Czechoslovak Mathematical Journal

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Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 3, 839–867

Persistent URL: http://dml.cz/dmlcz/152584

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Czechoslovak Mathematical Journal, 74 (149) (2024), 839–867

LEFT EM RINGS

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Received February 18, 2024. Published online July 15, 2024.

Dedicated to Professor Il Yoo for his retirement from Yonsei University

Abstract. Let $R[x]$ be the polynomial ring over a ring R with unity. A polynomial $f(x) \in R[x]$ is referred to as a left annihilating content polynomial (left ACP) if there exist an element $r \in R$ and a polynomial $g(x) \in R[x]$ such that $f(x) = rg(x)$ and $g(x)$ is not a right zero-divisor polynomial in $R[x]$. A ring R is referred to as left EM if each polynomial $f(x) \in R[x]$ is a left ACP. We observe the structure of left EM rings with various properties, and study the relationships between the one-sided EM condition and other standard ring theoretic conditions. Moreover, several extensions of EM rings are investigated, including polynomial rings, matrix rings, and Ore localizations.

Keywords: EM ring; annihilating content polynomial; polynomial ring; uniserial ring; generalized morphic ring; zero-divisor

MSC 2020: 16U80, 16W99, 16E50, 16P40

1. Introduction

In multiple specific areas of ring theory, zero-divisors are known to have adverse effects. In particular, zero-divisors impose many restrictions on providing the solutions of many equations. As a straightforward example, let R be a domain and $f(x)$ be a nonzero polynomial in the polynomial ring $R[x]$. If $f(x) = f_1(x)f_2(x)$ with the degrees of $f_1(x)$ and $f_2(x)$ being less than the degree of $f(x)$, then $f(x) = 0$ implies that $f_1(x) = 0$ or $f_2(x) = 0$. Thus, when we solve the equation $f(x) = 0$, we can solve the lesser degree equations $f_1(x) = 0$ and $f_2(x) = 0$. However, if R contains zero-divisor elements, this method should be used with caution. Clearly, zero-divisors are important and have been studied extensively by several researchers

This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (NRF-2021R1I1A1A01041451).

DOI: 10.21136/CMJ.2024.0071-24

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using a variety of methods. In recent decades, many algebraists have studied the relationships between the zero-divisors of based rings and the zero-divisor polynomials of polynomial rings. Armendariz, McCoy, and ZPZC rings are representative examples of this area of study. A ring R with unity is called:

▷ *Armendariz* if $f(x)g(x) = 0$ implies $a_ib_j = 0$, see Definition 1.1 of [38],

▷ *left McCoy* if $f(x)g(x) = 0$ implies $cg(x) = 0$, see Definition 4.1 of [38],

▷ *left ZPZC* if $f(x)g(x) = 0$ implies $c_jb_j = 0$, see Definition 1.1 of [8]

for each i, j and a nonzero $c, c_j \in R$, where $f(x) = \sum_{i=0}^{m} a_ix^i$ and $g(x) = \sum_{j=0}^{n} b_jx^j$ are in the polynomial ring $R[x]$ over R. Armendariz rings are left-right symmetric. *Right McCoy* rings and *right ZPZC* rings are defined symmetrically. Of course, a ring is called *McCoy* if it is both left and right McCoy, and *ZPZC* if it is both left and right ZPZC. Clearly, all Armendariz rings are (left) McCoy and (left) ZPZC, and all (left) McCoy rings are (left) ZPZC. To explain the importance of this trend, let R be a left McCoy ring and $f(x)$ be a nonzero polynomial in the polynomial ring $R[x]$. If $f(x) = f_1(x)x^n f_2(x)$ for some $n \in \mathbb{N}$ and $f_1(x), f_2(x) \in R[x]$, then $f(c) = 0$, which means that the left zero-divisor element $c \in R$ is a solution of the equation $f(x) = 0$. The interested readers can investigate a number of more practical and important properties of the above rings in the literatures. Representatively, the author recommends the papers [8], [9], [17], [23], [29], [34], [35], [37], and [38] for greater detail on these rings.

Recently, Abuosba and Ghanem defined commutative EM rings in line with this trend. They showed that, while it is included in the class of Armendariz rings, the class of commutative EM rings includes Bézout rings, principal ideal rings, and p.p. rings. It was also shown that a commutative Noetherian ring is EM if and only if the maximal prime ideals of zero-divisors are principal. Commutative EM rings have been further studied, and ideal theoretic characterizations and some equivalent conditions of commutative EM rings are provided in [1] and [4].

The purpose of this article is to extend the notion of EM rings within the noncommutative setting, and to further study one-sided EM rings. Let us define the followings for rigor.

Definition 1.1. Let $R[x]$ be the polynomial ring over a ring R with unity, and $\text{l.ann}_{R[x]}(g(x))$ be the set of all left annihilator polynomials of $g(x)$ in $R[x]$.

(1) A polynomial $f(x) \in R[x]$ is called a *left annihilating content polynomial* (left ACP) if there exist an element $r \in R$ and a polynomial $g(x) \in R[x]$ such that $f(x) = rg(x)$ and $\text{l.ann}_{R[x]}(g(x)) = \{0\}$. A *right annihilating content polynomial* (right ACP) is defined symmetrically. A polynomial is called an *annihilating content polynomial* if it is both left and right ACP.

(2) A ring R is called *left EM* if each polynomial in $R[x]$ is a left ACP. Right EM ring and EM ring are defined similarly.

The word "left" in "left ACP" and "left EM" comes from the "left annihilator" of $g(x)$. Observing the "right zero-divisor" instead of the "left annihilator", one may think that "right ACP" and "right EM" are more suitable. Additionally, considering the twisted results in Section 4 of this paper, we may use "right" instead of "left". However, the "left" of left McCoy rings and left ZPZC rings comes from the "left annihilator" of $g(x)$. Further, based on the definition of EM modules in Definition 5.1 of [4], we should use the word "left" instead of "right".

Clearly, when R is a left EM ring and $f(x) \in R[x]$, if $\mathrm{l.ann}_{R[x]}(f(x)) \neq \{0\}$, then $\mathrm{l.ann}_{R[x]}(f(x)) = \mathrm{l.ann}_{R[x]}(rg(x)) = \mathrm{l.ann}_{R[x]}(r) \neq \{0\}$ and thus, $\mathrm{l.ann}_R(r) = \mathrm{l.ann}_R(f(x)) \neq \{0\}$. This means that every right zero-divisor polynomial is factorized by $r \in R$ and $g(x) \in R[x]$, where r is a right zero-divisor in R and $g(x)$ is not a right zero-divisor in $R[x]$.

In this paper, one studies the various properties and extensions of one-sided EM rings, as well as the relationships between the one-sided EM condition and other standard ring theoretic conditions. In Section 2, specific examples of left EM rings are provided. Among other things, it is shown in Example 2.1 that the one-sided EM condition is not left-right symmetric. Subsequently, we ascertain several extensions of left EM rings. Specifically, the one-sided EM condition of based rings can be elevated to the polynomial rings (see Theorem 2.10), and be penetrated into Ore localizations, see Theorem 2.23. In Section 3, one can view the diagram that contains all the implications among some related rings. Representatively, we prove that a left or right EM ring is Armendariz, see Proposition 3.1. In Section 4, several useful properties are presented. More precisely, it is proven that every right uniserial ring is left EM (see Theorem 4.2), and every left EM ring has the left (a.c.), see Theorem 4.15. Moreover, it is also proven that if R is a right Noetherian and left EM ring, then each right ideal A of R can be written as $A = aB$, where $a \in R$ and B is a right ideal of R with $\mathrm{l.ann}_R(B) = \{0\}$, see Corollary 4.28. Overall, one can give several examples and counterexamples for answers to questions that occur naturally in the process.

Throughout this manuscript, all rings are associative (not necessarily commutative) with unity, unless specified otherwise. For a ring R, $Z_l(R) = \{r \in R\colon\ ra = 0$ for a nonzero $a \in R\}$, $Z_r(R) = \{r \in R\colon\ br = 0$ for a nonzero $b \in R\}$, and $Z(R) = \{r \in R\colon\ ra = 0 = br$ for a nonzero $a, b \in R\}$ refer to the set of all left zero-divisors, right zero-divisors of R, and the set of all left and right zero-divisors of R, respectively. For a nonempty subset A of a ring E, we also denote the set of all left annihilators of A by $\mathrm{l.ann}_E(A) = \{e \in E\colon\ eA = 0\}$ and the set of all right annihilators of A by $\mathrm{r.ann}_E(A) = \{e \in E\colon\ Ae = 0\}$. It is easy to see that $\mathrm{l.ann}_{R[x]}(p) = \{0\} \Leftrightarrow p \notin Z_{\mathrm{r}}(R[x]) \Leftrightarrow \mathrm{l.ann}_R(p) = \{0\} \Leftrightarrow p \notin Z_{\mathrm{r}}(R)$ for $p \in R$.

Additionally, for a polynomial $f(x) \in R[x]$, if $f(x) = rg(x)$ for some $r \in R$ and $g(x) \in R[x]$, then the degree of $f(x)$ is less than or equal to the degree of $g(x)$. We will henceforth freely use these facts without any particular mention of them.

2. Examples of left EM rings

In this section, we provide various examples of left EM rings. Let D be a domain. If $f(x) \in D[x] \setminus \{0\}$, then $f(x) = 1_D \cdot f(x) = f(x) \cdot 1_D$ and $\mathrm{l.ann}_{D[x]}(f(x)) = \{0\} = \mathrm{r.ann}_{D[x]}(f(x))$. Thus, every domain is EM. We begin with constructing a nontrivial left EM ring which is not right EM, using an argument similar to Example 5.1 of [9].

Example 2.1. Let $\mathcal{R} = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra with unity and noncommuting indeterminates a, b over $\mathbb{Z}_2$. Set $R = \mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathcal{R}$ generated by the relations:

$$ab = 0, \quad b^2 = 0.$$

We suppress all the bar notations for simplicity. Applying Bergman's diamond lemma (see [15]), we can write each $r \in R$ in the following reduced form:

$$r = \alpha_0 + \sum_{\lambda=1}^{\eta} \alpha_\lambda a^\lambda + \sum_{\mu=0}^{\kappa} \beta_\mu b a^\mu$$

for some $\eta \in \mathbb{N}$ and $\kappa \in \mathbb{N} \cup \{0\}$, where $\alpha_0, \ldots, \alpha_\eta, \beta_0, \ldots, \beta_\kappa \in \mathbb{Z}_2$ (under assuming $a^0 = 1$). Clearly, the ring R is not a domain.

Claim 2.2. *R is not right EM.*

Proof. Let $f(x) = a + bx$. If $f(x) = g(x)r$ for some $r \in R$, then r must be the unity element by the relations given above. So, $g(x)$ must be $f(x)$. Since $g(x)b = (a + bx)b = 0$ by the relations, $g(x) \in Z_{\mathrm{l}}(R[x])$. Thus, R is not right EM. □

Claim 2.3. *R is left EM.*

Proof. If $f(x) \notin Z_{\mathrm{r}}(R[x])$, then since $f(x) = 1_R \cdot f(x)$ with $f(x) \notin Z_{\mathrm{r}}(R[x])$, $f(x)$ is a left ACP. If $f(x) = r$ is a constant polynomial, then $f(x) = r = r \cdot 1_R$ with $1_R \notin Z_{\mathrm{r}}(R[x])$. So, it suffices to show that every nonconstant polynomial in $Z_{\mathrm{r}}(R[x])$ is a left ACP. Let $f(x) = \sum_{i=0}^{m} p_i x^i \in Z_{\mathrm{r}}(R[x])$ with $p_0 \neq 0$ because if $f(x) = x^k f_1(x)$,

then we can identify $f(x) = f_1(x)$. We observe that

$$\text{(2.1)} \qquad \mathrm{l.ann}_R(a) = \{0\} = \mathrm{l.ann}_R(1+a) = \ldots = \mathrm{l.ann}_R\Big(\sum_{j=1}^{\text{finite}} \alpha_j a^j\Big)$$
$$= \mathrm{l.ann}_R\Big(\sum_{j=1}^{\text{finite}} \alpha_j a^j + \sum_{k=0}^{\text{finite}} \beta_k b a^k\Big) \quad \text{for a nonzero } \alpha_j.$$

$$\text{(2.2)} \qquad \mathrm{l.ann}_R(b) = \Big\{\sum_{j=1}^{\text{finite}} \alpha_j a^j + \sum_{k=0}^{\text{finite}} \beta_k b a^k\Big\} = \mathrm{l.ann}_R\Big(\sum_{k=0}^{\text{finite}} \beta_k b a^k\Big)$$
$$\text{for a nonzero } \beta_k.$$

$$\text{(2.3)} \qquad \mathrm{r.ann}_R(a) = \Big\{\sum_{k=0}^{\text{finite}} \beta_k b a^k\Big\} = \mathrm{r.ann}_R\Big(\sum_{j=1}^{\text{finite}} \alpha_j a^j\Big)$$
$$= \mathrm{r.ann}_R\Big(\sum_{j=1}^{\text{finite}} \alpha_j a^j + \sum_{k=0}^{\text{finite}} \beta_k b a^k\Big)$$
$$= \mathrm{r.ann}_R(b) = \mathrm{r.ann}_R\Big(\sum_{k=0}^{\text{finite}} \beta_k b^k\Big) \quad \text{for a nonzero } \alpha_j, \beta_k.$$

We will show that no coefficient of $f(x)$ contains any term of the form $\sum_{j=0}^{\text{finite}} \alpha_j a^j$ in its support. First, we will show that p_0 satisfies this stipulation. Let $h(x) = \sum_{l=0}^{n} q_l x^l$ be a nonzero polynomial in $R[x]$ with $q_0 \neq 0$ and $h(x)f(x) = 0$. Assume that a^t appears in the support of p_0 for some $t \in \mathbb{N} \cup \{0\}$. Then we can find the smallest power s such that a^s is contained in the support of p_0. Since $\mathrm{l.ann}_R(a^s) = \{0\}$ by the relations and s is minimally chosen, q_0 must be zero, which is a contradiction. Therefore, a^t (and also 1) is not contained in the support of p_0 for any t. This implies that $p_0 = \sum_{k_0=0}^{\text{finite}} \beta_{k_0} b a^{k_0}$.

Second, we will show that another coefficient of $f(x)$ does not contain any term of the form $\sum_{j=0}^{\text{finite}} \alpha_j a^j$ in its support. Assume that a^t appears in the support of one of $\{p_1, \ldots, p_m\}$ for some $t \in \mathbb{N} \cup \{0\}$. Then we can find the smallest index u such that a^t is contained in the support of p_u for some t, and not contained in the support of each $p_0, \ldots, p_{u-1}$ for any t. Now, we have the following two cases.

Case 1: There exists a coefficient of $h(x)$ such that a^z appears in its support for some $z \in \mathbb{N} \cup \{0\}$. In this case, we let v be the smallest index such that a^z appears in the support of q_v for some z, and a^z does not appear in the support of any of $q_0, \ldots, q_{v-1}$ for any z. Then the $(v+u)$-degree coefficient of $h(x)f(x)$ is not zero. This contradicts to the assumption that $h(x)f(x) = 0$.

Case 2: There is no any coefficient of $h(x)$ such that a^z appears in its support for some $z \in \mathbb{N} \cup \{0\}$. In this case, the u-degree coefficient of $h(x)f(x)$ is not zero, which is a contradiction.

Consequently, we have shown that no coefficient of $f(x)$ contains any term of the form $\sum\limits_{j=0}^{\text{finite}} \alpha_j a^j$ in its support. Hence, we can write $p_i = \sum\limits_{k_i=0}^{\text{finite}} \beta_{k_i} b a^{k_i}$ for each $0 \leqslant i \leqslant m$. This implies that there exists a polynomial $f'(x) \in R[x]$ such that $f(x) = \Big(\sum\limits_{w=0}^{\text{finite}} \beta_w b a^w\Big) f'(x)$ and a^t appears in the support of a coefficient of $f'(x)$ for some $t \in \mathbb{N} \cup \{0\}$. Finally, by a similar approach to that above, we can conclude that $f'(x) \notin Z_{\mathrm{r}}(R[x])$, confirming that every polynomial in $Z_{\mathrm{r}}(R[x])$ is a left ACP. □

Next, we examine the direct products of left EM rings.

Theorem 2.4 (cf. [2], Theorem 3.12)**.** *Let Λ be any index set and R_λ be a ring for each $\lambda \in \Lambda$. Then the direct product $\mathcal{D} = \prod\limits_{\lambda} R_\lambda$ is a left EM ring if and only if each R_λ is a left EM ring.*

P r o o f. ($\Rightarrow$) We first identify $\mathcal{D}[x] = \Big(\prod\limits_{\lambda} R_\lambda\Big)[x] \cong \prod\limits_{\lambda}(R_\lambda[x])$. Fix λ and let $f_\lambda(x) \in R_\lambda[x]$. Consider $F(x) \in \mathcal{D}[x]$, where the λth component of $F(x)$ is $f_\lambda(x)$ and the other ωth component of $F(x)$ is 1_{R_ω} for each $\lambda \neq \omega \in \Lambda$. Since $\mathcal{D}$ is left EM, there exists an element $A \in \mathcal{D}$ and a polynomial $G(x) \in \mathcal{D}[x]$ such that $F(x) = AG(x)$ and $G(x) \notin Z_{\mathrm{r}}(\mathcal{D}[x])$. Take the λth component a_λ of A and $g_\lambda(x)$ of $G(x)$. Then $f_\lambda(x) = a_\lambda g_\lambda(x)$ with $g_\lambda(x) \notin Z_{\mathrm{r}}(R_\lambda[x])$, confirming that $f_\lambda(x)$ is a left ACP in $R_\lambda[x]$.

($\Leftarrow$) We also identify $\mathcal{D}[x] = \prod\limits_{\lambda}(R_\lambda[x])$. Let $F(x) = (f_\lambda(x))_{\lambda\in\Lambda} \in \mathcal{D}[x]$. Since each $f_\lambda(x)$ is a left ACP, there exist an element $a_\lambda \in R_\lambda$ and a polynomial $g_\lambda(x) \in R_\lambda[x]$ such that $f_\lambda(x) = a_\lambda g_\lambda(x)$ and $g_\lambda(x) \notin Z_{\mathrm{r}}(R_\lambda[x])$. Now, let $A = (a_\lambda)_{\lambda\in\Lambda}$ and $G(x) = (g_\lambda(x))_{\lambda\in\Lambda}$. Then $F(x) = AG(x)$ and $G(x) \notin Z_{\mathrm{r}}(\mathcal{D}[x])$. Thus, $F(x)$ is a left ACP. □

Remark 2.5. Even though each R_λ is a left EM ring, the infinite direct sum $\bigoplus\limits_{\lambda} R_\lambda$ must not be a left EM ring.

Let R be a ring and consider the binary operation $*$ on $\mathbb{Z} \times R$ defined by

$$(z_1, r_1) * (z_2, r_2) = (z_1 z_2, z_1 r_2 + z_2 r_1 + r_1 r_2).$$

Then $(\mathbb{Z} \times R, +, *)$ is an associative ring with unity $(1, 0)$, where $+$ is the usual addition. We call this ring the *Dorroh extension* of R denoted by $\mathbb{Z} * R$. According to Corollary 2.3 of [18], $\mathbb{Z} * R$ is isomorphic to $\mathbb{Z} \times R$. By Theorem 2.4, we obtain the following corollary.

Corollary 2.6. *A ring R is left EM if and only if the Dorroh extension $\mathbb{Z} * R$ is left EM.*

Every subring of a domain is clearly a left and right EM ring. However, the class of left EM rings is not closed under subrings by the following example.

Example 2.7. In Example 2.1, consider the subring $S = \left\{\alpha_0 + \sum\limits_{i=0}^{\text{finite}} \beta_i b a^i : \alpha_0, \beta_i \in \mathbb{Z}_2\right\}$ of the left EM ring R. Let $f(x) = ba + ba^2x \in S[x]$. Since $a \notin S$ and we can only write $f(x) = sf(x)$ for some $s \in S$, the element $b \in S$ annihilates $f(x)$ on the left side. Thus, the subring S is not left EM.

By Theorem 2.4, we can find the types of subrings to which the one-sided EM condition is passed.

Corollary 2.8. *Let e be a central idempotent in a ring R. Then R is left EM if and only if eR and $(1-e)R$ are both left EM.*

P r o o f. This follows from Theorem 2.4 and the fact that $R \cong eR \oplus (1-e)R$. □

Corollary 2.9. *If e is a central idempotent in a left EM ring R, then the corner ring eRe is a left EM ring.*

P r o o f. It follows from Corollary 2.8 and the fact that $eR = eRe$. □

The one-sided EM condition can be lifted up to the polynomial ring.

Theorem 2.10 (cf. [2], Theorem 3.13)**.** *If a ring R is left EM, then so is the polynomial ring $R[x]$.*

P r o o f. Let $F(y) = \sum\limits_{j=0}^{n} f_j y^j \in (R[x])[y]$, where $f_j = \sum\limits_{i=0}^{m_j} r_{i,j} x^i \in R[x]$ for each i, j. Set $m = \sum\limits_{j=0}^{n} m_j \in \mathbb{N}$. Consider the polynomial $f(x) = F(x^m) \in R[x]$ by inputting x^m to y. Since R is left EM, there exist an element $a \in R$ and a polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$ and $g(x) \notin Z_{\mathrm{r}}(R[x])$. Replacing x^m by y in $g(x)$, we can obtain that $G(y) \in (R[x])[y]$ such that $F(y) = aG(y)$. Now, assume that $G(y) \in Z_{\mathrm{r}}((R[x])[y])$. Then there exists $H(y) \in (R[x])[y]$ such that $H(y)G(y) = 0$. This implies that $H(x^k)G(x^k) = H(x^k)g(x) = 0$ for some $k \in \mathbb{N}$ which contradicts to $g(x) \notin Z_{\mathrm{r}}(R[x])$. Thus, $G(y) \notin Z_{\mathrm{r}}((R[x])[y])$. Hence, $R[x]$ is left EM. □

Theorem 2.10 permits us to legitimately state that the one-sided EM condition can be also elevated to the multivariable polynomial rings.

Corollary 2.11 (cf. [2], Corollary 3.14)**.** *If a ring R is left EM, then the polynomial ring $R[x_1, \ldots, x_k]$ is left EM.*

P r o o f. This proof follows from Theorem 2.10 and the fact that $R[x_1, \ldots, x_k] \cong (R[x_1, \ldots, x_{k-1}])[x_k]$. □

Let $R \subseteq E$ be rings and $r \in R$. If $r \notin Z_{\mathrm{r}}(E) \Leftrightarrow \mathrm{l.ann}_E(r) = \{0\}$, then $r \notin Z_{\mathrm{r}}(R) \Leftrightarrow \mathrm{l.ann}_R(r) = \{0\}$. However, $r \notin Z_{\mathrm{r}}(R)$ does not imply that $r \notin Z_{\mathrm{r}}(E)$ in general. Nevertheless, if $E = R[x]$, then $r \notin Z_{\mathrm{r}}(R)$ implies that $r \notin Z_{\mathrm{r}}(E)$. Even though we already stated that we will freely use this fact at the end of Section 1, a perusal of the proof of the next corollary reveals why this fact holds true.

Corollary 2.12 (cf. [2], Corollary 3.16)**.** *Assume that R is a left EM ring and $f(x_1, \ldots, x_k) \in R[x_1, \ldots, x_k]$. Then there exist a nonzero $r \in R$ and polynomials $g_i(x_1, \ldots, x_i) \in R[x_1, \ldots, x_i]$ such that $f(x_1, \ldots, x_k) = rg_1(x_1) \ldots g_k(x_1, \ldots, x_k)$, where $g_i(x_1, \ldots, x_i) \notin Z_{\mathrm{r}}(R[x_1, x_2, \ldots])$ for every $i = 1, \ldots, k$.*

P r o o f. If $f(x_1, \ldots, x_k) \notin Z_{\mathrm{r}}(R[x_1, \ldots, x_k])$, there is nothing to prove. So, we may assume that $f(x_1, \ldots, x_k) \in Z_{\mathrm{r}}(R[x_1, \ldots, x_k])$. From Corollary 2.11 and the fact that $R[x_1, \ldots, x_i] \cong (R[x_1, \ldots, x_{i-1}])[x_i]$ for each i, it is very clear that $f(x_1, \ldots, x_k) = rg_1(x_1) \ldots g_k(x_1, \ldots, x_k)$, where $r \in R$ and $g_i(x_1, \ldots, x_i) \notin Z_{\mathrm{r}}(R[x_1, \ldots, x_i])$ for each $i = 1, \ldots, k$. Now, suppose that $p(x_1, x_2)g_1(x_1) = 0$ for some $p(x_1, x_2) \in R[x_1, x_2]$. Since $R[x_1, x_2] \cong (R[x_1])[x_2]$, we can write $p(x_1, x_2) = q_0(x_1) + q_1(x_1)x_2 + \ldots + q_m(x_1)x_2^m$ for some $q_0(x_1), \ldots, q_m(x_1) \in R[x_1]$. Hence,

$$p(x_1, x_2)g_1(x_1) = q_0(x_1)g_1(x_1) + q_1(x_1)g_1(x_1)x_2 + \ldots + q_m(x_1)g_1(x_1)x_2^m = 0$$

implies that $q_0(x_1)g_1(x_1) = \ldots = q_m(x_1)g_1(x_1) = 0$. By the condition that $g_1(x_1) \notin Z_{\mathrm{r}}(R[x_1])$, we obtain that $q_0(x_1) = \ldots = q_m(x_1) = 0$ and thus, $p(x_1, x_2) = 0$, which means that $g_1(x_1) \notin Z_{\mathrm{r}}(R[x_1, x_2])$. In an entirely similar manner, we can reach that $g_1(x_1) \notin Z_{\mathrm{r}}(R[x_1, x_2, \ldots])$. Continuing almost the same process, we can eventually reach that

$$g_2(x_1, x_2), g_3(x_1, x_2, x_3), \ldots, g_k(x_1, \ldots, x_k) \notin Z_{\mathrm{r}}(R[x_1, x_2, \ldots]),$$

which is our conclusion. □

Corollary 2.13 (cf. [2], Theorem 3.17)**.** *If R is a left EM ring, then so is the polynomial ring $R[x_1, x_2, \ldots]$.*

P r o o f. Let $f \in Z_{\mathrm{r}}(R[x_1, x_2, \ldots])$. Because the nonzero coefficients of f are only finitely many, $f \in R[x_1, \ldots, x_k]$ for sufficiently large $k \in \mathbb{N}$. Now, Corollary 2.12 will work. □

Based on Theorem 2.10, one may suspect that if R is a left EM ring, then the Ore extension $R[x;\sigma,\delta]$ is left EM. However, the following example removes this possibility.

Example 2.14. Recall the left EM ring R given in Example 2.1. Consider the endomorphism σ of R induced by $\sigma(a) = 0$, $\sigma(b) = b$, and the polynomial $F(y) = ay + bay^2$ in the polynomial ring $(R[x;\sigma,\delta])[y]$ over the Ore extension $R[x;\sigma,\delta]$ of R, where δ is the zero map. If $F(y) = f(x)G(y)$ with $f(x) \in R[x;\sigma,\delta]$ and $G(y) \in (R[x;\sigma,\delta])[y]$, then one can show that $f(x) = 1$ and $G(y) = F(y)$ by an argument similar to the one given in Claim 2.3 of Example 2.1. However,

$$axG(y) = ax(ay + bay^2) = (a\sigma(a)x)y + (a\sigma(b)\sigma(a)x)y^2 = 0.$$

Thus, $F(y)$ is not a left ACP in $(R[x;\sigma,\delta])[y]$.

Next, we consider the converse statement of Theorem 2.10.

Proposition 2.15. *If the polynomial ring $R[x]$ is left EM and $f(x)$ is a nonzero polynomial in $R[x]$, then $f(x) = rg(x)$ for a nonzero $r \in R$ and $g(x) \in R[x]$.*

P r o o f. Write $f(x) = \sum_{i=0}^{l} r_i x^i$ and take the polynomial $F(y) = \sum_{i=0}^{l} r_i y^i \in (R[x])[y]$. Since $R[x]$ is left EM, there exist $A(x) = \sum_{j=0}^{m} a_j x^j \in R[x]$ and $G(y) = \sum_{k=0}^{n} g_k y^k \in (R[x])[y]$ such that $F(y) = A(x)G(y)$ and $G(y) \notin Z_{\mathrm{r}}((R[x])[y])$. Now, let $g_i = b_{i,0} + b_{i,1}x + \ldots + b_{i,n_i}x^{n_i}$ for each i. From the fact that $r_i = A(x)g_i \in R$ for each i, we obtain that

$$\begin{aligned} r_i &= (a_0 + \ldots + a_m x^m)(b_{i,0} + b_{i,1}x + \ldots + b_{i,n_i}x^{n_i}) \\ &= a_0 b_{i,0} + a_0(b_{i,1}x + \ldots + b_{i,n_i}x^{n_i}) + (a_1 x + \ldots + a_m x^m)(b_{i,0} + \ldots + b_{i,n_i}x^{n_i}). \end{aligned}$$

This means that $a_0 b_{i,0} = r_i$ and

$$a_0(b_{i,1}x + \ldots + b_{i,n_i}x^{n_i}) + (a_1 x + \ldots + a_m x^m)(b_{i,0} + b_{i,1}x + \ldots + b_{i,n_i}x^{n_i}) = 0.$$

Thus, $f(x) = a_0(b_{0,0} + b_{1,0}x + \ldots + b_{l,0}x^l)$. □

If we can show that $b_{0,0} + b_{1,0}x + \ldots + b_{l,0}x^l \notin Z_{\mathrm{r}}(R[x])$ in the proof of Proposition 2.15, then the converse statement of Theorem 2.10 holds true. However, the author has not uncovered any clues regarding this objective thus far. Hence, we leave the following as an open question:

Question 2.16. Does the converse statement of Theorem 2.10 hold true?

As seen in Theorems 2.4 and 2.10, the one-sided EM condition can be elevated to some overrings. However, this is untrue in general. For instance, in Example 2.14, the ring R is left EM, but the overring $R[x;\sigma,\delta]$ of R is not left EM. Considering the S-variant of the Eakin-Nagata-Eisenbud theorem (see Theorem 5.10 of [11], and Theorem 2.9 of [33]), we provide a condition under which the one-sided EM condition passes between subrings and overrings.

Theorem 2.17. *Let $R \subseteq E$ be rings with the same unity and e be an element in E such that $eR = Re$, $\mathrm{r.ann}_R(e) = \{0\}$, $e^2 = 0$, and $er \notin R$ for each $0 \neq r \in R$. Then the following statements hold true:*
(1) *If R is a domain and $E = R + eR$, then E is left EM.*
(2) *If E is left EM and there exists an element $s \in R$ such that s is regular in R, central in E, and $Es \subseteq Rs + eRs$, then R is left EM.*

P r o o f. (1) Let $f(x)$ be an arbitrary polynomial in $E[x]$. Then we can write $f(x) = f_1(x) + f_2(x)$ for some $f_1(x) \in R[x]$ and $f_2(x) \in eR[x]$. Because R is a domain, we can easily notice that if $f(x) \in Z_{\mathrm{r}}(E[x])$, then $f_1(x) = 0$. Since R is left EM, there exist $a_2 \in R$ and $g_2(x) \in R[x]$ such that $f(x) = f_2(x) = ea_2g_2(x)$ and $g_2(x) \notin Z_{\mathrm{r}}(R[x])$. Now, let $h(x) \in E[x]$ and suppose that $h(x)g_2(x) = 0$. Then $h(x) = h_1(x) + h_2(x)$ for some $h_1(x) \in R[x]$ and $h_2(x) \in eR[x]$. Thus, $h(x)g_2(x) = h_1(x)g_2(x) + h_2(x)g_2(x) = 0$. By the hypothesis and $g_2(x) \notin Z_{\mathrm{r}}(R[x])$, we obtain that $h_1(x) = h_2(x) = 0$. Hence, we are forced to conclude that $g_2(x) \notin Z_{\mathrm{r}}(E[x])$, confirming that $f(x)$ is a left ACP.

(2) Note that eR is a subring of E with $R \cap eR = \{0\}$, and $E[x]s \subseteq R[x] \oplus eR[x] \subseteq E[x]$, and without unity. Let $f(x)$ be a nonzero polynomial in $Z_{\mathrm{r}}(R[x])$. Then $f(x) \in Z_{\mathrm{r}}(E[x])$. So, there exist an element $a \in E$ and a polynomial $g(x) \in E[x]$ such that $f(x) = ag(x)$ and $g(x) \notin Z_{\mathrm{r}}(E[x])$. Because $sE = Es \subseteq Rs + eRs$, there exist $a_1, a_2 \in R$ and $g_1(x), g_2(x) \in R[x]$ such that $as = a_1s + ea_2s$ and $g(x)s = g_1(x)s + eg_2(x)s$. Thus, we obtain that

$$\begin{aligned} f(x)s^2 = asg(x)s &= (a_1s + ea_2s)(g_1(x)s + eg_2(x)s) \\ &= a_1sg_1(x)s + a_1seg_2(x)s + ea_2sg_1(x)s + ea_2seg_2(x)s \\ &= a_1g_1(x)s^2 + ea_1'g_2(x)s^2 + ea_2g_1(x)s^2 + e^2a_2'g_2(x)s^2 \\ &= a_1g_1(x)s^2 + e(a_1'g_2(x)s^2 + a_2g_1(x)s^2) \end{aligned}$$

for some $a_1', a_2' \in R$ with $a_1e = ea_1'$ and $a_2e = ea_2'$. This implies that

$$f(x)s^2 - a_1g_1(x)s^2 = e(a_1'g_2(x)s^2 + a_2g_1(x)s^2).$$

Since $R[x] \cap eR[x] = \{0\}$ and s is regular in $R[x]$, we obtain that $f(x) = a_1g_1(x)$. Finally, in anticipation of a contradiction, let us assume that $g_1(x) \in Z_{\mathrm{r}}(R[x])$.

Then $h(x)g_1(x) = 0$ for a nonzero $h(x) \in R[x]$. From the fact that $h(x)e = eh'(x)$ for a nonzero $h'(x) \in R[x]$, we obtain that $eh(x)sg(x) = eh(x)(g_1(x)s + eg_2(x)s) = eh(x)g_1(x)s + e^2h'(x)g_2(x)s = 0$. This contradicts to the condition that $g(x) \notin Z_{\mathrm{r}}(E[x])$. Hence, $g_1(x) \notin Z_{\mathrm{r}}(R[x])$, completing the proof. □

Applying $S = \{1\}$ in Theorem 2.17 (2), we can directly obtain:

Corollary 2.18. *Let $R \subseteq E$ be rings with the same unity, and let e be an element in E such that $eR = Re$, $\mathrm{r.ann}_R(e) = \{0\}$, $e^2 = 0$, and $er \notin R$ for each $r \in R$. If $E = R + eR$ is left EM, then R is left EM.*

Recall that a ring R is *reduced* if R has no nonzero nilpotent element. Equivalently, R is reduced if for every $r \in R$, $r^2 = 0$ implies $r = 0$. According to Theorem 5 of [6], if $n \in \mathbb{N}$ with $n \geqslant 2$ and (x^n) is the ideal of $R[x]$ generated by x^n, then $R[x]/(x^n)$ is Armendariz if and only if R is reduced. In the proof of Theorem 5 of [6], the commutative condition of R is not used. Thus, we can obtain the same result even when R is noncommutative. From Theorem 2.17 and Corollary 2.18, we can provide many examples of left EM rings.

Corollary 2.19 (cf. [2], Theorems 3.19 and 3.20)**.** *Let σ be an endomorphism of a ring R, and (x^2) be the ideal generated by x^2 of $R[x]$ or $R[x;\sigma]$. Then the following statements hold true:*
(1) *If R is a domain, then $R[x;\sigma]/(x^2)$ is a left EM ring.*
(2) *If R is a domain, then $R[x]/(x^2)$ is a left EM ring.*
(3) *If $R[x]/(x^2)$ is a left EM ring, then R is both reduced and left EM.*

P r o o f. (1) Clearly, the subring $\overline{R} = R + (x^2)$ of the ring $\overline{E} = R[x;\sigma]/(x^2)$ is a domain, and $\overline{E} = R[x;\sigma]/(x^2) \cong R + R\bar{x} \cong \overline{R} + \overline{R}\bar{x}$. Now, Theorem 2.17 (1) applies.

(2) This follows from (1) and the fact that $R[x] = R[x;\sigma]$ when σ is the identity endomorphism of R.

(3) By Proposition 3.1, the ring $R[x]/(x^2)$ is Armendariz. So, R is reduced by Theorem 5 of [6]. Now, let $\overline{R} = R + (x^2)$, $\overline{E} = \overline{R} + \overline{R}\bar{x}$, and $e = \bar{x}$. From the facts that $R \cong \overline{R}$ and $E = R[x]/(x^2) \cong R + R\bar{x} \cong \overline{R} + \overline{R}\bar{x} = \overline{E}$, we can arrive at the conclusion by Corollary 2.18. □

Next, we consider matrix rings over a left EM ring. For a ring R and a fixed positive integer n, we denote the $n \times n$ full matrix ring by $\mathbb{M}_n(R)$ and the $n \times n$ upper triangular matrix ring by $\mathbb{T}_n(R)$. We also denote the subrings of $\mathbb{T}_n(R)$ by

$$\mathbb{H}_n(R) = \{[a_{i,j}] \in \mathbb{T}_n(R) \colon\ a_{1,1} = \ldots = a_{n,n}\} \quad \text{and}$$
$$\mathbb{V}_n(R) = \{[a_{i,j}] \in \mathbb{H}_n(R) \colon\ a_{u,v} = a_{(u+1),(v+1)} \text{ for } u = 1, \ldots, n-2 \text{ and } v = 2, \ldots, n-1\}.$$

Furthermore, I_n refers to the identity matrix, and $E_{i,j}$ refers to the matrix unit, which is the matrix with (i,j)-entry 1 and elsewhere 0. The next examples show that each $\mathbb{M}_n(R)$, $\mathbb{T}_n(R)$, $\mathbb{H}_n(R)$, and $\mathbb{V}_n(R)$ need not be left EM even though R is left EM.

Example 2.20.

(1) See Remark 3.9 of [8]. Fix an integer $n \geqslant 2$ and let $H(x) = E_{n,n} + E_{n,1}x$ and $F(x) = -E_{1,n} + I_n x - E_{n,1}x^2$ be nonzero polynomials in $\mathbb{M}_n(R)[x]$. Then $H(x)F(x) = 0$ and so $F(x) \in Z_{\mathrm{r}}(\mathbb{M}_n(R)[x])$. From the middle coefficient of $F(x)$, one can easily notice that if $F(x) = AG(x)$ for some $A \in \mathbb{M}_n(R)$ and $G(x) \in \mathbb{M}_n(R)[x]$, then A must be right invertible in $\mathbb{M}_n(R)$. Now, assume that $G(x) \notin Z_{\mathrm{r}}(\mathbb{M}_n(R)[x])$. Since $0 = H(x)F(x) = H(x)AG(x)$ and A is right invertible, $H(x)A = 0$ and thus, $H(x) = 0$, which is a contradiction. Hence, $G(x) \in Z_{\mathrm{r}}(\mathbb{M}_n(R)[x])$. Consequently, $F(x)$ is not a left ACP, which implies that $\mathbb{M}_n(R)$ must not be left EM for any integer $n \geqslant 2$ and ring R with unity.

(2) Let F be a nonzero field. Then F is clearly EM. By Example 1 of [29], $\mathbb{T}_2(F)$ is not Armendariz. This guarantees that $\mathbb{T}_2(F)$ is not (both left and right) EM by Proposition 3.1 (to follow). Additionally, by Example 3 of [29], $\mathbb{H}_4(F)$ is not Armendariz. Thus, $\mathbb{H}_4(F)$ is also not (both left and right) EM.

(3) Note that $\mathbb{Z}_4$ is a commutative EM ring by Corollary 2.3 of [2]. As a well-known result, $\mathbb{Z}_4[x]/(x^2)$ is (ring) isomorphic to $\mathbb{V}_2(\mathbb{Z}_4)$, where (x^2) is the ideal of $\mathbb{Z}_4[x]$ generated by x^2, see page 2288 of [34]. By Example 2.6 of [2], we can conclude that $\mathbb{Z}_4[x]/(x^2) \cong \mathbb{V}_2(\mathbb{Z}_4)$ is not (both left and right) EM.

Remark 2.21.

(1) By Example 2.20, it turns out that the one-sided EM condition is badly behaved with regards to Morita invariance.

(2) The class of left EM rings is not closed under factor rings and homomorphic images. Precisely, let R be the ring of quaternions with integer coefficients. Then R is a domain and thus, (left) EM. However, R/pR is isomorphic to $\mathbb{M}_2(F)$ by the argument in Exercise 2A of [21], where p is an odd prime integer and F is the Galois field of order p. Hence, R/pR is not a left EM ring by Example 2.20 (1).

(3) Note that the ring $\mathbb{T}_2(F)$ over a field F is left and right Noetherian. Thus, Example 2.20 (2) shows that a left or right Noetherian ring may not be left EM.

(4) Because the trivial extension $\mathcal{T}(R,R)$ is (ring) isomorphic to $\mathbb{V}_2(R)$, $\mathcal{T}(R,R)$ does not have to be left EM by Example 2.20 (3), even though R is a commutative EM ring.

Combining Theorem 2.4, Example 2.20 (1), and the Wedderburn-Artin theorem (see Theorem 3.5 of [30]), we obtain the following result.

Proposition 2.22. *If a ring R is semisimple, then R is left EM if and only if R is a product of division rings.*

Next, we consider Ore localizations, see Section 10 of [31].

Theorem 2.23 (cf. [2], Theorem 3.6)**.** *Let T be a left denominator set of a ring R.*
(1) *If R is left EM, then $T^{-1}R$ is left EM ring.*
(2) *If $T^{-1}R$ is left EM and T consists of only left invertible elements of R, then R is left EM.*

P r o o f. (1) Let $F(x) \in (T^{-1}R)[x]$. Then we can write $F(x) = u_0/t + u_1x/t + \ldots + u_nx^n/t$ for some $t \in T$. Consider $f(x) = u_0 + u_1x + \ldots + u_nx^n \in R[x]$. Since R is left EM, there exist an element $a \in R$ and a polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$ and $g(x) \notin Z_{\mathrm{r}}(R[x])$. Thus, we can rewrite $F(x) = (a/t) \cdot (g(x)/1)$. Now, assume that $g(x)/1 \in Z_{\mathrm{r}}((T^{-1}R)[x])$. Then there exists a nonzero polynomial $H(x) = v_0/s + v_1x/s + \ldots + v_mx^m/s \in (T^{-1}R)[x]$ such that

$$H(x) \cdot \frac{g(x)}{1} = \Big(\frac{v_0 + v_1x + \ldots + v_mx^m}{s}\Big) \cdot \frac{g(x)}{1} = \frac{(v_0 + v_1x + \ldots + v_mx^m)g(x)}{s} = \frac{0}{1},$$

where $s \in T$. By the left version of (10.9) of [31], we obtain that

$$s'(v_0 + \ldots + v_mx^m) \cdot g(x) = 0$$

for some $s' \in T$. Because $H(x) = (1/s) \cdot (v_0 + v_1x + \ldots + v_mx^m)/1 \neq 0$, we also obtain that $(v_0 + v_1x + \ldots + v_mx^m)/1 \neq 0$. Thus, $s'(v_0 + v_1x + \ldots + v_mx^m) \neq 0$, which contradicts that $g(x) \notin Z_{\mathrm{r}}(R[x])$. Hence, $g(x)/1 \notin Z_{\mathrm{r}}((T^{-1}R)[x])$, confirming that $T^{-1}R$ is left EM.

(2) Let $f(x)$ be arbitrary in $R[x]$. Since $f(x)/1 \in (T^{-1}R)[x]$, there exist the element $c/s \in T^{-1}R$ and the polynomial $G(x) = d_0/t + d_1x/t + \ldots + d_nx^n/t \in (T^{-1}R)[x]$ such that $f(x)/1 = (c/s)G(x)$ and $G(x) \notin Z_{\mathrm{r}}((T^{-1}R)[x])$. By the left version of [31], (10.10), we obtain that

$$\frac{f(x)}{1} = \frac{c}{s} \cdot \frac{(d_0 + d_1x + \ldots + d_nx^n)}{t} = \frac{r(d_0 + d_1x + \ldots + d_nx^n)}{t's}$$

for some $r \in R$ and $t' \in T$ with $t'c = rt \in Tc \cap Rt \neq \emptyset$ (keeping in mind that $(s^{-1}c)(t^{-1}d)$ should be $s^{-1}(ct^{-1})d = s^{-1}(t'^{-1}r)d = (t's)^{-1}rd$). This implies that $r(d_0 + d_1x + \ldots + d_nx^n) = t'sf(x)$. Thus, $f(x) = ur(d_0 + d_1x + \ldots + d_nx^n)$, where u is the left inverse of $t's$ in R. Now, let $a = ur$ and $g(x) = d_0 + d_1x + \ldots + d_nx^n \in R[x]$. Because $G(x) = (d_0 + d_1x + \ldots + d_nx^n)/t \notin Z_{\mathrm{r}}((T^{-1}R)[x])$, $(v/1)G(x) = (d_0 + d_1x + \ldots + d_nx^n)/1 \notin Z_{\mathrm{r}}((T^{-1}R)[x])$, where v is the left inverse of t. This guarantees that for any $h(x) \in R[x] \setminus \{0\}$, $h(x)/1 \cdot (v/1)G(x) = h(x)g(x)/1 \neq 0/1$. Hence, $g(x) \notin Z_{\mathrm{r}}(R[x])$ which leads us to the conclusion. □

In accordance with Theorem 2.23, we can possess the following corollaries.

Corollary 2.24. *Let R be a left Ore ring with the classical left quotient ring $Q^l_{cl}(R)$. If R is left EM, then $Q^l_{cl}(R)$ is left EM.*

As a well-known result, a ring R is semiprime left Goldie if and only if there exists the classical left quotient ring of R which is semisimple Artinian, see Theorem 11.13 of [31].

Corollary 2.25. *Let R be a semiprime left Goldie ring. If R is left EM, then $Q^l_{cl}(R)$ is left EM and a product of division rings.*

P r o o f. This proof follows from Proposition 2.22 and Corollary 2.24. □

Let Δ be a multiplicative subset of a ring R consisting of central regular elements. Then Δ is a (left and right) denominator set of R and thus, $\Delta^{-1}R \cong R\Delta^{-1}$ forms a ring with unity. Hence, if R is a left EM ring, then $R\Delta^{-1}$ is a left EM ring by Theorem 2.23 (1). Clearly, $\Delta = \{1, x, x^2, \ldots\}$ is a multiplicative subset consisting of central regular elements of the polynomial ring $R[x]$ over a ring R. From the fact that $R[x; x^{-1}] \cong \Delta^{-1}(R[x])$, where $R[x; x^{-1}]$ is the Laurent polynomial ring over R, the one-sided EM condition of a ring R can be ascent to the Laurent polynomial ring of R.

Corollary 2.26. *If R is a left EM ring, then $R[x; x^{-1}]$ is a left EM ring.*

P r o o f. This proof is clear by Theorems 2.10 and 2.23, and the fact that $R[x; x^{-1}] \cong \Delta^{-1}(R[x])$. □

Question 2.27. Does the converse statement of Corollary 2.26 hold also true?

We close this section with the following result.

Proposition 2.28 (cf. [23], Proposition 2.6 (1))**.** *The class of left EM rings is closed under direct limits with injective maps.*

P r o o f. Let U be a directed partially ordered set and $D = \langle M_u, \alpha_{u,v} \rangle$ be a direct system of left EM rings R_u for $u \in U$ and homomorphisms $\alpha_{u,v} \colon R_u \to R_v$ for each $u \leqslant v$ satisfying $\alpha_{u,v}(1) = 1$. Set $\mathcal{R} = \varinjlim R_u$ to be the direct limit of D with $\beta_u \colon R_u \to \mathcal{R}$ and $\beta_v \circ \alpha_{u,v} = \beta_u$. As a well-known result, $\mathcal{R}$ is a unital ring, see Theorem 3.10.1 of [32]. Now, let $f(x) \in \mathcal{R}[x]$. Then there exist an element $w \in U$ and a polynomial $f_w(x) \in R_w[x]$ such that $\beta_w(f_w(x)) = f(x)$, extending the map β_w from R_w to $R_w[x]$ in the obvious manner. Inasmuch as R_w is left EM, we can find an element $a_w \in R_w$ and a polynomial $g_w(x) \in R_w[x]$ such that $f_w(x) = a_w g_w(x)$ and $g_w(x) \notin Z_{\mathrm{r}}(R_w[x])$. This guarantees that $f(x) = \beta_w(f_w(x)) = \beta_w(a_w)\beta_w(g_w(x)) = ag(x)$ with $a \in \mathcal{R}$ and $g(x) \notin Z_{\mathrm{r}}(\mathcal{R}[x])$ by the injectivity hypothesis. With this result, the proof that $f(x)$ is a left ACP is complete. □

3. Implications

In this section, we focus on the relationships between left EM rings and other known rings. The readers can refer to the implication diagrams in [9], [13], [17] for more details, and can view the extended implication diagram at the end of this section. First, we extend Theorem 3.18 of [2] by weakening the commutative condition to the noncommutative condition.

Proposition 3.1 (cf. [2], Theorem 3.18)**.** *If R is a left or right EM ring, then R is Armendariz.*

P r o o f. Without loss of generality, we assume that R is left EM. Let $a(x) = \sum_{i=0}^{l} a_i x^i$ and $b(x) = \sum_{j=0}^{m} b_j x^j$ be nonzero polynomials in $R[x]$ with $a(x)b(x) = 0$. Since R is a left EM ring, there exist a nonzero element $b \in R$ and polynomial $c(x) \in R[x]$ such that $b(x) = bc(x)$ and $c(x) = \sum_{k=0}^{n} c_k x^k \notin Z_{\mathrm{r}}(R[x])$ with $b_j = bc_j$ for each j. This implies that $a(x)b = 0$ and thus, $a_i b = 0$ for each i. Hence, $a_i b_j = a_i b c_j = 0$ for each i and j, completing the proof. □

By the implications (see [8], [13], [17], [23]), we have the following corollary.

Corollary 3.2. *If R is a left or right EM ring, then R is strongly McCoy, McCoy, ZPZC, abelian, Camillo, and Dedekind finite.*

Remark 3.3.

(1) Example 2.1 shows that there exists a left EM ring which is not reduced because $b^2 = 0$, but $b \neq 0$. Conversely, the commutative ring $R = \mathbb{Z}_6[x, y]/(xy)$ is reduced, but not EM, see Example 3.11 of [2]. Thus, none implies the other. Further, from the fact that every reduced ring is Armendariz, the converse of Proposition 3.1 is not true in general.

(2) A ring R is called *symmetric* if $abc = 0$ implies $bac = 0$, and *reversible* if $ab = 0$ implies $ba = 0$ for all $a, b, c \in R$. Obviously, symmetric rings must be reversible. Example 2.1 also shows that left EM rings need not be symmetric and reversible in general because $ab \cdot 1 = 0$, but $ba \cdot 1 \neq 0$ in Example 2.1.

(3) We say that a ring R is *left duo* if each left ideal of R is two-sided (equivalently, $rR \subseteq Rr$ for each $r \in R$). *Right duo* rings are defined symmetrically. If a ring R is left and right duo, then R is called *duo*. Example 2.1 also guarantees that left EM rings do not have to be left and right duo in general. Indeed, $\left\{ \sum_{j=0}^{\text{finite}} \beta_j b a^j \right\} = bR \not\subseteq Rb = \{0, b\}$ in Example 2.1, which means that R is not left

duo. Also, $\Big\{ \sum_{i=1}^{\text{finite}} \alpha_i a^i + \sum_{j=1}^{\text{finite}} \beta_j b a^j \Big\} = Ra \not\subseteq aR = \Big\{ \sum_{i=1}^{\text{finite}} \alpha_i a^i \Big\}$ in Example 2.1, and so R is not right duo.

Recall that a ring R is called *NI* if the set of all nilpotent elements is an ideal of R. The next example shows that there exists an EM ring that is not an NI ring. Thus, EM rings do not have to be semicommutative or 2-primal in general by the implications.

Example 3.4. Let $\mathcal{R} = \mathbb{Z}_2\langle a, b\rangle$ be the free algebra with unity and noncommuting indeterminates a, b over $\mathbb{Z}_2$. Set $R = \mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathcal{R}$ generated by the relation $b^2 = 0$. We denote $r = r + \mathcal{I}$ for simplicity. Then the ring R is not NI since b is nilpotent, but ba is not nilpotent.

To show that the ring R is EM, let $f(x)$ be a nonzero polynomial in $R[x]$. If $f(x) \notin Z_{\mathrm{r}}(R[x])$, then we are done since $f(x) = 1 \cdot f(x)$ with $f(x) \notin Z_{\mathrm{r}}(R[x])$. So, we may assume that $f(x) \in Z_{\mathrm{r}}(R[x])$. Then, by the same argument as in the proof of Example 2.11 of [13], the elements $1, a^k, \ldots,$ and $a^{k_1} b a^{k_2} b \ldots a^{k_n} b$ do not belong to the support of any coefficient of $f(x)$ for any $k, k_1, \ldots, k_n \in \mathbb{N}$. This implies that each coefficient of $f(x)$ can be written in the form br for some $r \in R$. Therefore, we can write $f(x) = bg(x)$ for some $g(x) \in R[x]$. Because one of $1, a^k, \ldots,$ and $a^{k_1} b a^{k_2} b \ldots a^{k_n} b$ belongs to the support of a coefficient of $g(x)$, we obtain that $g(x) \notin Z_{\mathrm{r}}(R[x])$, confirming that $f(x)$ is a left ACP. Further, by the same argument as above, one can conclude that R is a right EM ring.

Remark 3.5.

(1) Example 3.4 shows that strongly McCoy rings need not be duo, semicommutative, 2-primal in general.

(2) A ring R is said to be *left eversible* if $Z_{\mathrm{l}}(R) \subseteq Z_{\mathrm{r}}(R)$. Of course, R is *right eversible* if $Z_{\mathrm{r}}(R) \subseteq Z_{\mathrm{l}}(R)$, and *eversible* if R is left and right eversible. Example 3.4 leads us to the fact that EM rings do not have to be left eversible in general since $ab \in Z_{\mathrm{l}}(R)$, but $ab \notin Z_{\mathrm{r}}(R)$.

Recall that a ring R is called *left Gaussian* if $c_{\mathrm{l}}(f(x)g(x)) = c_{\mathrm{l}}(f(x))c_{\mathrm{l}}(g(x))$ for any $f(x), g(x) \in R[x]$, where $c_{\mathrm{l}}(f(x))$ is the left ideal of R generated by the coefficients of $f(x)$. The *right Gaussian* rings are defined analogously, and Gaussian rings are both left and right Gaussian. Every left Gaussian ring is left duo and Armendariz, see page 132 of [36]. As stated in Remark 2.2 (1) of [9], the commutative ring $\mathbb{Z}_4$ is Gaussian, but not reduced. Conversely, the polynomial ring $\mathbb{Z}[x]$ is reduced, but not Gaussian, see Remark 2.2 (2) of [9]. Thus, none implies the other between reduced rings and one-sided Gaussian rings. Further, EM rings do not have to be Gaussian in general because $\mathbb{Z}[x]$ is (domain and so) EM.

Example 3.6.

(1) According to Example 3.7 of [1], commutative Gaussian rings may not be (both left and right) EM.

(2) See Theorem 4.2. A ring R is said to be *right uniserial* (or *right chain*) if all right ideals of R are totally ordered by inclusion. Left uniserial rings are defined symmetrically. Certainly, a uniserial ring denotes a left and right uniserial ring. The ring $\overline{R}$ in Example 2.5 of [13] is a duo uniserial ring, but not left eversible. Since $\overline{R}$ is Gaussian by Corollary 2.4 of [36], we can notice that Gaussian rings may not be left eversible.

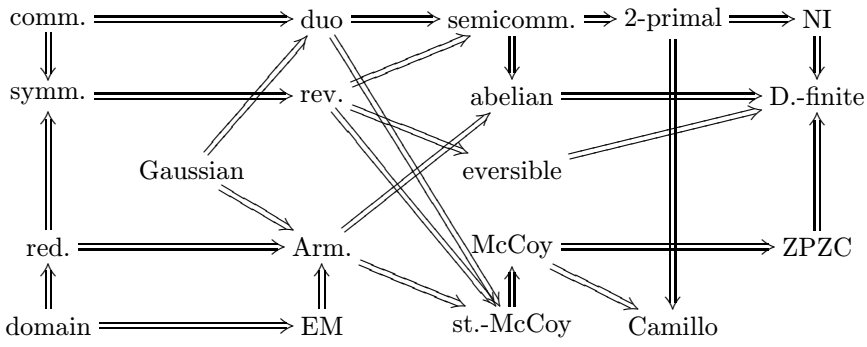
The "abelian" condition and the one-sided "Camillo" condition are independent conditions by the following example.

Example 3.7. Let $\mathcal{R} = \mathbb{Z}_2\langle a_0, a_1, b_0, b_1\rangle$ be the free algebra with unity and noncommuting indeterminates labeled above over $\mathbb{Z}_2$. Set $R = \mathcal{R}/\mathcal{I}$, where $\mathcal{I}$ is the ideal of $\mathcal{R}$ generated by the following relations:

$$a_0b_0 = 0, \quad a_0b_1 = a_1b_0, \quad a_1b_1 = 0.$$

We suppress all the bar notations for simple expression. It can be easily shown that the ring R has only two idempotent elements 0 and 1. So, R is abelian. Now, consider the nonzero polynomials $a(x) = a_0+a_1x$ and $b(x) = b_0+b_1x$. Then $a(x)b(x) = 0$, but there is no $r \in R$ such that $ra(x) = 0$ or $rb(x) = 0$. Further, there is no $s \in R$ such that $a(x)s = 0$ or $b(x)s = 0$. Thus, the abelian ring R is neither left nor right Camillo.

Finally, one can see at a glance all implications among some rings related to EM rings in the following diagram. From the results of this paper or the diagrams in [13], [17], no other implications hold except for transitivity. To create a simpler diagram, we avoided using the words "left" and "right". Of course, one can notice that all implications among the one-sided rings (if they exist) in the diagram also hold. For instance, left EM $\Rightarrow$ left st. McCoy $\Rightarrow$ left McCoy $\Rightarrow$ left ZPZC.



4. Properties of left EM rings

In this section, we study various properties of left EM rings. We first introduce some twisted properties. Recall that a ring R is called *left Bézout* if every finitely generated left ideal of R is principal. Based on Corollary 2.3 of [2], one may suspect that if R is a left Bézout ring, then R is a left EM ring. However, the following example shows that this suspicion is unsupported.

Example 4.1. See Example 2.8 (1) of [19]. Note that the 2×2 matrix ring $\mathbb{M}_2(D)$ over a division ring D is left Bézout. However, $\mathbb{M}_2(D)$ is never left EM by Example 2.20 (1). Further, notice that $\mathbb{M}_2(D)$ is a semisimple, right Bézout, and principal (left and right) ideal ring. Thus, an ordinary semisimple or right Bézout or principal (left or right) ideal ring need not be left EM.

It is easy to show that every right uniserial ring is right Bézout. With the help of Lemma 2.3 of [5], and Corollary 6.3 of [35], we obtain a twisted result.

Theorem 4.2. *If a ring R is right uniserial, then R must be left EM.*

P r o o f. Consider a nonzero polynomial $f(x) = \sum\limits_{i=0}^{m} a_i x^i \in Z_{\mathrm{r}}(R[x])$. Since R is right uniserial, $a_0R+\ldots+a_mR = cR$ for a nonzero $c = a_k \in R$. Then for each i there exist $s_i, t_i \in R$ such that $a_i = cs_i$ and $c = \sum\limits_{i=0}^{m} a_i t_i$. Hence, $c = c\Big(\sum\limits_{i=0}^{m} s_i t_i\Big)$ and thus, $c\Big(1 - \sum\limits_{i=0}^{m} s_i t_i\Big) = 0$. Now, let $g(x) = s_0 + \ldots + s_m x^m + \Big(1 - \sum\limits_{i=0}^{m} s_i t_i\Big)x^{m+1} \in R[x]$. Then $f(x) = cg(x)$. Because $1 \in s_0R + \ldots + s_mR + \Big(1 - \sum\limits_{i=0}^{m} s_i t_i\Big)R$, $R = s_0R + \ldots + s_mR + \Big(1 - \sum\limits_{i=0}^{m} s_i t_i\Big)R$. Finally, we only need to show that $g(x) \notin Z_{\mathrm{r}}(R[x])$. Assume that there exists a nonzero polynomial $h(x) \in R[x]$ such that $h(x)g(x) = 0$. From the fact that R is Armendariz by Corollary 6.3 of [35], there exists a nonzero element $d \in R$ such that $dg(x) = 0$. This implies that $d\Big(s_0R+\ldots+s_mR+\Big(1-\sum\limits_{i=0}^{m} s_i t_i\Big)R\Big) = dR = 0$. However, this is absurd. Therefore, $g(x) \notin Z_{\mathrm{r}}(R[x])$, which indicates that $f(x)$ is a left ACP, as required. □

Question 4.3. When a ring R is left uniserial, is it true that R is left EM?

By applying almost the same reasoning as that used in the proof of Theorem 4.2, we obtain a particular answer to this question.

Corollary 4.4. *Let a left uniserial ring R be given. The following assertions hold true.*

(1) *If R is right duo, then R is left EM.*

(2) *If R is right Gaussian, then R is left EM.*

Following the same approach as that used in the proof of Theorem 4.2, we provide sufficient conditions that right Bézout rings or principal right ideal rings imply left EM.

Corollary 4.5 (cf. [2], Corollary 2.3)**.** *A ring R is necessarily left EM if one of the following statements holds true:*
(1) *R is a right Bézout and left McCoy ring.*
(2) *R is a principal right ideal ring and left McCoy ring.*

Remark 4.6.
(1) As the symmetric result of Theorem 4.2, every left uniserial ring is right EM. Thus, we already extend Corollary 6.3 of [35].
(2) Example 4.1 guarantees that the "left McCoy" condition is essential in Corollary 4.5.
(3) By the implications, we can replace the "left McCoy" condition with one of the conditions "left strongly McCoy", "left duo", "reversible", "Armendariz", "left Gaussian", "right Gaussian", "symmetric" or "reduced" in Corollary 4.5.

Let S be a multiplicative subset and A be a right ideal of a ring R. Recall that A is called *S-principal* if there exist an element $s \in S$ and a principal right ideal P of R such that $As \subseteq P \subseteq A$. Recall also that R is an *S-principal right ideal ring* if every right ideal is S-principal, see Definition 1.1 of [10]. Trivially, every principal right ideal ring is always an S-principal right ideal ring for every multiplicative subset S. Based on Corollary 4.5 (2), one may suspect that if an S-principal right ideal ring R is left McCoy, then R is left EM. However, the following example shows that there exists an S-principal right ideal ring which is left McCoy, but not left EM.

Example 4.7. See Example 2.6 of [10]. Recall the commutative S-principal ideal ring R in Example 2.6 of [10]. By commutativity, R is (left) McCoy. Let $f(x) = a_1 + a_2x \in R[x]$. Since $a_1sf(x) = 0$ by the relations ${a_1}^2 = {a_2}^2 = 0$ and $a_1s = a_2s$, $f(x)$ is a (right) zero-divisor in $R[x]$. By the relations, we can notice that if $f(x) = rg(x)$ for some $r \in R$ and $g(x) \in R[x]$, then $r = 1_R$ and $g(x) = f(x)$. This indicates that the (right) zero-divisor polynomial $f(x)$ is not (left) ACP. Further, the finitely generated ideal $a_1R + a_2R$ is not principal, which indicates that R is not (right) Bézout. Thus, the "right Bézout" or "right principal ideal ring" condition is not superfluous in Corollary 4.5.

In reference to Definition 1.1 of [25], we say that a ring R has *right Property* (A) if for every finitely generated (two-sided) ideal $I \subseteq Z_l(R)$, there exists a nonzero element $r \in R$ such that $Ir = \{0\}$. We do not know whether the polynomial ring $R[x]$ has right Property (A) even if the ring R has right Property (A), see

Question 1 (2) of [25] and Question 1.2 of [22]. According to Remark 2 of Section 2 of [25], and Corollary 1 of [26], there exists a commutative ring R which does not have Property (A) whose polynomial ring $R[x]$ has Property (A).

Proposition 4.8. *For a left EM ring R, the following assertions hold true:*
(1) *If R has right Property* (A)*, then $R[x]$ has right Property* (A).
(2) *If $R[x]$ has right Property* (A) *and every polynomial whose entire coefficients are left zero-divisors in R is a left zero-divisor polynomial, then R has right Property* (A).

P r o o f. (1) This follows immediately from Proposition 2.9 of [25], and Corollary 3.2.

(2) Let $p_1, \ldots, p_k \in R$ such that $P = Rp_1R + \ldots + Rp_kR \subseteq Z_l(R)$. To find a nonzero element $r \in R$ such that $Pr = 0$, consider the ideal $\mathcal{P} = R[x]p_1R[x] + \ldots + R[x]p_kR[x]$ in $R[x]$. By the hypothesis that every polynomial whose entire coefficients are left zero-divisors in R is a left zero-divisor polynomial, we can notice that $\mathcal{P} \subseteq Z_{\mathrm{l}}(R[x])$. Because $R[x]$ has right Property (A), there exists a nonzero polynomial $f(x) \in R[x]$ such that $\mathcal{P}f(x) = 0$. Since R is left EM, we can find a nonzero element $a \in R$ and a nonzero polynomial $g(x) \in R[x]$ such that $f(x) = ag(x)$ and $g(x) \notin Z_{\mathrm{r}}(R[x])$. Thus, $Pf(x) = 0$, which implies $Pa = 0$. □

Remark 4.9. See Remark 4.31 (4). Rings with right Property (A) might not be left EM. For instance, let D be a domain. Obviously, D satisfies right Property (A). Thus, $\mathbb{M}_2(D)$ has right Property (A) by Theorem 2.1 of [25]. However, $\mathbb{M}_2(D)$ is not left EM by Example 2.20 (1).

Recall that R is called a *right p.p. ring* if the right annihilator of each element of R is generated by an idempotent. Right p.p. rings are often called *right Rickart rings*. A *left p.p. ring* is defined symmetrically, and R is called a *p.p. ring* if it is a left and right p.p. ring. Following Lemma 2 of [20], if a left p.p. ring R is abelian, then for every element c of R, there exist an idempotent e and a right non zero-divisor a of R such that $c = ea$. Additionally, if a ring R is abelian, then R is a left p.p. ring if and only if R is a right p.p. ring by Proposition 2 of [20]. Therefore, we have the following equivalence conditions.

Theorem 4.10 (cf. [2], Theorem 3.21)**.** *Let R be a left or right p.p. ring. The following statements are equivalent:*

(1) *R is reduced.* (2) *R is symmetric.* (3) *R is reversible.*
(4) *R is semicommutative.* (5) *R is Armendariz.* (6) *R is abelian.*
(7) *R is (left and right) EM.*

P r o o f. The equivalences (1)⇔(2)⇔(3)⇔(4)⇔(5)⇔(6) follow from Theorem 2.14 of [3], and page 483 of [29]. (7)⇒(5) follows from Proposition 3.1. Lastly, the proof of (5)⇒(7) is identical to that of Theorem 3.21 of [2]. □

We say that a ring R is *Baer* if for every nonempty subset X, the right annihilator of X, $\mathrm{r.ann}_R(X) = eR$ for an idempotent e, see page 3 of [28]. By Theorem 3 of [28] the "Baer" condition is left-right symmetric. Clearly, every Baer ring is a (left and right) p.p. ring and every semisimple ring is Baer, see Example 7.47 of [31]. Hence, Theorem 4.10 ensures that the following corollary holds true.

Corollary 4.11. *For a semisimple or Baer ring R, the following statements are equivalent:*

(1) *R is reduced.* (2) *R is symmetric.* (3) *R is reversible.*
(4) *R is semicommutative.* (5) *R is Armendariz.* (6) *R is abelian.*
(7) *R is (left and right) EM.*

A ring R is called (*von Neumann*) *regular* if for each $a \in R$, there exists $b \in R$ such that $a = aba$. It is well-known that every regular ring is a right p.p. ring, see (7.50) of [31]. A ring R is called *right APIP* if the right annihilator of any nonzero element $r \in R$ contains an ideal generated by some power of any right zero-divisor of the element r, see [12]. Even though, for regular rings, there are many equivalent conditions to being reduced, we add the one-sided EM condition into the list.

Corollary 4.12. *The following conditions are equivalent for a regular ring R:*

(1) *R is reduced.* (2) *R is abelian.*
(3) *R is (left and right) APIP.* (4) *R is (left and right) EM.*
(5) *R is (left and right outer) McCoy.* (6) *R is (left and right) ZPZC.*

P r o o f. These equivalences are made clear by Proposition 2.17 of [8], the diagram in page 1958 of [12], Proposition 2.12 of [13], and Theorem 4.10. □

A ring R is called *strongly (von Neumann) regular* if for any $a \in R$ there exists $x \in R$ such that $a = a^2x$. Due to Theorem 9.30 of [31], a ring R is strongly regular if and only if R is regular and reduced. Thus, we have:

Corollary 4.13. *For a ring R, R is strongly regular if and only if R is regular and one of the conditions in the previous corollary holds.*

P r o o f. This follows directly from Theorem 9.30 of [31], and Corollary 4.12. □

According to Definition 1.1 of [24], a ring R is said to have the *left annihilator condition* (simply, *left (a.c.)*) if for any finitely generated two-sided ideal I of R, there exists $p \in R$ such that $\mathrm{l.ann}_R(I) = \mathrm{l.ann}_R(RpR)$. Rings having the *right (a.c.)* are defined analogously, and a ring has the (a.c.) if it has the left and right (a.c.). For a ring R, $\mathrm{Spec}(R)$ means the set of all prime ideals of R and $\mathrm{Min}(R)$ means the set of all minimal prime ideals of R. Thus, $\mathrm{Min}(R)$ is a subspace of the hull-kernel topology $\mathrm{Spec}(R)$, see Section 3 of [25]. Application of Theorem 4.10 to Theorem 3.4 of [24] gives:

Corollary 4.14. *If a left or right p.p. ring R is left or right EM with its classical right quotient ring $Q^r_{cl}(R)$, then the following statements are equivalent:*
(1) *$Q^r_{cl}(R)$ is strongly regular.*
(2) *R has the (a.c.) and $\mathrm{Min}(R)$ is compact.*
(3) *R has Property* (A) *and $\mathrm{Min}(R)$ is compact.*

Clearly, $\mathrm{l.ann}_R(Ra) = \mathrm{l.ann}_R(RaR)$ and $\mathrm{l.ann}_R(Ra + Rb) = \mathrm{l.ann}_R(Ra) \cap \mathrm{l.ann}_R(Rb)$ for $a, b \in R$, see Remark 1 (1) of [24]. Based on these basic facts, and Theorem 3.22 of [2], we obtain the following result.

Theorem 4.15 (cf. [2], Theorem 3.22)**.** *If R is a left EM ring, then it has the left (a.c.).*

P r o o f. Let $A = Ra_0R + \ldots + Ra_kR$ be a finitely generated ideal of R. We wish to show that $\mathrm{l.ann}_R(A) = \mathrm{l.ann}_R(RpR)$ for some $p \in R$. If $\mathrm{l.ann}_R(A) = \{0\}$, then $\mathrm{l.ann}_R(A) = \{0\} = \mathrm{l.ann}_R(R1R)$. Thus, we may assume that $\mathrm{l.ann}_R(A) \neq \{0\}$. Now, consider the polynomial $f(x) = \sum_{i=0}^{k} a_i x^i \in R[x]$. Since $\mathrm{l.ann}_R(A) \neq \{0\}$, $f(x) \in Z_{\mathrm{r}}(R[x])$. Because R is left EM, there exist $c \in R$ and $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $f(x) = cg(x)$, $c \in Z_{\mathrm{r}}(R)$, and $g(x) \notin Z_{\mathrm{r}}(R[x])$. Consequently, $\mathrm{l.ann}_R(b_0, \ldots, b_n) = \{0\}$.

To show that $\mathrm{l.ann}_R(A) = \mathrm{l.ann}_R(RcR)$, let $d \in \mathrm{l.ann}_R(A)$ and $e \in \mathrm{l.ann}_R(RcR)$. Then for arbitrary $r \in R$ and every i, we obtain $dra_i = drcb_i = 0$, which implies that $drc = 0$. Hence, $d \in \mathrm{l.ann}_R(Rc) = \mathrm{l.ann}_R(RcR)$. Conversely, from the fact that $e(r_0a_0 + \ldots + r_ka_k) = e(r_0cb_0 + \ldots + r_kcb_k) = 0$ for arbitrary $r_0, \ldots, r_k \in R$, we see that $e \in \mathrm{l.ann}_R(Ra_0 + \ldots + Ra_k) = \mathrm{l.ann}_R(Ra_0) \cap \ldots \cap \mathrm{l.ann}_R(Ra_k) = \mathrm{l.ann}_R(Ra_0R) \cap \ldots \cap \mathrm{l.ann}_R(Ra_kR) = \mathrm{l.ann}_R(Ra_0R + \ldots + Ra_kR) = \mathrm{l.ann}_R(A)$. Therefore, $\mathrm{l.ann}_R(A) = \mathrm{l.ann}_R(RcR)$ as required. □

Remark 4.16.
(1) In Example 2.1, let I be a finitely generated ideal of the ring R. Write $r \in R$ of the form $\sum_{\lambda=0}^{m} \alpha_\lambda a^\lambda + \sum_{\mu=0}^{n} \beta_\mu b a^\mu$ for a nonzero α_λ. Then one can easily

notice that $\mathrm{l.ann}_R(r) = \{0\}$. Thus, if I contains an element r of the form noted above, then $\mathrm{l.ann}_R(I) = \{0\}$. If I does not contain any element of the form, then I consists only of elements of the form $\sum_{\mu=0}^{n} \beta_\mu b a^\mu$. Hence, $\mathrm{l.ann}_R(I) = \mathrm{l.ann}_R(Rba^kR)$ for some $k \in \mathbb{N} \cup \{0\}$. Therefore, the result of Theorem 4.15 is already exemplified.

(2) Consider the element $r \in R$ of the form $r = 1 + \sum_{\lambda=1}^{m} \alpha_\lambda a^\lambda + \sum_{\mu=0}^{n} \beta_\mu b a^\mu$ in Example 2.1. One can easily notice that $\mathrm{r.ann}_R(r) = \{0\}$ by the relations. Thus, if a finitely generated ideal I of R contains an element r of the aforementioned form, then $\mathrm{r.ann}_R(I) = \{0\}$. Additionally, if I does not contain any element of the aforementioned form, then $\mathrm{r.ann}_R(I) = \mathrm{r.ann}_R(RaR) = RbR$. Hence, the ring R has the right (a.c.). Therefore, although the opposite ring R^{op} has the left (a.c.), it is not a left EM ring by Claim 2.2 of Example 2.1. This means that the converse statement of Theorem 4.15 does not hold true in general. By Example 2.20 (1), one can find another ring $\mathbb{M}_2(\mathbb{Z}_2)$, see Proposition 1.17 of [24].

The following result can be deduced immediately from Theorems 4.10 and 4.15.

Corollary 4.17. *Let R be a right p.p. ring. If R is symmetric, semicommutative, Armendariz or abelian, then R has the left and right (a.c.).*

We still do not know whether the polynomial ring $R[x]$ has the left (a.c.), even if a ring R has the left (a.c.), see Question 1 of [24]. Theorems 2.10 and 4.15 enable a particular resolution of this matter.

Corollary 4.18. *If R is a left EM ring, then $R[x]$ has the left (a.c.).*

In [7], Anderson and Dumitrescu introduced the commutative S-Noetherian notion, which is an extended notion of commutative Noetherian rings. Baeck et al. in [14] and Bilgin et al. in [16] generalized this concept to one-sided S-Noetherian rings and modules, independently. Let S be a multiplicative subset of a ring R. For a right ideal A of R, A is called *S-finite* if there exist an element $s \in S$ and a finitely generated right ideal F of R such that $As \subseteq F \subseteq A$, and R is called *right S-Noetherian* if every right ideal is S-finite. Trivially, every finitely generated right ideal is S-finite, and all right Noetherian rings are right S-Noetherian for any multiplicative subset S. Now, we totally extend Proposition 3.23 of [2].

Theorem 4.19 (cf. [2], Proposition 3.23)**.** *Let S be a multiplicative subset of a left EM ring R, and $t \in R$. If $\mathrm{r.ann}_R(t)$ is S-finite, then $\mathrm{r.ann}_R(t)$ is S-principal.*

P r o o f. If $\mathrm{r.ann}_R(t)s = \{0\}$ for some $s \in S$, then we are done. So, we may assume that $\mathrm{r.ann}_R(t)s \neq \{0\}$ for every $s \in S$. Since $\mathrm{r.ann}_R(t)$ is S-finite, there exist $s \in S$ and $a_0, \ldots, a_n \in R$ such that $\mathrm{r.ann}_R(t)s \subseteq a_0R + \ldots + a_nR \subseteq \mathrm{r.ann}_R(t)$. Consider the polynomial $f(x) = \sum_{i=0}^{n} a_i x^i \in R[x]$. Then $tf(x) = 0$ and thus, $f(x) = cg(x)$, where $c \in R$ and $g(x) = \sum_{j=0}^{k} b_j x^j \notin Z_{\mathrm{r}}(R[x])$. Hence, $a_i = cb_i$ for each i and so $a_0R + \ldots + a_nR \subseteq cR$. Additionally, because $ta_i = tcb_i = 0$ and $\mathrm{l.ann}_R(b_0, \ldots, b_k) = \{0\}$, we can notice that $tc = 0$, which means that $c \in \mathrm{r.ann}_R(t)$. Therefore, $cR \subseteq \mathrm{r.ann}_R(t)$. Consequently, we obtain that $\mathrm{r.ann}_R(t)s \subseteq a_0R + \ldots + a_nR \subseteq cR \subseteq \mathrm{r.ann}_R(t)$, which is the desired conclusion. □

Theorem 4.19 has the following corollaries that are worthy of separate statements.

Corollary 4.20. *Let S be a multiplicative subset of a right S-Noetherian ring R. If R is left EM, then the set of all the right annihilators of a single element is necessarily S-principal.*

By Theorem 2.9 of [33], Theorem 2.17 (2), and Corollary 4.20, we can possess the following.

Corollary 4.21. *Let $R \subseteq E$ be rings with the same unity and e be an element in E such that $eR = Re$, $\mathrm{r.ann}_R(e) = \{0\}$, $e^2 = 0$, and $er \notin R$ for each $0 \neq r \in R$. Then the following statements hold true:*

(1) *If E is right S-Noetherian and $Es \subseteq R + eR$ for some $s \in S$, then R is right S-Noetherian.*

(2) *If E is both right S-Noetherian, left EM, and there exists an element $s \in R$ such that s is regular in R, central in E, and $Es \subseteq Rs + eRs$, then $\mathrm{r.ann}_R(t)$ is S-principal for each $t \in R$.*

According to Definition 2.1 of [39], a ring R is called *right generalized morphic* if $\mathrm{r.ann}_R(r)$ is a principal right ideal for each $r \in R$. *Left generalized morphic* rings are defined similarly. Applying $S = \{1\}$ in Corollary 4.20, we obtain:

Corollary 4.22. *If R is a right Noetherian and left EM ring, then R is a right generalized morphic ring.*

Corollary 4.23. *Let $R \subseteq E$ be rings with the same unity, and let e be an element in E such that $eR = Re$, $\mathrm{r.ann}_R(e) = \{0\}$, $e^2 = 0$, and $er \notin R$ for each $r \in R$. If E is both right Noetherian and left EM, and $E = R + eR$, then R is a right generalized morphic ring.*

Remark 4.24.

(1) There exists a left Noetherian and right EM ring which is not right Noetherian, see [27]. Observing the opposite ring, there also exists a right (S)-Noetherian and left EM ring that is not left (S)-Noetherian.

(2) See Example 3.11 of [2]. Let $R = \mathbb{Z}_6[x,y]/(xy)$ and $S = \{\bar{1}\}$. Then R is a commutative (S)-Noetherian ring, but not a (left) EM ring. Since $\mathrm{r.ann}_R(\bar{2}\bar{y}) = \bar{3}R + \bar{x}R$ is not (S)-principal, the "(left) EM" condition is not superfluous in Theorem 4.19, Corollaries 4.20, and 4.22.

(3) Recall the ring R given in Example 2.1 once more. If R is right Noetherian, then $\mathcal{R}$ is right Noetherian by Proposition 1.2 of [21]. But this is absurd by Exercise 1E (b) of [21]. Thus, R is not right (S)-Noetherian, where $S = \{1\}$. Hence, the "(S)-Noetherian" condition and the "EM" condition are independent.

(4) Obviously, every principal right ideal ring is right generalized morphic. From this fact, one may suspect that every S-principal right ideal ring is right generalized morphic. For a counterexample, recall the commutative S-principal ideal ring R in Example 4.6 of [11]. Here, one can easily observe that $\mathrm{ann}_R(s) = (1+s)R + a_2R + a_3R + \ldots$, confirming that R is not a generalized morphic ring.

(5) Example 4.1 tells us that the converse statement is generally not true in Corollaries 4.20 and 4.22.

Question 4.25.

(1) Is there a right Noetherian and left EM ring which is not left generalized morphic?

(2) When R is a right Noetherian and right EM ring, is it also true that R is right generalized morphic?

Let $f(x) = \sum_{i=0}^{n} a_i x^i$ be a polynomial in the polynomial ring $R[x]$ over a ring R. We recall that the right ideal $c_{\mathrm{r}}(f(x)) = a_0R + \ldots + a_nR$ of R is called the *right content* of $f(x)$. The proof of the next theorem is nearly identical to that of Theorem 2.5 of [4], but we insert it here for the sake of completeness.

Theorem 4.26 (cf. [4], Theorem 2.5)**.** *Let $f(x)$ be a polynomial in the polynomial ring $R[x]$ over a ring R.*

(1) *If $f(x)$ is a left ACP, then $c_{\mathrm{r}}(f(x)) = aB$ for some $a \in R$ and finitely generated right ideal B of R with $\mathrm{l.ann}_R(B) = \{0\}$.*

(2) *If R is a left McCoy ring, then the converse of* (1) *holds true.*

P r o o f. (1) Since $f(x) = \sum_{i=0}^{m} a_i x^i$ is a left ACP, there exist an element $a \in R$ and a polynomial $g(x) = \sum_{j=0}^{n} b_j x^j \in R[x]$ such that $f(x) = ag(x)$ and $\mathrm{l.ann}_{R[x]}(g(x)) = \{0\}$.

Consider the right ideal $B = b_0R + \ldots + b_nR$. Then $\mathrm{l.ann}_R(B) = \{0\}$ by the condition $\mathrm{l.ann}_{R[x]}(g(x)) = \{0\}$. From the fact that $a_i = ab_i$ for each i, we obtain that

$$c_{\mathrm{r}}(f(x)) = a_0R + \ldots + a_mR = ab_0R + \ldots + ab_mR + \ldots + ab_nR = aB.$$

(2) Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$, $a \in R$, and $B = b_1R + \ldots + b_kR$ be a right ideal of R such that $c_{\mathrm{r}}(f(x)) = aB$ and $\mathrm{l.ann}_R(B) = \{0\}$. Then $a_i = ad_i$ for some $d_i \in B$ and each i. So, $aB = c_{\mathrm{r}}(f(x)) = ad_0R + \ldots + ad_mR$. Now, let $D = d_0R + \ldots + d_mR$ and $e \in B \setminus D$. Then $ae = ad$ for some $d \in D$, and thus, $d - e \in \mathrm{r.ann}_R(a)$. This forces $B \subseteq D + \mathrm{r.ann}_R(a)$. Hence, for each $j = 1, \ldots, k$, $b_j = d'_j + e_j$, where $d'_j \in D$ and $e_j \in \mathrm{r.ann}_R(a)$. Finally, consider the polynomial $g(x) = d_0 + d_1x + \ldots + d_mx^m + e_1x^{m+1} + \ldots + e_kx^{m+k}$. By the construction, we are assured that $f(x) = ag(x)$ and $\mathrm{l.ann}_R(c_{\mathrm{r}}(g(x))) = \mathrm{l.ann}_R(d_0R + \ldots + d_mR + e_1R + \ldots + e_kR) \subseteq \mathrm{l.ann}_R(B) = \{0\}$. Therefore, the left McCoy condition leads us to the fact that $\mathrm{l.ann}_{R[x]}(g(x)) = \{0\}$, completing the proof. □

From Theorem 4.26, we can provide practical properties of left EM rings.

Corollary 4.27 (cf. [4], Corollary 2.6)**.** *If a ring R is left EM, then for each finitely generated right ideal A of R, there exist an element $a \in R$ and a finitely generated right ideal B of R such that $A = aB$ and $\mathrm{l.ann}_R(B) = \{0\}$.*

Corollary 4.28. *Let a right Noetherian ring R be given. If R is left EM, then for each right ideal A of R, there exist an element $a \in R$ and a right ideal B of R such that $A = aB$ and $\mathrm{l.ann}_R(B) = \{0\}$.*

A proper ideal P of a ring R is called *completely prime* if for $a, b \in R$, $ab \in P$ implies that $a \in P$ or $b \in P$. Equivalently, P is completely prime if and only if the factor ring R/P is a domain. We can prove the following corollary by adapting the proof of Corollary 2.7 (1) in [4].

Corollary 4.29. *Let R be a left EM ring. If a finitely generated right ideal A is completely prime of R with $\mathrm{l.ann}_R(A) \neq \{0\}$, then A is a principal ideal.*

P r o o f. By Corollary 4.27, there exist an element $a \in R$ and a finitely generated right ideal B of R such that $A = aB$ and $\mathrm{l.ann}_R(B) = \{0\}$. If $B \subseteq A$, then $tB = 0$ for a nonzero $t \in \mathrm{l.ann}_R(A) \neq \{0\}$. So, this case does not arise. Thus, $B \nsubseteq A$. Now, let $c \in B \setminus A$. From the fact that $ac \in A$ and $c \notin A$, we obtain $a \in A$. Hence, $A = aB \subseteq aR \subseteq A$. By virtue of the fact that A is a two-sided ideal of R, the equation $A = aB \subseteq aR \subseteq A$ translates into $A \subseteq RA = RaB \subseteq RaR \subseteq RA \subseteq A$. Therefore, the proper ideal $A = RaB = RaR$ is principal. □

Due to Lemma 12.6 of [30], all minimal prime ideals of a reduced ring are completely prime. By Corollary 4.29, we have:

Corollary 4.30. *Let R be a reduced and left EM ring. If a minimal prime ideal A of R is finitely generated with $\mathrm{l.ann}_R(A) \neq \{0\}$, then A is a principal ideal.*

Remark 4.31.

(1) Of course, if R is a left McCoy ring, then the converse statement of Corollary 4.27 holds true by Theorem 4.26 (2). This fact also provides a condition under which one of the following rings implies a left EM ring:

(left) duo, symmetric, reversible, (left) Gaussian, reduced, Armendariz, (left) strongly McCoy or (left) McCoy.

(2) We can exemplify the result of Corollary 4.27. For instance, consider the right ideal $A = ba^3R + (ba^2 + ba^5)R$ of the ring R in Example 2.1. Then $(ba^3)a^2 + (ba^2 + ba^5) = ba^2 \in A$. So, easy calculation shows that $A = ba^2R$ and $\mathrm{l.ann}_R(a^2R) = \{0\}$.

(3) There is a reduced and (left) EM ring. For instance, $\mathbb{Z}_6$ is such a ring. Of course, $\mathbb{Z}_6$ is a principal ideal ring.

(4) Following Theorem 82 of [28], every commutative Noetherian ring has Property (A). However, there exists a commutative EM ring that does not have Property (A), see Example 4.1 of [1].

Based on Remark 4.31 (3) and Remark 4.31 (4), we can suspect that there are noncommutative rings satisfying the conditions in the following questions. Nevertheless, the author failed to find such rings. Therefore, we end this article by formally raising these open questions.

Question 4.32.

(1) Is there a reduced and left EM ring that is not both domain and right EM?

(2) Is there a noncommutative left EM ring that does not satisfy left nor right Property (A)?

Acknowledgments. The author would like to express his gratitude to the editor and anonymous referee for their careful reading of the manuscript, many helpful suggestions, and valuable comments. The author also wishes to express his sincere appreciation to Professor Nam Kyun Kim for his suggestion, and to Professor Il Yoo for his encouraging comments and supports which were helpful for finishing this paper.

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