big\|\frac{\partial(zpf)}{\partial z}\Big\|^2_{L^2(\mathbb{D})} - \Big\|\frac{\partial(pf)}{\partial z}\Big\|^2_{L^2(\mathbb{D})}\\ &= \Big\|f\frac{\partial(zp)}{\partial z} + (zp)\frac{\partial f}{\partial z}\Big\|^2_{L^2(\mathbb{D})} - \Big\|f\frac{\partial p}{\partial z} + p\frac{\partial f}{\partial z}\Big\|^2_{L^2(\mathbb{D})}\\ &= \|fT^{\mathbb{D}}_{\varphi_{zp}}(zp) + (zp)T^{\mathbb{D}}_{\varphi_f}(f)\|^2_{L^2(\mathbb{D})} - \|fT^{\mathbb{D}}_{\varphi_p}(p) + pT^{\mathbb{D}}_{\varphi_f}(f)\|^2_{L^2(\mathbb{D})}.\end{aligned}$$

Since φ_f is holomorphic, the second term in the first norm after the equality, we have $zpT^{\mathbb{D}}_{\varphi_f}(f) = zpP^{\mathbb{D}}(\varphi_f f) = zp(\varphi_f f) = f(\varphi_f zp) = fP^{\mathbb{D}}(\varphi_f zp)$. Similarly, the second term in the second norm after the equality, we have $pT^{\mathbb{D}}_{\varphi_f}(f) = fP^{\mathbb{D}}(\varphi_f p)$. Thus, the right hand side of equation (4.3) becomes

$$\begin{aligned}(4.4)\qquad &\|fP^{\mathbb{D}}(\varphi_{zp}zp) + fP^{\mathbb{D}}(\varphi_f zp)\|^2_{L^2(\mathbb{D})} - \|fP^{\mathbb{D}}(\varphi_p p) + fP^{\mathbb{D}}(\varphi_f p)\|^2_{L^2(\mathbb{D})}\\ &\quad= \|fP^{\mathbb{D}}(\varphi_{zp}zp + \varphi_f zp)\|^2_{L^2(\mathbb{D})} - \|fP^{\mathbb{D}}(\varphi_p p + \varphi_f p)\|^2_{L^2(\mathbb{D})}\\ &\quad= \|fT^{\mathbb{D}}_{\varphi_{zp}+\varphi_f}(zp)\|^2_{L^2(\mathbb{D})} - \|fT^{\mathbb{D}}_{\varphi_p+\varphi_f}(p)\|^2_{L^2(\mathbb{D})}\\ &\quad= \|T^{\mathbb{D}}_f T^{\mathbb{D}}_{\varphi_{zp}+\varphi_f}(zp)\|^2_{L^2(\mathbb{D})} - \|T^{\mathbb{D}}_f T^{\mathbb{D}}_{\varphi_p+\varphi_f}(p)\|^2_{L^2(\mathbb{D})}.\end{aligned}$$

The equality between (4.4) and $\pi\|Sh\|^2_{L^2(\partial\mathbb{D})}$ follows from identity (4.1) and the discussion in the following paragraph. □

Next, we continue generalizing identity (4.2). To do this, we consider Toeplitz operators with antiholomorphic symbols, φ, to generalize the holomorphic function h in (4.2). In the proof, we need to verify that the derivative of the Toeplitz operator remains square-integrable on the unit disc. This can be found in [6], Theorem 6.2.2. In addition, in subsection 6.1 of the appendix, we offer an elementary proof for the regularity of the Toeplitz operator with a symbol $\bar{\varphi}$, where $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$, acting on a function $h \in A^{1,2}(\mathbb{D})$; no originality is claimed.

Proposition 4.1. *Let $h \in A^{1,2}(\mathbb{D})$ and $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. Then*

$$(4.5)\qquad \Big\|\frac{\partial}{\partial z}(T^{\mathbb{D}}_z T^{\mathbb{D}}_{\bar{\varphi}}h)\Big\|^2_{L^2(\mathbb{D})} - \Big\|\frac{\partial}{\partial z}(T^{\mathbb{D}}_{\bar{\varphi}}(h))\Big\|^2_{L^2(\mathbb{D})} = \pi\|ST^{\mathbb{D}}_{\bar{\varphi}}ht\|^2_{L^2(\partial\mathbb{D})}.$$

P r o o f of Proposition 4.1. Let $h \in A^{1,2}(\mathbb{D})$ and $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$, then $T^{\mathbb{D}}_{\bar{\varphi}} h = P^{\mathbb{D}}(\bar{\varphi} h)$. Bergman projection on the unit disc maps $A^{1,2}(\mathbb{D})$ to itself, see [6], Theorem 6.2.2. An elementary proof of the case can be found in Subsection 6.1 of the appendix. Equation (4.5) then follows by replacing the function h in equation (4.2) with $T^{\mathbb{D}}_{\bar{\varphi}} h$. □

Let us point out that in Chapter 5 of [9], D'Angelo proves a much deeper generalization of identity (4.2) for certain maps of the unit ball in $\mathbb{C}^n$.

5. Remarks

Consider the Hankel operator with a symbol $\varphi \in L^\infty(\mathbb{D})$, defined as $H^{\mathbb{D}}_{\varphi} = (I - P^{\mathbb{D}}) M_\varphi \colon A^2(\mathbb{D}) \to L^2(\mathbb{D})$, where $M_\varphi \colon A^2(\mathbb{D}) \to L^2(\mathbb{D})$. There is a connection between Toeplitz and Hankel operators,

$$T^{\mathbb{D}}_{\varphi_1} T^{\mathbb{D}}_{\varphi_2} = T^{\mathbb{D}}_{\varphi_1 \varphi_2} - (H^{\mathbb{D}}_{\bar{\varphi}_1})^* H^{\mathbb{D}}_{\varphi_2}, \tag{5.1}$$

where $H^{\mathbb{D}}_{\bar{\varphi}_1}$ and $H^{\mathbb{D}}_{\varphi_2}$ are Hankel operators with symbols $\bar{\varphi}_1, \varphi_2 \in L^\infty(\mathbb{D})$, respectively. The connection is also true for more general domains.

A relevant question to consider is whether Theorem 2.1 can be extended to a broader class of holomorphic functions on the unit disc. For instance, can all $A^2(\mathbb{D})$ functions be included in the theorem instead of just polynomials? We wonder whether the connection between Toeplitz and Hankel operators in (5.1) can be used to extend the results to a wider class of holomorphic functions than polynomials. For example, when the symbol φ_2 is holomorphic on the domain, then $T^{\mathbb{D}}_{\varphi_1} T^{\mathbb{D}}_{\varphi_2} = T^{\mathbb{D}}_{\varphi_1 \varphi_2}$ since $(H^{\mathbb{D}}_{\bar{\varphi}_1})^* H^{\mathbb{D}}_{\varphi_2}$ vanishes. Alternatively, is it possible to approximate a Toeplitz operator with an antiholomorphic symbol that acts on $A^2(\mathbb{D})$ functions using a sequence of Toeplitz operators with symbols generated, as in the proof of Theorem 2.1, for polynomials?

Remark 5.1. Let $h(z) = \mathrm{e}^{z^2}$. Then as in Proposition 2.1 the symbol $\phi_h(z) = 2z$ gives

$$(T^{\mathbb{D}}_{\phi_h}(h))(z) = (T^{\mathbb{D}}_{2z}(h))(z) = T^{\mathbb{D}}_{2z}(\mathrm{e}^{z^2}) = P^{\mathbb{D}}(2z\mathrm{e}^{z^2}) = 2z\mathrm{e}^{z^2} = h'(z).$$

Based on the algorithm in Theorem 2.1 for a polynomial $p(z) = p_0 + p_1 z + \ldots + p_n z^n$ we construct a symbol ϕ_p that satisfies $T^{\mathbb{D}}_{\phi_p}(p) = \partial p / \partial z$ as

$$\phi_p(z) = (n+1)\overline{z} - \Big(\frac{(n+1)p_{n-1}}{np_n} \overline{z}^2 + \frac{(n-1)^2}{n^2 p_n^2} (n^2 p_{n-2} p_n - (n-1)^2 p_{n-1}^2) \Big). \tag{5.2}$$

For $h(z) = \sum_{j=0}^{\infty} z^{2j}/j!$, consider the partial sum $S_n(h)(z) = \sum_{j=0}^{n} z^{2j}/j!$ and the derivative of its partial sum $\partial/\partial z(S_n(h)(z)) = \sum_{j=0}^{n-1} 2z^{2j+1}/j!$. To construct a symbol $\phi_{S_n(h)}$ which satisfies

$$(T^{\mathbb{D}}_{\phi_{S_n(h)}}(S_n(h)))(z) = \frac{\partial}{\partial z}(S_n(h)(z))$$

we use (5.2), which yields the symbol

$$\phi_{S_n(h)}(z) = (2n+1)\overline{z} - \Big(\Big(n+\frac{1}{2}\Big)\overline{z}^2 - n^2 + n - \frac{1}{4}\Big).$$

It can be shown that as n approaches infinity, the norm of the difference between $S_n(h)$ and h in the $A^2(\mathbb{D})$ space tends to zero. Similarly, the norm of the difference between the partial derivative of $S_n(h)$ with respect to z and the partial derivative of h with respect to z in the $A^2(\mathbb{D})$ space also tends to zero. However, it should be noted that the sequence $\{\phi_{S_n(h)}(z)\}_{n\in\mathbb{N}}$ is not a Cauchy sequence in the sup-norm on the unit disc, so $\{\phi_{S_n(h)}(z)\}_{n\in\mathbb{N}}$ does not converge in the sup-norm on the unit disc.

Our work in this paper primarily focuses on the unit disc, except Theorem 2.2. In this context, $\mathbb{D}$ is a domain of convergence and $\|z^k\|^2_{L^2(\mathbb{D})} = \pi/(k+1)$. However, if we move to an annulus, we need to include nonnegative degree monomials in our work. Ravisankar and Zeytuncu have investigated the Bergman projection of conjugate holomorphic functions on a class of domains, including arbitrary annulus domains, as detailed in Section 2.B of [17]. Thus, it would be interesting to further explore variations of identity (4.2) and the idea in Theorem 2.1 in annulus domains.

It would be interesting to explore the generalization of extending generating polynomials as stated in Corollary 3.1. This could be achieved by removing the restriction that the highest degree coefficient must be a monomial. Moreover, some of the results may be generalized to an n-ball or any domain Ω in $\mathbb{C}^n$, where the monomials form a complete orthonormal set for $A^2(\Omega)$.

6. Appendix

6.1. On the regularity of the Toeplitz operator on the unit disc. In this section, we discuss the preservation of 1-Sobolev square-integrable holomorphic functions on the unit disc under Toeplitz operators with a symbol $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. This is equivalent to studying the regularity of the Bergman projection, which has been previously investigated for a more general class of domains in $\mathbb{C}^n$, see, for example, [5], [18]. For Bergman projection of conjugate holomorphic functions on Reinhardt domains, refer to [17] and the references in it. The regularity of the Bergman projection has been extensively studied as it plays a crucial role in determining the boundary regularity of holomorphic mappings, see [3], [4], [10].

We are not claiming any new results in this section. The calculations are elementary since we are working on the unit disk.

Let $h \in A^{1,2}(\mathbb{D})$ and $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. We will show that $T^{\mathbb{D}}_{\bar{\varphi}} h \in A^{1,2}(\mathbb{D})$. To bound the derivatives of $T^{\mathbb{D}}_{\bar{\varphi}} h$, we use the convergence of the Taylor coefficients of h and its derivatives, which implies that the $A^{1,2}(\mathbb{D})$-norm of $T^{\mathbb{D}}_{\bar{\varphi}} h$ is finite.

Lemma 6.1. *Let $h \in A^{1,2}(\mathbb{D})$ and $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. Then $T^{\mathbb{D}}_{\bar{\varphi}} h \in A^{1,2}(\mathbb{D})$.*

We have mainly only been able to consider sequences of Taylor coefficients of h that are of the form $|h_k|^2$ due to the convergence facts that the following relation yields:

$$\left\| \frac{\partial h}{\partial z} \right\|^2_{L^2(\mathbb{D})} = \pi \sum_{k=1}^{\infty} k |h_k|^2 < \infty. \tag{6.1}$$

In addition to useful convergence facts, the above equality also shows the relation between the $L^2(\mathbb{D})$-norm of ∂h and the l^2-norm of the Taylor coefficients of h. In particular, $h \in A^{1,2}(\mathbb{D})$ implies that the sequence of Taylor coefficients of h is in l^2. As we attempt to generalize equation (6.1) to $h \in A^{k,2}(\mathbb{D})$ for $k \in \mathbb{N} \setminus \{1\}$, we hope to relate the $L^2(\mathbb{D})$-norm of $\partial^k h / \partial z^k$ to the series of the Taylor coefficients in l^p for $p > 2$.

To prove that Toeplitz operators, with a symbol $\varphi \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$, preserve 1-Sobolev square-integrable holomorphic property on the unit disc, we introduce some tools. We will then demonstrate how these tools can be applied to our proof.

Proposition 6.1 ([12], Theorem 2.15)**.** *If $(f_n \colon \mathbb{C} \to [0, \infty])$ is a finite or infinite sequence of Lebesgue integrable functions and $f = \sum\limits_n f_n$, then $\int f = \sum\limits_n \int f$.*

Proposition 6.2 ([15], Corollary 12.33)**.** *Let $(f_n)_{n=1}^{\infty}$ be a sequence of complex-valued $\mathrm{d}A$-measurable functions on $\mathbb{D}$ such that $\sum\limits_{n=1}^{\infty} \int_{\mathbb{D}} |f_n| \, \mathrm{d}A < \infty$. Then*

$$\left| \int_{\mathbb{D}} \sum_{n=1}^{\infty} f_n \, \mathrm{d}A \right| < \infty \quad \text{and} \quad \int_{\mathbb{D}} \sum_{n=1}^{\infty} f_n \, \mathrm{d}A = \sum_{n=1}^{\infty} \int_{\mathbb{D}} f_n \, \mathrm{d}A.$$

The following Lemma 6.2 is used in proving Proposition 6.3.

Lemma 6.2. *Suppose $h \in A^{1,2}(\mathbb{D})$ and let h_k denote its Taylor coefficients. Then the series $\sum\limits_{k=1}^{\infty} |h_k| / \sqrt{k+1}$ converges.*

P r o o f of Lemma 6.2. Suppose $h \in A^{1,2}(\mathbb{D})$ and let h_k denote its Taylor coefficients. By equation (6.1), we know that $\sum_{k=1}^{\infty}(1+k)|h_k|^2$ is finite since it can be bounded above by a multiple of $\|\partial h/\partial z\|^2_{L^2(\mathbb{D})}$ as follows:

$$\frac{2}{\pi}\Big\|\frac{\partial h}{\partial z}\Big\|^2_{L^2(\mathbb{D})} = 2\sum_{k=1}^{\infty} k|h_k|^2 \geqslant \sum_{k=1}^{\infty}(1+k)|h_k|^2 = \sum_{k=2}^{\infty} k|h_{k-1}|^2.$$

Then we can apply

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{|h_k|}{\sqrt{k+1}} &= \sum_{k=1}^{\infty}\frac{\sqrt{k+1}|h_k|}{k+1} \leqslant \Big(\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\Big)\Big(\sum_{k=1}^{\infty} k|h_k|^2\Big)\\ &= \Big(\sum_{k=1}^{\infty}\frac{1}{(k+1)^2}\Big)\|\partial h\|^2\end{aligned}$$

and obtain

$$\sum_{k=2}^{\infty}\frac{\sqrt{k|h_{k-1}|}}{k} = \sum_{k=2}^{\infty}\frac{|h_{k-1}|}{\sqrt{k}} = \sum_{k=1}^{\infty}\frac{|h_k|}{\sqrt{k+1}}.$$

□

Proposition 6.3 (Proof of applicability)**.** *Let $h \in A^{1,2}(\mathbb{D})$ and $f \in \mathcal{O}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$. Then*

$$P^{\mathbb{D}}(h(w)\overline{f(w)}) = P^{\mathbb{D}}\Big(\sum_{k=0}^{\infty} h_k w^k\overline{f(w)}\Big) = \sum_{k=0}^{\infty} P^{\mathbb{D}}(h_k w^k\overline{f(w)}).$$

P r o o f of Proposition 6.3. Let $h \in A^{1,2}(\mathbb{D})$ and $f \in \mathcal{O}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$. It will suffice to show the sufficient conditions stated within Proposition 6.2 for all $z \in \mathbb{D}$, i.e.,

$$\sum_{k=0}^{\infty}\int_{\mathbb{D}} |K_{\mathbb{D}}(z,w)h_k w^k\overline{f(w)}|\,\mathrm{d}A(w) < \infty$$

for any fixed $z \in \mathbb{D}$. By direct computation, we can see that

$$\begin{aligned}&\sum_{k=1}^{\infty}\int_{\mathbb{D}} |K_{\mathbb{D}}(z,w)h_k w^k\overline{f(w)}|\,\mathrm{d}A(w)\\ &\quad\leqslant \sum_{k=1}^{\infty}|h_k|\int_{\mathbb{D}}|K_{\mathbb{D}}(z,w)w^k|\,\mathrm{d}A(w) && |\overline{f(w)}| \text{ is bounded},\\ &\quad\leqslant \sum_{k=1}^{\infty}|h_k|\Big(\int_{\mathbb{D}}|w^k|^2\,\mathrm{d}A(w)\Big)^{1/2}\Big(\int_{\mathbb{D}}|K_{\mathbb{D}}(z,w)|^2\,\mathrm{d}A(w)\Big)^{1/2}\\ &&& \text{Cauchy-Schwarz inequality},\\ &\quad= \sqrt{\pi}\sum_{k=1}^{\infty}\frac{|h_k|}{\sqrt{k+1}}\Big(\int_{\mathbb{D}}|K_{\mathbb{D}}(z,w)|^2\,\mathrm{d}A(w)\Big)^{1/2} && \text{direct calculation},\end{aligned}$$

$$\leqslant \sqrt{\pi}C\left(\int_{\mathbb{D}} |K_{\mathbb{D}}(z,w)|^2 \,\mathrm{d}A(w)\right)^{1/2} \qquad \text{by Lemma 6.2, where } C \in [0,\infty),$$

$$= \sqrt{\pi}C(K_{\mathbb{D}}(z,z))^{1/2} = \frac{C}{1-|z|^2} \qquad \text{by definition (1.1) of } K_{\mathbb{D}}(z,w).$$

Therefore, by Proposition 6.2 we have that

$$P^{\mathbb{D}}\left(\sum_{k=0}^{\infty} h_k w^k \overline{f(w)}\right) = \sum_{k=0}^{\infty} P^{\mathbb{D}}(h_k w^k \overline{f(w)}).$$

□

Now, we are ready to prove that $A^{1,2}(\mathbb{D})$ functions are preserved under a Toeplitz operator with a symbol $\varphi \in \mathcal{O} \cap C^1(\overline{\mathbb{D}})$. To begin, we calculate $T^{\mathbb{D}}_{\overline{f}}(h)$ along with its derivative, using the Taylor series of h and f. We then establish an upper bound on the $A^{1,2}(\mathbb{D})$-norm of $T^{\mathbb{D}}_{\overline{f}}(h)$ through the $L^2(\mathbb{D})$-norms of $T^{\mathbb{D}}_{\overline{f}}(h)$ and $(\partial/\partial z)(T^{\mathbb{D}}_{\overline{f}}h)$, which implies that $T^{\mathbb{D}}_{\overline{f}}(h) \in A^{1,2}(\mathbb{D})$.

In order to verify the accuracy of our computations and bounds, we make use of Lemma 2.1 as well as the previous three propositions: 6.1, 6.2, and 6.3.

P r o o f of Lemma 6.1. Let $h \in A^{1,2}(\mathbb{D})$ and $f \in \mathcal{O}(\mathbb{D}) \cap C^1(\overline{\mathbb{D}})$. Then we can find the series expansion of $T_{\overline{f}}(h)$ as follows.

$$(6.2)\quad T_{\overline{f}}(h) = P^{\mathbb{D}}(\overline{f}h) = P^{\mathbb{D}}\left(\sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_k z^k \overline{f}_l \overline{z}^l\right) \qquad \text{expanding } h \text{ and } \overline{f} \text{ through their Taylor series,}$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_k \overline{f}_l P^{\mathbb{D}}(z^k \overline{z}^l) \qquad \text{by Proposition 6.3 and since the Taylor series of } f \text{ converges uniformly on } \overline{\mathbb{D}}\,,$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} h_k \overline{f}_l T^{\mathbb{D}}_{\overline{z}^l}(z^k) \qquad \text{by the definition of } T^{\mathbb{D}},$$

$$= \sum_{k=0}^{\infty}\sum_{l=0}^{k} h_k \overline{f}_l \frac{k-l+1}{k+1} z^{k-l} \qquad \text{by Lemma 2.1 that implies } T_{\overline{z}^l}(z^k) = 0 \text{ for } l > k.$$

Now, calculating the derivative of equation (6.2)

$$\frac{\partial}{\partial z}((T^{\mathbb{D}}_{\overline{f}}h)(z)) = \frac{\partial}{\partial z}\sum_{k=0}^{\infty}\sum_{l=0}^{k} h_k \overline{f}_l \frac{k-l+1}{k+1} z^{k-l} \qquad \text{by equation (6.2),}$$

$$= \sum_{k=1}^{\infty}\sum_{l=0}^{k} h_k \overline{f}_l \frac{k-l+1}{k+1}(k-l) z^{k-l-1} \qquad \text{by linearity and Proposition 3.2.5 in [7].}$$

We will now estimate an upper bound for the $L^2(\mathbb{D})$-norm of $T^{\mathbb{D}}_{\overline{f}}(h)$:

$$
\begin{aligned}
(6.3)\ \|T^{\mathbb{D}}_{\overline{f}}(h)\|^2_{L^2(\mathbb{D})} & \\
&= \int_{\mathbb{D}} \Big| \sum_{k=0}^{\infty} \sum_{l=0}^{k} h_k \overline{f}_l \frac{k-l+1}{k+1} z^{k-l} \Big|^2 \,\mathrm{d}A(z) && \text{using (6.2),} \\
&\leqslant \int_{\mathbb{D}} \sum_{k=0}^{\infty} \sum_{l=0}^{k} |h_k|^2 |f_l|^2 \Big[\frac{k-l+1}{k+1} \Big]^2 |z^{k-l}|^2 \,\mathrm{d}A(z) && \text{by the triangle inequality.}
\end{aligned}
$$

Since the summands of equation (6.3) are positive, they satisfy the requirements of Proposition 6.1; thus, the sums and integrals commute:

$$
\begin{aligned}
(6.4) \qquad &= \sum_{k=0}^{\infty} \sum_{l=0}^{k} |h_k|^2 |f_l|^2 \Big[\frac{k-l+1}{k+1} \Big]^2 \int_{\mathbb{D}} |z^{k-l}|^2 \,\mathrm{d}A(z) && \text{by Proposition 6.1,} \\
&\leqslant \pi \sum_{k=0}^{\infty} \sum_{l=0}^{k} \frac{|h_k|^2 |f_l|^2}{k+1} && \text{since } k-l+1 \leqslant k+1.
\end{aligned}
$$

Next, we will estimate an upper bound for the $L^2(\mathbb{D})$-norm of the derivative of $T^{\mathbb{D}}_{\overline{f}}(h)$, which we use the triangle inequality for:

$$
\begin{aligned}
(6.5) \quad \Big\| \frac{\partial}{\partial z} T^{\mathbb{D}}_{\overline{f}}(h) \Big\|^2_{L^2(\mathbb{D})} &= \int_{\mathbb{D}} \Big| \sum_{k=1}^{\infty} \sum_{l=0}^{k} h_k \overline{f}_l \frac{k-l+1}{k+1} (k-l) z^{k-l-1} \Big|^2 \,\mathrm{d}A(z) \\
&\leqslant \int_{\mathbb{D}} \sum_{k=1}^{\infty} \sum_{l=0}^{k} |h_k|^2 |f_l|^2 \Big(\frac{k-l+1}{k+1} \Big)^2 (k-l)^2 |z^{k-l-1}|^2 \,\mathrm{d}A(z).
\end{aligned}
$$

By the same reasoning that follows equation (6.3), we can apply Proposition 6.1:

$$
\begin{aligned}
&\leqslant \sum_{k=1}^{\infty} \sum_{l=0}^{k} |h_k|^2 |f_l|^2 \Big(\frac{k-l+1}{k+1} \Big)^2 (k-l)^2 \int_{\mathbb{D}} |z^{k-l-1}|^2 \,\mathrm{d}A(z) \\
&= \pi \sum_{k=1}^{\infty} \sum_{l=0}^{k} \frac{|h_k|^2 |f_l|^2 (k-l)^2}{k-l} \Big(\frac{k-l+1}{k+1} \Big)^2 \\
&\leqslant \pi \sum_{k=1}^{\infty} \sum_{l=0}^{k} |h_k|^2 |f_l|^2 (k-l).
\end{aligned}
$$

To obtain an upper bound for the $A^{1,2}(\mathbb{D})$-norm of $T^{\mathbb{D}}_{\overline{f}}h$, we use the estimates from equation (6.4):

$$
\begin{aligned}
&\frac{1}{\pi}\|T^{\mathbb{D}}_{\overline{f}(h)}\|^2_{A^{1,2}(\mathbb{D})} \\
&\quad\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{|h_k|^2|f_l|^2}{k+1}+\sum_{k=1}^{\infty}\sum_{l=0}^{k}|h_k|^2|f_l|^2(k-l) && \text{by equations (6.4)},\\
&\quad\leqslant \sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{|h_k|^2|f_l|^2}{k+1}+\sum_{k=1}^{\infty}\sum_{l=0}^{k}k|h_k|^2|f_l|^2 && \text{since } l>0\ \forall\, l\in\mathbb{N},\\
&\quad< |h_0|^2|f_0|^2+2\sum_{k=1}^{\infty}\sum_{l=0}^{k}k|h_k|^2|f_l|^2 && \text{since } 1/(k+1)<k\ \forall\, k\in\mathbb{N},\\
&\quad\leqslant |h_0|^2|f_0|^2+2\sum_{k=1}^{\infty}k|h_k|^2\sum_{l=0}^{\infty}|f_l|^2 && \text{since all } |f_l|^2 \text{ are nonnegative},\\
&\quad\leqslant |h_0|^2|f_0|^2+2\sum_{k=1}^{\infty}k|h_k|^2\Big(|f_0|^2+\sum_{l=1}^{\infty}l|f_l|^2\Big) && \text{since } l\geqslant 1,\\
&\quad= |h_0|^2|f_0|^2+\frac{2}{\pi}\Big\|\frac{\partial h}{\partial z}\Big\|^2_{L^2(\mathbb{D})}\Big(|f_0|^2+\frac{1}{\pi}\Big\|\frac{\partial f}{\partial z}\Big\|^2_{L^2(\mathbb{D})}\Big) && \text{by equation (6.1)},\\
&\quad< \infty && \text{since } \partial h/\partial z, \partial f/\partial z\in L^2(\mathbb{D}).
\end{aligned}
$$

Since $f\in\mathcal{O}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$, we know that $\partial f/\partial z\in L^2(\mathbb{D})$ which implies $\partial h/\partial z$ is bounded and consequentially is in $L^2(\mathbb{D})$. Therefore, $\|T^{\mathbb{D}}_{\overline{f}}h\|_{A^{1,2}(\mathbb{D})}<\infty$, that is $T^{\mathbb{D}}_{\overline{f}}h\in A^{1,2}(\mathbb{D})$ for arbitrary $h\in A^{1,2}(\mathbb{D})$ and $f\in\mathcal{O}(\mathbb{D})\cap C^1(\overline{\mathbb{D}})$. □

Remark 6.1. The above proof can be extended to $A^{k,2}(\mathbb{D})$ for arbitrary $k\in\mathbb{N}$. It is the same as the one for the $A^{1,2}(\mathbb{D})$ version, which is a consequence of the fact that any property that $\partial h/\partial z$ has for $h\in A^{1,2}(\mathbb{D})$, $\partial^k g/\partial z^k$ also has for $g\in A^{k,2}(\mathbb{D})$.

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References

[1] *L. Ahlfors*: Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable. AMS Chelsea Publishing 385. AMS, Providence, 2021. zbl MR

[2] *H. K. Bambico, M. Çelik, S. T. Gross, F. Hall*: Generalization of the excess area and its geometric interpretation. New York J. Math. *28* (2022), 1230–1255. zbl MR

[3] *S. R. Bell*: Biholomorphic mappings and the $\overline{\partial}$-problem. Ann. Math. (2) *114* (1981), 103–113. zbl MR doi

[4] *S. Bell, D. Catlin*: Boundary regularity of proper holomorphic mappings. Duke Math. J. *49* (1982), 385–396. zbl MR doi
[5] *H. P. Boas*: Holomorphic reproducing kernels in Reinhardt domains. Pac. J. Math. *112* (1984), 273–292. zbl MR doi
[6] *S.-C. Chen, M.-C. Shaw*: Partial Differential Equations in Several Complex Variables. AMS/IP Studies in Advanced Mathematics 19. AMS, Providence, 2001. zbl MR doi
[7] *J. B. Conway*: Functions of One Complex Variable. Graduate Texts in Mathematics 11. Springer, Berlin, 1978. zbl MR doi
[8] *J. P. D'Angelo*: Inequalities from Complex Analysis. The Carus Mathematical Monographs 28. Mathematical Association of America, Washington, 2002. zbl MR doi
[9] *J. P. D'Angelo*: Hermitian Analysis: From Fourier Series to Cauchy-Riemann Geometry. Cornerstones. Springer, Cham, 2019. zbl MR doi
[10] *K. Diederich, J. E. Fornaess*: Boundary regularity of proper holomorphic mappings. Invent. Math. *67* (1982), 363–384. zbl MR doi
[11] *G. B. Folland*: Introduction to Partial Differential Equations. Princeton University Press, Princeton, 1995. zbl MR doi
[12] *G. B. Folland*: Real Analysis: Modern Techniques and Their Applications. Pure and Applied Mathematics. John Wiley & Sons, New York, 1999. zbl MR
[13] *R. E. Greene, S. G. Krantz*: Function Theory of One Complex Variable. Graduate Studies in Mathematics 40. AMS, Providence, 2006. zbl MR doi
[14] *H. Hedenmalm, B. Korenblum, K. Zhu*: Theory of Bergman Spaces. Graduate Texts in Mathematics 199. Springer, Berlin, 2000. zbl MR doi
[15] *E. Hewitt, K. Stromberg*: Real and Abstract Analysis: A Modern Treatment of the Theory of Functions of a Real Variable. Graduate Texts in Mathematics 25. Springer, Berlin, 1975. zbl MR doi
[16] *S. G. Krantz*: Function Theory of Several Complex Variables. AMS, Providence, 2001. zbl MR doi
[17] *S. Ravisankar, Y. E. Zeytuncu*: A note on smoothing properties of the Bergman projection. Int. J. Math. *27* (2016), Article ID 1650087, 10 pages. zbl MR doi
[18] *E. J. Straube*: Exact regularity of Bergman, Szegő and Sobolev space projections in nonpseudoconvex domains. Math. Z. *192* (1986), 117–128. zbl MR doi

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