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THE GROTHENDIECK RING OF QUANTUM DOUBLE OF QUATERNION GROUP

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Abstract. Let $\Bbbk$ be an algebraically closed field of characteristic $p \neq 2$, and let Q_8 be the quaternion group. We describe the structures of all simple modules over the quantum double $D(\Bbbk Q_8)$ of group algebra $\Bbbk Q_8$. Moreover, we investigate the tensor product decomposition rules of all simple $D(\Bbbk Q_8)$-modules. Finally, we describe the Grothendieck ring $G_0(D(\Bbbk Q_8))$ by generators with relations.

Keywords: Grothendieck ring; simple module; quantum double; quaternion group

MSC 2020: 16G30, 16T99

1. Introduction

The quasitriangular Hopf algebra was introduced by Drinfeld. The representation category of a quasitriangular Hopf algebra is a braided tensor category. The braided structure of a quasitriangular Hopf algebra can provide solutions for the Yang-Baxter equation. Drinfeld gave a general method, by which one can construct a quasitriangular Hopf algebra from a finite dimensional Hopf algebra, i.e., the quantum double of a Hopf algebra.

The representation theory of finite dimensional Hopf algebras attracts many researchers' interests. Many important results have been achieved. For example, Liu, Van Oystaeyen and Zhang studied the representations of small quasi-quantum groups $Qu_q(sl_2)$, see [9]. They defined Quasi-Frobenius-Luszting Kernel for any simple Lie algebra, which generalizes that of sl_2, and obtained many new quasitriangular Hopf algebras, see [10]. Suter and Xiao studied the representations of the restricted quantum universal enveloping algebra associated to sl_2, see [14], [16]. Kondo and

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Saito gave the decomposition rules for the tensor products of modules over the restricted quantum universal enveloping algebra associated to sl_2, see [8]. Chen studied the representation theory of the Drinfeld doubles $D(H_n(q))$ of Taft algebras $H_n(q)$, and classified all simple modules and finite dimensional indecomposable modules over $D(H_n(q))$, see [2], [3], [4]. Dong and Chen studied the representation theory of the Drinfeld doubles $D(\Bbbk D_n)$ of dihedral group algebras $\Bbbk D_n$, see [5], [6].

In this paper, we study the representation theory of the quantum double $D(\Bbbk Q_8)$ of quaternion group Q_8. In Section 2, we recall some basic results and some concepts including Grothendieck ring, quantum double of Hopf algebras and the Yetter-Drinfeld modules. In Section 3, we classify all simple $D(\Bbbk Q_8)$-modules. In Section 4, we investigate the tensor product of any two simple $D(\Bbbk Q_8)$-modules. In Section 5, we describe the Grothendieck ring of $D(\Bbbk Q_8)$ by generators with relations.

2. Preliminaries

Throughout, let $\Bbbk$ be an algebraically closed field of characteristic $p \neq 2$, $\mathbb{Z}$ the ring of integers. Unless otherwise stated, all algebras and Hopf algebras are defined over $\Bbbk$; all modules are finite dimensional and left modules; dim and $\otimes$ mean $\dim_{\Bbbk}$ and $\otimes_{\Bbbk}$, respectively. Our references for basic concepts and notations about representation theory and Hopf algebras are [1], [7], [13]. In particular, for a Hopf algebra, we will use ε, Δ and S to denote the counit, comultiplication and antipode, respectively. For a group G and $g \in G$, let $\langle g \rangle$ denote the cyclic subgroup of G generated by g.

2.1. Grothendieck ring. The Grothendieck ring $G_0(H)$ of a Hopf algebra H is defined to be the abelian group generated by the isomorphism classes $[V]$ of V in $\operatorname{mod} H$ modulo the relations $[V] = [U] + [W]$ for all short exact sequences $0 \to U \to V \to W \to 0$ in $\operatorname{mod} H$. The multiplication of $G_0(H)$ is defined by $[U][V] = [U \otimes V]$, the tensor product of H-modules. The ring $G_0(H)$ is associative and has identity. Further, $G_0(H)$ has a $\mathbb{Z}$-basis $\{[V_i] \colon i \in I\}$, where $\{V_i \colon i \in I\}$ are all non-isomorphic simple modules. Moreover, for each $V \in \operatorname{mod} A$ we have $[V] = \sum\limits_i [V : V_i][V_i]$ in $G_0(H)$, where $[V : V_i]$ denotes the multiplicity of V_i in a composition series of V.

2.2. Quantum double of a Hopf algebra. In this subsection, we recall the quantum double of a Hopf algebra H and the Yetter-Drinfeld module over H.

In the next, we use the sigma notation: for $x \in H$, H being a coalgebra,

$$\Delta(x) = \sum_{(x)} x_1 \otimes x_2.$$

Definition 2.1 ([7], Definition IX. 4.1)**.** Let H be a finite dimensional Hopf algebra. The quantum double $D(H)$ of H is the bicrossed product $D(H) := H^{*\mathrm{cop}} \bowtie H$, also the vector space $D(H) = H^{*\mathrm{cop}} \otimes H$. Denote $f \otimes a$ by $f \bowtie a$ in $D(H)$. Then the multiplication, comultiplication and counit can be described as follows:

$$(f \otimes a)(g \otimes b) = \sum_{(a)} fg(S^{-1}(a_3)a_1) \otimes a_2 b, \quad \varepsilon(f \otimes a) = \varepsilon(a)f(1),$$
$$\Delta(f \otimes a) = \sum_{(a)(f)} (f_1 \otimes a_1)(f_2 \otimes a_2),$$

where $f, g \in H^{*\mathrm{cop}}$ and $a, b \in H$, S is the antipode of H. The sum $\sum f_1 \otimes f_2$ is determined by $\sum f_1(x)f_2(y) = f(yx)$ for all $x, y \in H$.

Definition 2.2. Let H be a Hopf algebra with a bijective antipode S. A Yetter-Drinfeld H-module M is both a left H-module and a right H-comodule satisfying the following two equivalent compatibility conditions

$$\sum h_1 \cdot m_0 \otimes h_2 \cdot m_1 = \sum (h_2 \cdot m)_0 \otimes (h_2 \cdot m)_1 h_1,$$
$$\sum (h \cdot m)_0 \otimes (h \cdot m)_1 = \sum h_2 \cdot m_0 \otimes h_3 m_1 S^{-1}(h_1).$$

Theorem 2.1 ([11])**.** *Let H be a finite dimensional Hopf algebra. Then the category ${}_H\mathcal{YD}^H$ of Yetter-Drinfeld H-modules can be identified with the category ${}_{D(H)}\mathcal{M}$ of left modules over the quantum double $D(H)$.*

Let $H = \Bbbk G$ be a finite dimensional group algebra. Let $\mathcal{K}(G)$ be the set of conjugate classes of G. For any $g \in G$, let $C_G(g) = \{x \in G\colon xg = gx\}$ be the centralizer of g in G. For any $C \in \mathcal{K}(G)$, fix a $g_C \in C$. Then $\{g_C\colon C \in \mathcal{K}(G)\}$ is a set of representatives of conjugate classes of G. Now let N be a $\Bbbk C_G(g_C)$-module. Then $N \uparrow^G = \Bbbk G \otimes_{\Bbbk C_G(g_C)} N$ is a $\Bbbk G$-module. Define a $\Bbbk$-linear map $\varphi\colon N \uparrow^G \to N \uparrow^G \otimes \Bbbk G$ by $\varphi(g \otimes n) = (g \otimes n) \otimes gg_Cg^{-1}$ for all $g \in G$ and $n \in N$. Then we have the following lemma.

Lemma 2.1 ([6])**.** *Let $C \in \mathcal{K}(G)$ and N be a $\Bbbk C_G(g_C)$-module. Then $(N \uparrow^G, \varphi)$ is a Yetter-Drinfeld $\Bbbk G$-module, denoted by $D(N)$.*

Let M be a Yetter-Drinfeld $\Bbbk G$-module with coaction $\varphi\colon M \to M \otimes \Bbbk G$. For $C \in \mathcal{K}(G)$, let

$$M_C = \bigoplus_{g \in C} M_g,$$

where $M_g = \{m \in M\colon \varphi(m) = m \otimes g\}$. One can check that M_g is a $\Bbbk C_G(g)$-module, and M_C is a Yetter-Drinfeld $\Bbbk G$-submodule of M. By [12], [15], we have the following conclusions.

Theorem 2.2. *Let $C \in \mathcal{K}(G)$ and N be a $\Bbbk C_G(g_C)$-module. Then $D(N)$ is indecomposable (or simple) Yetter-Drinfeld $\Bbbk G$-module if and only if N is an indecomposable (or simple) $\Bbbk C_G(g_C)$-module.*

Theorem 2.3. *Let M be an indecomposable (or simple) Yetter-Drinfeld $\Bbbk G$-module. Then there exists a conjugate class $C \in \mathcal{K}(G)$ such that $M \cong M_C \cong D(M_{g_C})$.*

Corollary 2.1. *Let N_1 and N_2 be an indecomposable (or simple) $\Bbbk C_G(g_C)$-module. Then $D(N_1) \cong D(N_2)$ if and only if $N_1 \cong N_2$.*

3. Simple $D(\Bbbk Q_8)$-modules

In this section, we investigate the simple $D(\Bbbk Q_8)$-modules. Since $\mathrm{char}(\Bbbk) \neq 2$, $\Bbbk Q_8$ is semisimple, and hence $D(\Bbbk Q_8)$ is semisimple.

Let $Q_8 = \langle a, b \colon a^4 = 1, a^2 = b^2, ba = a^3 b \rangle$ be the quaternion group. It is easy to check that $\{1\}$, $\{a, a^3\}$, $\{a^2\}$, $\{b, a^2 b\}$, $\{ab, a^3 b\}$ are all conjugate classes of Q_8. Let 1, a, a^2, b and ab be the representatives of conjugate classes of Q_8. Their centralizers are $C_{Q_8}(1) = Q_8$, $C_{Q_8}(a) = \{1, a, a^2, a^3\} = \langle a \rangle \cong \mathbb{Z}_4$, $C_{Q_8}(a^2) = \{1, a, a^2, a^3, b, ab, a^2 b, a^3 b\} = Q_8$, $C_{Q_8}(b) = \{1, a^2, b, a^2 b\} = \langle b \rangle \cong \mathbb{Z}_4$, $C_{Q_8}(ab) = \{1, a^2, ab, a^3 b\} = \langle ab \rangle \cong \mathbb{Z}_4$. By the discussion above, one knows that there is a one-one correspondence between the simple $\Bbbk C_{Q_8}(g_C)$-modules and simple Yetter-Drinfeld $\Bbbk Q_8$-modules, where $g_C \in \{1, a, a^2, b, ab\}$. Thus, it is enough to classify simple $\Bbbk C_{Q_8}(g_C)$-modules.

In the next, let ω be a primitive 4th root of unity. We give the classification of simple $\Bbbk Q_8$-modules and simple $\Bbbk \mathbb{Z}_4$-modules. We know that there are only 5 simple $\Bbbk Q_8$-modules up to isomorphism, and there are only 4 simple $\Bbbk \mathbb{Z}_4$-modules up to isomorphism; we describe these simple modules as follows.

For any $1 \leqslant i \leqslant 4$, let V_i be a vector space of dimension 1 with a $\Bbbk$-basis $\{v_i\}$. Then V_i is a $\Bbbk Q_8$-module with the action determined by

$$V_1 \colon av_1 = v_1,\ bv_1 = v_1; \quad V_2 \colon av_2 = v_2,\ bv_2 = -v_2;$$
$$V_3 \colon av_3 = -v_3,\ bv_3 = v_3; \quad V_4 \colon av_4 = -v_4,\ bv_4 = -v_4.$$

Let V_5 be a vector space of dimension 2 with a $\Bbbk$-basis $\{m_1, m_2\}$. Then V_5 is a $\Bbbk Q_8$-module with the action determined by

$$a(m_1, m_2) = (m_1, m_2) \begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}, \quad b(m_1, m_2) = (m_1, m_2) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Theorem 3.1. *V_1, V_2, V_3, V_4 and V_5 are all simple $\Bbbk Q_8$-modules up to isomorphism.*

P r o o f. It is easy to check that V_1, V_2, V_3, V_4 and V_5 are simple $\Bbbk Q_8$-modules, moreover they are non-isomorphic to each other. □

For any $j \in \mathbb{Z}$, let U_j be a vector space of dimension 1 with a $\Bbbk$-basis $\{u_j\}$. Then U_j is a $\mathbb{Z}_4 = \langle g \rangle$-module with the action determined by

$$gu_j = \omega^j u_j.$$

It is easy to prove the following propositions and theorems.

Proposition 3.1. *Let $i, j \in \mathbb{Z}$. Then $U_i \cong U_j$ if and only if $i \equiv j \pmod 4$.*

Theorem 3.2. *The set $\{U_j \colon 0 \leqslant j \leqslant 3\}$ is a representative set of simple $\mathbb{Z}_4 = \langle g \rangle$-modules.*

Note that $C_{Q_8}(1) = C_{Q_8}(a^2) = Q_8$. Let V_m^1 denote the simple $C_{Q_8}(1)$-module for $1 \leqslant m \leqslant 5$, and let V_m^2 denote the simple $C_{Q_8}(a^2)$-module for $1 \leqslant m \leqslant 5$. In this case, let $\{v_i^1\}$, $\{v_i^2\}$, $\{m_1^1, m_2^1\}$ and $\{m_1^2, m_2^2\}$ be the basis of V_i^1, V_i^2, V_5^1 and V_5^2 as stated before, respectively, where $1 \leqslant i \leqslant 4$.

Proposition 3.2. *For any $1 \leqslant m \leqslant 5$, let V_m^1 be the simple $\Bbbk C_{Q_8}(1)$-module. Then the induced $\Bbbk Q_8$-modules $D(V_m^1) = \Bbbk Q_8 \otimes_{\Bbbk Q_8} V_m^1 \cong V_m^1$ are non-isomorphic simple Yetter-Drinfeld $\Bbbk Q_8$-modules, with the comodule structures given by*

$$\varrho_m^1 \colon V_m^1 \to V_m^1 \otimes \Bbbk Q_8, \quad \varrho(v_m^1) = v_m^1 \otimes 1.$$

Proposition 3.3. *For any $1 \leqslant m \leqslant 5$, let V_m^2 be the simple $\Bbbk C_{Q_8}(a^2)$-module. Then the induced $\Bbbk Q_8$-modules $D(V_m^2) = \Bbbk Q_8 \otimes_{\Bbbk Q_8} V_m^2 \cong V_m^2$ are non-isomorphic simple Yetter-Drinfeld $\Bbbk Q_8$-modules, with the comodule structures given by*

$$\varrho_m^2 \colon V_m^2 \to V_m^2 \otimes \Bbbk Q_8, \quad \varrho(v_m^2) = v_m^2 \otimes a^2.$$

Note that $C_{Q_8}(a) = \langle a \rangle \cong C_{Q_8}(b) = \langle b \rangle \cong C_{Q_8}(ab) = \langle ab \rangle$. Let U_j^a, U_j^b and U_j^{ab} denote the simple $\Bbbk C_{Q_8}(a)$-modules, simple $\Bbbk C_{Q_8}(b)$-modules and the simple $\Bbbk C_{Q_8}(ab)$-modules, respectively, where $0 \leqslant j \leqslant 3$. In this case, let u_j^a, u_j^b and u_j^{ab} be the basis of U_j^a, U_j^b and U_j^{ab} as stated before, respectively.

Proposition 3.4. *For any $0 \leqslant j \leqslant 3$, let U_j^a, U_j^b and U_j^{ab} be the simple modules over $\Bbbk C_{Q_8}(a)$, $\Bbbk C_{Q_8}(b)$ and $\Bbbk C_{Q_8}(ab)$, respectively. Then the induced $\Bbbk Q_8$-modules $D(U_j^a) = \Bbbk Q_8 \otimes_{\Bbbk C_{Q_8}(a)} U_j^a$, $D(U_j^b) = \Bbbk Q_8 \otimes_{\Bbbk C_{Q_8}(b)} U_j^b$, $D(U_j^{ab}) = \Bbbk Q_8 \otimes_{\Bbbk C_{Q_8}(ab)} U_j^{ab}$*

are non-isomorphic simple Yetter-Drinfeld $\Bbbk Q_8$-modules with the comodule structures given by

$$\varrho_j^a\colon\ D(U_j^a)\to D(U_j^a)\otimes \Bbbk Q_8,\ 1\otimes u_j^a\to (1\otimes u_j^a)\otimes a,\ b\otimes u_j^a\to (b\otimes u_j^a)\otimes a^3,$$
$$\varrho_j^b\colon\ D(U_j^b)\to D(U_j^b)\otimes \Bbbk Q_8,\ 1\otimes u_j^b\to (1\otimes u_j^b)\otimes b,\ a\otimes u_j^b\to (a\otimes u_j^b)\otimes a^2b,$$
$$\varrho_j^{ab}\colon\ D(U_j^{ab})\to D(U_j^{ab})\otimes \Bbbk Q_8,\ 1\otimes u_j^{ab}\to (1\otimes u_j^{ab})\otimes ab,$$
$$b\otimes u_j^{ab}\to (b\otimes u_j^{ab})\otimes a^3b.$$

Theorem 3.3. *The set*

$$\{D(V_m^1),\, D(V_m^2),\, D(U_j^a),\, D(U_j^b),\, D(U_j^{ab})\colon\ 1\leqslant m\leqslant 5,\, 0\leqslant j\leqslant 3\}$$

is a representative set of simple Yetter-Drinfeld $\Bbbk Q_8$-modules. Next, $\dim(D(V_m^1)) = \dim(D(V_m^2)) = \dim(V_m)$, *and* $\dim(D(U_j^a)) = \dim(D(U_j^b)) = \dim(D(U_j^{ab})) = 2$.

Thus, we have classified all simple $D(\Bbbk Q_8)$-modules up to isomorphism.

4. Tensor products of simple $D(\Bbbk Q_8)$-modules

In this section, we investigate the tensor products of simple $D(\Bbbk Q_8)$-modules. Since $D(\Bbbk Q_8)$ is a quasitriangular Hopf algebra, $M\otimes N\cong N\otimes M$ for any two $D(\Bbbk Q_8)$-modules M and N.

Theorem 4.1 ([5], Theorem 3.1)**.** *Let G be a finite group, $\Bbbk$ an arbitrary field, and M a Yetter-Drinfeld $\Bbbk G$-module. Let $D(N) = \Bbbk G\otimes_{\Bbbk C_G(g_C)} N$ be a simple Yetter-Drinfeld $\Bbbk G$-module, where g_C is a representative of some conjugate class C and N is a simple $\Bbbk C_G(g_C)$-module. Then the multiplicity of $D(N)$ in M is equal to the multiplicity of $N = D(N)_{g_C}$ in M_{g_C} as a $\Bbbk C_G(g_C)$-module.*

Let C_1, C_2 be conjugate classes in G. Take $x\in C_1$, $y\in C_2$, and let M, N be $\Bbbk C_G(x)$- and $\Bbbk C_G(y)$-modules, respectively. Let $\{g_1, g_2, \dots, g_m\}$ and $\{h_1, h_2, \dots h_l\}$ be the sets of left coset representatives of $C_G(x)$ and $C_G(y)$ in G, respectively. For any $z\in G$ and simple $\Bbbk C_G(z)$-module P, we can determine the multiplicity of $D(P)$ in $D(M)\otimes D(N)$ by Theorem 4.1. Consider the tensor product $D(M)\otimes D(N)$ as $\Bbbk G$-module. It contains the subspace

$$S(z) = \bigoplus_{i,j}(g_i\otimes_{\Bbbk C_G(x)} M)\otimes (h_j\otimes_{\Bbbk C_G(y)} N),$$

where the sum is taken over all indices i, j such that $(h_j y h_j^{-1})(g_i x g_i^{-1}) = z$. Then $S(z) = (D(M)\otimes D(N))_z$ is a $\Bbbk C_G(z)$-module, and the multiplicity of $D(P)$ in $D(M)\otimes D(N)$ is equal to the multiplicity of P in $S(z)$ by Theorem 4.1.

Firstly, we consider the tensor products of simple $\Bbbk Q_8$-modules. By the above discussion, one knows that there are 5 simple $\Bbbk Q_8$-modules up to isomorphism, V_1, V_2, V_3, V_4 and V_5, respectively.

Lemma 4.1. *Let V_1, V_2, V_3, V_4 and V_5 be the simple $\Bbbk Q_8$-modules. Then we have*

$$V_1 \otimes V_i \cong V_i \otimes V_1 \cong V_i,\ 1 \leqslant i \leqslant 5; \quad V_j \otimes V_j \cong V_1,\ 1 \leqslant j \leqslant 4;$$
$$V_2 \otimes V_3 \cong V_3 \otimes V_2 \cong V_4; \quad V_2 \otimes V_4 \cong V_4 \otimes V_2 \cong V_3; \quad V_2 \otimes V_5 \cong V_5 \otimes V_2 \cong V_5;$$
$$V_3 \otimes V_4 \cong V_4 \otimes V_3 \cong V_2; \quad V_3 \otimes V_5 \cong V_5 \otimes V_3 \cong V_5; \quad V_4 \otimes V_5 \cong V_5 \otimes V_4 \cong V_5;$$
$$V_5 \otimes V_5 \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4.$$

P r o o f. We only prove the decomposition of $V_5 \otimes V_5$ since other decompositions are obvious. Let $\{m_1, m_2\}$ be the basis of V_5 as stated in last section. Then $\{m_1 \otimes m_1, m_1 \otimes m_2, m_2 \otimes m_1, m_2 \otimes m_2\}$ is a $\Bbbk$-basis of $V_5 \otimes V_5$. Moreover, $\{m_1 \otimes m_1 + m_2 \otimes m_2, m_1 \otimes m_1 - m_2 \otimes m_2, m_1 \otimes m_2 + m_2 \otimes m_1, m_1 \otimes m_2 - m_2 \otimes m_1\}$ is also a $\Bbbk$-basis of $V_5 \otimes V_5$. One can check that under the basis $\{m_1 \otimes m_1 + m_2 \otimes m_2, m_1 \otimes m_1 - m_2 \otimes m_2, m_1 \otimes m_2 + m_2 \otimes m_1, m_1 \otimes m_2 - m_2 \otimes m_1\}$ the matrices of a and b are

$$\begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & 1 \end{pmatrix},$$

respectively. Consequently, $V_5 \otimes V_5 \cong V_1 \oplus V_2 \oplus V_3 \oplus V_4$. □

4.1. Decompositions of the tensor products $D(V_k^i) \otimes D(V_l^j)$. In this subsection, we consider the tensor products $D(V_k^1) \otimes D(V_l^1)$, $D(V_k^1) \otimes D(V_l^2)$ and $D(V_k^2) \otimes D(V_l^2)$, $1 \leqslant k, l \leqslant 5$.

Proposition 4.1. *Let $1 \leqslant k, l \leqslant 5$. Then we have*

$$D(V_1^1) \otimes D(V_i^1) \cong D(V_i^1),\ 1 \leqslant i \leqslant 5; \quad D(V_j^1) \otimes D(V_j^1) \cong D(V_1^1),\ 1 \leqslant j \leqslant 4;$$
$$D(V_2^1) \otimes D(V_3^1) \cong D(V_4^1); \quad D(V_2^1) \otimes D(V_4^1) \cong D(V_3^1); \quad D(V_2^1) \otimes D(V_5^1) \cong D(V_5^1);$$
$$D(V_3^1) \otimes D(V_4^1) \cong D(V_2^1); \quad D(V_3^1) \otimes D(V_5^1) \cong D(V_5^1); \quad D(V_4^1) \otimes D(V_5^1) \cong D(V_5^1);$$
$$D(V_5^1) \otimes D(V_5^1) \cong D(V_1^1) \oplus D(V_2^1) \oplus D(V_3^1) \oplus D(V_4^1).$$

P r o o f. It follows from $S(1) = (1 \otimes_{\Bbbk C_{Q_8}(1)} V_k^1) \otimes (1 \otimes_{\Bbbk C_{Q_8}(1)} V_l^1) \cong V_k^1 \otimes V_l^1$, Lemma 4.1 and the discussion above. □

Proposition 4.2. *Let $1 \leqslant k, l \leqslant 5$. Then we have*

$$\begin{aligned}
D(V_1^1) \otimes D(V_i^2) &\cong D(V_i^2),\ 1 \leqslant i \leqslant 5; \quad D(V_j^1) \otimes D(V_j^2) \cong D(V_1^2),\ 1 \leqslant j \leqslant 4;\\
D(V_2^1) \otimes D(V_1^2) &\cong D(V_2^2); \quad D(V_2^1) \otimes D(V_3^2) \cong D(V_4^2);\\
D(V_2^1) \otimes D(V_4^2) &\cong D(V_3^2); \quad D(V_2^1) \otimes D(V_5^2) \cong D(V_5^2);\\
D(V_3^1) \otimes D(V_1^2) &\cong D(V_3^2); \quad D(V_3^1) \otimes D(V_2^2) \cong D(V_4^2);\\
D(V_3^1) \otimes D(V_4^2) &\cong D(V_2^2); \quad D(V_3^1) \otimes D(V_5^2) \cong D(V_5^2);\\
D(V_4^1) \otimes D(V_1^2) &\cong D(V_4^2); \quad D(V_4^1) \otimes D(V_2^2) \cong D(V_3^2);\\
D(V_4^1) \otimes D(V_3^2) &\cong D(V_2^2); \quad D(V_4^1) \otimes D(V_5^2) \cong D(V_5^2);\\
D(V_5^1) \otimes D(V_1^2) &\cong D(V_5^2); \quad D(V_5^1) \otimes D(V_2^2) \cong D(V_5^2);\\
D(V_5^1) \otimes D(V_3^2) &\cong D(V_5^2); \quad D(V_5^1) \otimes D(V_4^2) \cong D(V_5^2);\\
D(V_5^1) \otimes D(V_5^2) &\cong D(V_1^2) \oplus D(V_2^2) \oplus D(V_3^2) \oplus D(V_4^2).
\end{aligned}$$

P r o o f. It follows from $S(a^2) = (1 \otimes_{\Bbbk C_{Q_8}(1)} V_k^1) \otimes (1 \otimes_{\Bbbk C_{Q_8}(a^2)} V_l^2) \cong V_k^1 \otimes V_l^2$, Lemma 4.1 and the discussion above. □

Proposition 4.3. *Let $1 \leqslant k, l \leqslant 5$. Then we have*

$$\begin{aligned}
D(V_1^2) \otimes D(V_i^2) &\cong D(V_i^1),\ 1 \leqslant i \leqslant 5; \quad D(V_j^2) \otimes D(V_j^2) \cong D(V_1^1),\ 1 \leqslant j \leqslant 4;\\
D(V_2^2) \otimes D(V_3^2) &\cong D(V_4^1); \quad D(V_2^2) \otimes D(V_4^2) \cong D(V_3^1);\\
D(V_2^2) \otimes D(V_5^2) &\cong D(V_5^1); \quad D(V_3^2) \otimes D(V_4^2) \cong D(V_2^1);\\
D(V_3^2) \otimes D(V_5^2) &\cong D(V_5^1); \quad D(V_4^2) \otimes D(V_5^2) \cong D(V_5^1);\\
D(V_5^2) \otimes D(V_5^2) &\cong D(V_1^1) \oplus D(V_2^1) \oplus D(V_3^1) \oplus D(V_4^1).
\end{aligned}$$

P r o o f. It follows from $S(1) = (1 \otimes_{\Bbbk C_{Q_8}(a^2)} V_k^2) \otimes (1 \otimes_{\Bbbk C_{Q_8}(a^2)} V_l^2) \cong V_k^2 \otimes V_l^2$, Lemma 4.1 and the discussion above. □

4.2. Decompositions of the tensor products $D(V_k^i) \otimes D(U_j^l)$. In this subsection, we consider the tensor products $D(V_k^1) \otimes D(U_j^a)$, $D(V_k^1) \otimes D(U_j^b)$, $D(V_k^1) \otimes D(U_j^{ab})$, $D(V_k^2) \otimes D(U_j^a)$, $D(V_k^2) \otimes D(U_j^b)$ and $D(V_k^2) \otimes D(U_j^{ab})$, $0 \leqslant j \leqslant 3$, $1 \leqslant k \leqslant 5$.

Proposition 4.4. *Let $1 \leqslant k \leqslant 5$, $1 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 3$. Then we have:*
(1) *If $1 \leqslant k \leqslant 2$, then $D(V_k^i) \otimes D(U_j^a) \cong D(U^a_{(2i-1)j})$.*
(2) *If $3 \leqslant k \leqslant 4$, then $D(V_k^i) \otimes D(U_j^a) \cong D(U^a_{(2i-1)j+2})$.*
(3) *If $k = 5$, then $D(V_5^i) \otimes D(U_j^a) \cong D(U^a_{(2i-1)j+1}) \oplus D(U^a_{(2i-1)j-1})$.*

P r o o f. We first prove the case of $i = 1$. Assume $i = 1$, it is easy to see that $S(a) = (1 \otimes_{\Bbbk C_{Q_8}(1)} V_k^1) \otimes (1 \otimes_{\Bbbk C_{Q_8}(a)} U_j^a) \cong V_k^1 \otimes U_j^a$. If $1 \leqslant k \leqslant 2$, then $a(v_k^1 \otimes u_j^a) = \omega^j v_k^1 \otimes u_j^a$, hence $S(a) \cong U_j^a$. By the discussion above, one knows that the multiplicity of $D(U_j^a)$ in $D(V_k^1) \otimes D(U_j^a)$ is 1. Hence, $D(V_k^1) \otimes D(U_j^a) \cong D(U_j^a)$ by comparing their dimensions. If $3 \leqslant k \leqslant 4$, then $a(v_k^1 \otimes u_j^a) = \omega^{j+2} v_k^1 \otimes u_j^a$, hence $S(a) \cong U_{j+2}^a$. Consequently, $D(V_k^1) \otimes D(U_j^a) \cong D(U_{j+2}^a)$. If $k = 5$, then $S(a) \cong V_5^1 \otimes U_j^a$. Let $\{m_1^1, m_2^1\}$ and $\{u_j^a\}$ be the basis of V_5^1 and U_j^a as stated in the last section, respectively. Then $\{m_1^1 \otimes u_j^a, m_2^1 \otimes u_j^a\}$ is a basis of $S(a)$. Moreover, one can easily check $a(m_1^1 \otimes u_j^a) = \omega^{j+1} m_1^1 \otimes u_j^a$ and $a(m_2^1 \otimes u_j^a) = \omega^{j-1} m_2^1 \otimes u_j^a$. Hence, $S(a) \cong U_{j+1}^a \oplus U_{j-1}^a$. Consequently, $D(V_5^1) \otimes D(U_j^a) \cong D(U_{j+1}^a) \oplus D(U_{j-1}^a)$ by comparing their dimensions.

Now, let $i = 2$. One can check that $S(a) = (1 \otimes_{\Bbbk C_{Q_8}(a^2)} V_k^2) \otimes (b \otimes_{\Bbbk C_{Q_8}(a)} U_j^a)$. If $1 \leqslant k \leqslant 2$, then $\{(1 \otimes v_k^2) \otimes (b \otimes u_j^a)\}$ is a $\Bbbk$-basis of $S(a)$, and $a((1 \otimes v_k^2) \otimes (b \otimes u_j^a)) = \omega^{3j}(1 \otimes v_k^2) \otimes (b \otimes u_j^a)$, hence $S(a) \cong U_{3j}^a$. Then $D(V_k^2) \otimes D(U_j^a) \cong D(U_{3j}^a)$. If $3 \leqslant k \leqslant 4$, then $\{(1 \otimes v_k^2) \otimes (b \otimes u_j^a)\}$ is a $\Bbbk$-basis of $S(a)$, and $a((1 \otimes v_k^2) \otimes (b \otimes u_j^a)) = \omega^{3j+2}(1 \otimes v_k^2) \otimes (b \otimes u_j^a)$, hence $S(a) \cong U_{3j+2}^a$. Consequently, $D(V_k^2) \otimes D(U_j^a) \cong D(U_{3j+2}^a)$. If $k = 5$, then $\{(1 \otimes m_1^2) \otimes (b \otimes u_j^a), (1 \otimes m_2^2) \otimes (b \otimes u_j^a)\}$ is a $\Bbbk$-basis of $S(a)$. Moreover, $a((1 \otimes m_1^2) \otimes (b \otimes u_j^a)) = \omega^{3j+1}(1 \otimes m_1^2) \otimes (b \otimes u_j^a)$ and $a((1 \otimes m_2^2) \otimes (b \otimes u_j^a)) = \omega^{3j-1}(1 \otimes m_2^2) \otimes (b \otimes u_j^a)$, and hence $S(a) \cong U_{3j+1}^a \oplus U_{3j-1}^a$. Consequently, $D(V_5^2) \otimes D(U_j^a) \cong D(U_{3j+1}^a) \oplus D(U_{3j-1}^a)$ by comparing their dimensions. □

Proposition 4.5. *Let $1 \leqslant k \leqslant 5$, $1 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 3$. Then we have:*
(1) *If $k = 1, 3$, then $D(V_k^i) \otimes D(U_j^b) \cong D(U_{(2i-1)j}^b)$.*
(2) *If $k = 2, 4$, then $D(V_k^i) \otimes D(U_j^b) \cong D(U_{(2i-1)j+2}^b)$.*
(3) *If $k = 5$, then $D(V_5^i) \otimes D(U_j^b) \cong D(U_{(2i-1)j+1}^b) \oplus D(U_{(2i-1)j-1}^b)$.*

P r o o f. It is similar to the proof of Proposition 4.4 □

Proposition 4.6. *Let $1 \leqslant k \leqslant 5$, $1 \leqslant i \leqslant 2$ and $0 \leqslant j \leqslant 3$. Then we have:*
(1) *If $k = 1, 4$, then $D(V_k^i) \otimes D(U_j^{ab}) \cong D(U_{(2i-1)j}^{ab})$.*
(2) *If $k = 2, 3$, then $D(V_k^i) \otimes D(U_j^{ab}) \cong D(U_{(2i-1)j+2}^{ab})$.*
(3) *If $k = 5$, then $D(V_5^i) \otimes D(U_j^{ab}) \cong D(U_{(2i-1)j+1}^{ab}) \oplus D(U_{(2i-1)j-1}^{ab})$.*

P r o o f. It is similar to the proof of Proposition 4.4 □

4.3. Decompositions of the tensor products $D(U_k^l) \otimes D(U_j^q)$. In this subsection, we consider the tensor products $D(U_k^a) \otimes D(U_j^a)$, $D(U_k^b) \otimes D(U_j^b)$, $D(U_k^{ab}) \otimes D(U_j^{ab})$, $D(U_k^a) \otimes D(U_j^b)$, $D(U_k^a) \otimes D(U_j^{ab})$ and $D(U_k^b) \otimes D(U_j^{ab})$, $0 \leqslant j, k \leqslant 3$.

Proposition 4.7. *Let $0 \leqslant j, k \leqslant 3$. Then we have:*

(1) *If* $k=j=1$ *or* $k=j=3$, *then*

$$D(U_k^a)\otimes D(U_j^a)\cong D(V_1^1)\oplus D(V_2^1)\oplus D(V_3^2)\oplus D(V_4^2).$$

(2) *If* $k=j=0$ *or* $k=j=2$, *then*

$$D(U_k^a)\otimes D(U_j^a)\cong D(V_1^1)\oplus D(V_2^1)\oplus D(V_1^2)\oplus D(V_2^2).$$

(3) *If* $k=1$,$j=3$ *or* $k=3$, $j=1$, *then*

$$D(U_k^a)\otimes D(U_j^a)\cong D(V_3^1)\oplus D(V_4^1)\oplus D(V_1^2)\oplus D(V_2^2).$$

(4) *If* $k=0$, $j=2$ *or* $k=2$, $j=0$, *then*

$$D(U_k^a)\otimes D(U_j^a)\cong D(V_3^1)\oplus D(V_4^1)\oplus D(V_3^2)\oplus D(V_4^2).$$

(5) *If* $\omega^{2k}\neq\omega^{2j}$, *then*

$$D(U_k^a)\otimes D(U_j^a)\cong D(V_5^1)\oplus D(V_5^2).$$

P r o o f. One can check that

$$S(1)=(1\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(b\otimes_{\Bbbk C_{Q_8}(a)}U_j^a)\oplus(b\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(1\otimes_{\Bbbk C_{Q_8}(a)}U_j^a),$$
$$S(a^2)=(b\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(b\otimes_{\Bbbk C_{Q_8}(a)}U_j^a)\oplus(1\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(1\otimes_{\Bbbk C_{Q_8}(a)}U_j^a).$$

Let $X=(1\otimes u_k^a)\otimes(b\otimes u_j^a)$, $Y=(b\otimes u_k^a)\otimes(1\otimes u_j^a)$, $Z=(1\otimes u_k^a)\otimes(1\otimes u_j^a)$ and $W=(b\otimes u_k^a)\otimes(b\otimes u_j^a)$. Then $\{X,Y\}$ is a $\Bbbk$-basis of $S(1)$, $\{Z,W\}$ is a $\Bbbk$-basis of $S(a^2)$. Moreover, we have

$$a(X,Y)=(X,Y)\begin{pmatrix}\omega^{3j+k}&0\\0&\omega^{3k+j}\end{pmatrix},\quad b(X,Y)=(X,Y)\begin{pmatrix}0&\omega^{2k}\\\omega^{2j}&0\end{pmatrix},$$

and

$$a(Z,W)=(Z,W)\begin{pmatrix}\omega^{j+k}&0\\0&\omega^{3(k+j)}\end{pmatrix},\quad b(Z,W)=(Z,W)\begin{pmatrix}0&\omega^{2(k+j)}\\1&0\end{pmatrix}.$$

If $\omega^{2k}=\omega^{2j}$, then $\omega^{3j+k}=\omega^{3k+j}$. It is easy to see that $\{X+Y,X-Y\}$ is a $\Bbbk$-basis of $S(1)$, and $\{Z+W,Z-W\}$ is a $\Bbbk$-basis of $S(a^2)$. Moreover, we have

$$a(X+Y,X-Y)=(X+Y,X-Y)\begin{pmatrix}\omega^{3j+k}&0\\0&\omega^{3j+k}\end{pmatrix},$$
$$b(X+Y,X-Y)=(X+Y,X-Y)\begin{pmatrix}(-1)^j&0\\0&(-1)^{j+1}\end{pmatrix},$$

and

$$a(Z+W,Z-W)=(Z+W,Z-W)\begin{pmatrix}\omega^{j+k}&0\\0&\omega^{j+k}\end{pmatrix},$$
$$b(Z+W,Z-W)=(Z+W,Z-W)\begin{pmatrix}1&0\\0&-1\end{pmatrix}.$$

Case 1: If $k = j = 1$ or $k = j = 3$, then we have $S(1) \cong V_1^1 \oplus V_2^1$ and $S(a^2) \cong V_3^2 \oplus V_4^2$. Consequently, we have

$$D(U_k^a) \otimes D(U_j^a) \cong D(V_1^1) \oplus D(V_2^1) \oplus D(V_3^2) \oplus D(V_4^2)$$

by comparing their dimensions.

Case 2: If $k = j = 0$ or $k = j = 2$, then we have $S(1) \cong V_1^1 \oplus V_2^1$ and $S(a^2) \cong V_1^2 \oplus V_2^2$. Consequently, we have

$$D(U_k^a) \otimes D(U_j^a) \cong D(V_1^1) \oplus D(V_2^1) \oplus D(V_1^2) \oplus D(V_2^2)$$

by comparing their dimensions.

Case 3: If $k = 1$, $j = 3$ or $k = 3$, $j = 1$, then we have $S(1) \cong V_3^1 \oplus V_4^1$ and $S(a^2) \cong V_1^2 \oplus V_2^2$. Consequently, we have

$$D(U_k^a) \otimes D(U_j^a) \cong D(V_3^1) \oplus D(V_4^1) \oplus D(V_1^2) \oplus D(V_2^2)$$

by comparing their dimensions.

Case 4: If $k = 0$, $j = 2$ or $k = 2$, $j = 0$, then we have $S(1) \cong V_3^1 \oplus V_4^1$ and $S(a^2) \cong V_3^2 \oplus V_4^2$. Consequently, we have

$$D(U_k^a) \otimes D(U_j^a) \cong D(V_3^1) \oplus D(V_4^1) \oplus D(V_3^2) \oplus D(V_4^2)$$

by comparing their dimensions.

If $\omega^{2k} \neq \omega^{2j}$, then $\omega^{3j+k} \neq \omega^{3k+j}$. In this case, we have

$$a(X,Y) = (X,Y)\begin{pmatrix} \omega^{3j+k} & 0 \\ 0 & \omega^{-(3j+k)} \end{pmatrix}, \quad b(X,Y) = (X,Y)\begin{pmatrix} 0 & (-1)^k \\ (-1)^j & 0 \end{pmatrix},$$

and

$$a(Z,W) = (Z,W)\begin{pmatrix} \omega^{k+j} & 0 \\ 0 & \omega^{-(k+j)} \end{pmatrix}, \quad b(Z,W) = (Z,W)\begin{pmatrix} 0 & (-1)^{k+j} \\ 1 & 0 \end{pmatrix}.$$

Since $\omega^{2k} \neq \omega^{2j}$, $|k-j| = 1$ or $|k-j| = 3$. Hence, $\omega^{3j+k} \neq \pm 1$ and $\omega^{j+k} \neq \pm 1$. Moreover, by the classification of simple $\Bbbk Q_8$-modules, we have

$$D(U_k^a) \otimes D(U_j^a) \cong D(V_5^1) \oplus D(V_5^2).$$

□

Proposition 4.8. *Let $0 \leqslant j, k \leqslant 3$. Then we have:*

(1) *If $k = j = 1$ or $k = j = 3$, then*

$$D(U_k^b) \otimes D(U_j^b) \cong D(V_1^1) \oplus D(V_3^1) \oplus D(V_2^2) \oplus D(V_4^2).$$

(2) *If $k = j = 0$ or $k = j = 2$, then*

$$D(U_k^b) \otimes D(U_j^b) \cong D(V_1^1) \oplus D(V_3^1) \oplus D(V_1^2) \oplus D(V_3^2).$$

(3) *If* $k=1$, $j=3$ *or* $k=3$, $j=1$, *then*

$$D(U_k^b)\otimes D(U_j^b)\cong D(V_2^1)\oplus D(V_4^1)\oplus D(V_1^2)\oplus D(V_3^2).$$

(4) *If* $k=0$, $j=2$ *or* $k=2$, $j=0$, *then*

$$D(U_k^b)\otimes D(U_j^b)\cong D(V_2^1)\oplus D(V_4^1)\oplus D(V_2^2)\oplus D(V_4^2).$$

(5) *If* $\omega^{2k}\neq\omega^{2j}$, *then*

$$D(U_k^b)\otimes D(U_j^b)\cong D(V_5^1)\oplus D(V_5^2).$$

P r o o f. It is similar to the proof of Proposition 4.7. □

Proposition 4.9. *Let* $0\leqslant j,\,k\leqslant 3$. *Then we have:*

(1) *If* $k=j=1$ *or* $k=j=3$, *then*

$$D(U_k^{ab})\otimes D(U_j^{ab})\cong D(V_1^1)\oplus D(V_4^1)\oplus D(V_2^2)\oplus D(V_3^2).$$

(2) *If* $k=j=0$ *or* $k=j=2$, *then*

$$D(U_k^{ab})\otimes D(U_j^{ab})\cong D(V_1^1)\oplus D(V_4^1)\oplus D(V_1^2)\oplus D(V_4^2).$$

(3) *If* $k=1$, $j=3$ *or* $k=3$, $j=1$, *then*

$$D(U_k^{ab})\otimes D(U_j^{ab})\cong D(V_2^1)\oplus D(V_3^1)\oplus D(V_1^2)\oplus D(V_4^2).$$

(4) *If* $k=0$, $j=2$ *or* $k=2$, $j=0$, *then*

$$D(U_k^{ab})\otimes D(U_j^{ab})\cong D(V_2^1)\oplus D(V_3^1)\oplus D(V_2^2)\oplus D(V_3^2).$$

(5) *If* $\omega^{2k}\neq\omega^{2j}$, *then*

$$D(U_k^{ab})\otimes D(U_j^{ab})\cong D(V_5^1)\oplus D(V_5^2).$$

P r o o f. It similar to the proof of Proposition 4.7. □

Proposition 4.10. *Let* $0\leqslant j,\,k\leqslant 3$. *Then we have*

$$D(U_k^a)\otimes D(U_j^b)\cong D(U_{j+k}^{ab})\oplus D(U_{j+k+2}^{ab}).$$

P r o o f. One can check that

$$S(ab)=(1\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(a\otimes_{\Bbbk C_{Q_8}(b)}U_j^b)\oplus(b\otimes_{\Bbbk C_{Q_8}(a)}U_k^a)\otimes(1\otimes_{\Bbbk C_{Q_8}(b)}U_j^b).$$

Let $X=(1\otimes u_k^a)\otimes(a\otimes u_j^b)$, $Y=(b\otimes u_k^a)\otimes(1\otimes u_j^b)$. Then $\{X,Y\}$ is a $\Bbbk$-basis of $S(ab)$, and $\{X+Y,X-Y\}$ is also a basis of $S(ab)$. Moreover, we have

$$ab(X+Y,X-Y)=(X+Y,X-Y)\begin{pmatrix}\omega^{3k+j} & 0\\ 0 & -\omega^{3k+j}\end{pmatrix}.$$

Hence, $S(ab) \cong U^{ab}_{3k+j} \oplus U^{ab}_{3k+j+2} \cong U^{ab}_{k+j} \oplus U^{ab}_{k+j+2}$. Consequently, we have

$$D(U^a_k) \otimes D(U^b_j) \cong D(U^{ab}_{j+k}) \oplus D(U^{ab}_{j+k+2})$$

by comparing their dimensions. □

Proposition 4.11. *Let $0 \leqslant j, k \leqslant 3$. Then we have*

$$D(U^a_k) \otimes D(U^{ab}_j) \cong D(U^b_{j+k}) \oplus D(U^b_{j+k+2}).$$

P r o o f. It is similar to the proof of Proposition 4.10. □

Proposition 4.12. *Let $0 \leqslant j, k \leqslant 3$. Then we have*

$$D(U^b_k) \otimes D(U^{ab}_j) \cong D(U^a_{j+k}) \oplus D(U^a_{j+k+2}).$$

P r o o f. It is similar to the proof of Proposition 4.10. □

5. The Grothendieck ring of $D(\Bbbk Q_8)$

In this section, we describe the Grothendieck ring $G_0(D(\Bbbk Q_8))$ of $D(\Bbbk Q_8)$ by generators with relations.

In $G_0(D(\Bbbk Q_8))$, let $x = [D(V^1_2)]$, $y = [D(V^1_4)]$, $z = [D(V^1_5)]$, $w = [D(V^2_1)]$, $z_0 = [D(U^a_0)]$, $z_1 = [D(U^a_1)]$, $t_0 = [D(U^b_0)]$, $t_1 = [D(U^b_1)]$, $p_0 = [D(U^{ab}_0)]$ and $p_1 = [D(U^{ab}_1)]$.

Proposition 5.1. *$G_0(D(\Bbbk Q_8))$ is generated as a ring by x, y, z, w, z_0, z_1, t_0, t_1, p_0 and p_1.*

P r o o f. By Proposition 4.1, one knows that $[D(V^1_1)] = x^2$, $[D(V^1_3)] = xy$. By Proposition 4.2, one gets $D(V^2_2) = xw$, $D(V^2_4) = yw$, $D(V^2_3) = xyw$ and $D(V^2_5) = zw$. By Proposition 4.4, one knows that $[D(U^a_2)] = yz_0$, $[D(U^a_3)] = yz_1$. By Proposition 4.5, $[D(U^b_2)] = xt_0$, $[D(U^b_3)] = xt_1$. By Proposition 4.6, we have $[D(U^{ab}_2)] = xp_0$ and $[D(U^{ab}_3)] = xp_1$. Hence, the proposition follows from Theorem 3.3. □

By Propositions 4.1, 4.2 and 4.3, one knows that $N_0 = \mathrm{span}\{D(V^1_k), D(V^2_k) | 1 \leqslant k \leqslant 5\}$ is a subring of $G_0(D(\Bbbk Q_8))$, which is generated as a ring by x, y, z, w. Moreover, it is easy to check (in N_0) $x^2 = 1$, $y^2 = 1$, $xz = z$, $yz = z$, $w^2 = 1$ and $z^2 = 1 + x + y + xy$.

Let $\mathbb{Z}[x,y,z,w]$ be the polynomial ring with the variables x, y, z, w. Let I_0 be the following set.

$$I_0 := \begin{cases} x^2-1,\ y^2-1,\ xz-z, \\ yz-z,\ w^2-1, \\ z^2-1-x-y-xy. \end{cases}$$

Let $\langle I_0\rangle$ be the ideal of $\mathbb{Z}[x,y,z,w]$ generated by I_0.

Theorem 5.1. *The subring N_0 of $G_0(D(\Bbbk Q_8))$ is isomorphic to $\mathbb{Z}[x,y,z,w]/\langle I_0\rangle$.*

P r o o f. By the proof of Proposition 5.1, one knows that there is an epimorphism $\psi\colon\ \mathbb{Z}[x,y,z,w] \to N_0$ such that $\psi(x) = [D(V_2^1)]$, $\psi(y) = [D(V_4^1)]$, $\psi(z) = [D(V_5^1)]$, and $\psi(w) = [D(V_1^2)]$. By the above discussion, $\psi(I_0) = 0$. Hence, ψ induces a ring epimorphism $\overline{\psi}\colon\ \mathbb{Z}[x,y,z,w]/\langle I_0\rangle \to N_0$ such that $\psi = \overline{\psi}\pi$, where $\pi\colon\ \mathbb{Z}[x,y,z,w] \to \mathbb{Z}[x,y,z,w]/\langle I_0\rangle$ is the canonical projection. Let $\bar{x} := \pi(x)$ for any $x \in \mathbb{Z}[x,y,z,w]$. One can easily check that $\mathbb{Z}[x,y,z,w]/\langle I_0\rangle$ is generated as a $\mathbb{Z}$-module, by $F := \{\bar{1}, \bar{x}, \overline{y}, \overline{z}, \overline{xy}, \overline{w}, \overline{xw}, \overline{yw}, \overline{zw}, \overline{xyw}\}$. By $\psi = \overline{\psi}\pi$, the set F is a linearly independent set over $\mathbb{Z}$. It follows that the set F is a $\mathbb{Z}$-basis of $\mathbb{Z}[x,y,z,w]/\langle I_0\rangle$. Consequently, $\overline{\psi}$ is a $\mathbb{Z}$-module isomorphism, and so it is a ring isomorphism. □

Proposition 5.2. *The following relations are satisfied in $G_0(D(\Bbbk Q_8))$.*

$$\begin{cases} z_0^2-1-x-w-xw,\ z_1^2-1-x-yw-xyw, \\ z_0z_1-z-zw,\ xz_0-z_0,\ zz_0-z_1-yz_1,\ wz_0-z_0, \\ xz_1-z_1,\ zz_1-z_0-yz_0,\ wz_1-yz_1. \end{cases}$$

Denote the set by I_1.

P r o o f. It follows from Propositions 4.4, 4.7 and the proof of Proposition 5.1. □

Proposition 5.3. *The following relations are satisfied in $G_0(D(\Bbbk Q_8))$.*

$$\begin{cases} t_0^2-1-xy-w-xyw,\ t_1^2-1-xy-xw-yw, \\ t_0t_1-z-zw, wt_0-t_0,\ zt_0-t_1-xt_1,\ xyt_0-t_0, \\ xwt_1-t_1,\ zt_1-t_0-xt_0,\ ywt_1-t_1. \end{cases}$$

Denote the set by I_2.

P r o o f. It follows from Propositions 4.5, 4.8 and the proof of Proposition 5.1. □

Proposition 5.4. *The following relations are satisfied in* $G_0(D(\Bbbk Q_8))$.

$$\begin{cases} p_0^2 - 1 - y - w - yw,\ p_1^2 - 1 - y - xw - xyw, \\ p_0p_1 - z - zw,\ yp_0 - p_0,\ zp_0 - p_1 - xp_1,\ wp_0 - p_0, \\ yp_1 - p_1,\ zp_1 - p_0 - xp_0,\ xwp_1 - p_1. \end{cases}$$

Denote the set by I_3.

P r o o f. It follows from Propositions 4.6, 4.9 and the proof of Proposition 5.1. □

Proposition 5.5. *The following relations are satisfied in* $G_0(D(\Bbbk Q_8))$.

$$\begin{cases} z_0t_0 - p_0 - xp_0,\ z_0t_1 - p_1 - xp_1, \\ z_1t_0 - z_0t_1,\ z_0t_0 - z_1t_1, \\ z_0p_0 - t_0 - xt_0,\ z_0p_1 - t_1 - xt_1, \\ z_1p_0 - z_0p_1,\ z_1p_1 - z_0p_0, \\ t_0p_0 - z_0 - yz_0,\ t_0p_1 - z_1 - yz_1, \\ t_1p_0 - t_0p_1,\ t_1p_1 - t_0p_0. \end{cases}$$

Denote the set by I_4.

P r o o f. It follows from Propositions 4.10–4.12 and the proof of Proposition 5.1. □

Let $H = \{x, y, z, w, z_0, z_1, t_0, t_1, p_0, p_1\}$ and let $\mathbb{Z}[H]$ be the corresponding polynomial ring. Denote by $\langle I_0, I_1, I_2, I_3, I_4\rangle$ the ideal of $\mathbb{Z}[H]$ generated by I_0, I_1, I_2, I_3, I_4. Obversely, $\mathbb{Z}[x, y, z, w] \subset \mathbb{Z}[H]$.

Theorem 5.2. *The Grothendieck ring* $G_0(D(\Bbbk Q_8))$ *of* $D(\Bbbk Q_8)$ *is isomorphic to* $\mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle$.

P r o o f. By Proposition 5.1, one knows that there is a ring epimorphism ϕ form $\mathbb{Z}[H]$ onto $G_0(D(\Bbbk Q_8))$ such that $\phi(x) = [D(V_2^1)]$, $\phi(y) = [D(V_4^1)]$, $\phi(z) = [D(V_5^1)]$, $\phi(w) = [D(V_1^2)]$, $\phi(z_0) = [D(U_0^a)]$, $\phi(z_1) = [D(U_1^a)]$, $\phi(t_0) = [D(U_0^b)]$, $\phi(t_1) = [D(U_1^b)]$, $\phi(p_0) = [D(U_0^{ab})]$ and $\phi(p_1) = [D(U_1^{ab})]$, and $\phi|_{N_0} = \psi$. By Theorem 5.1, Propositions 5.2, 5.3, 5.4 and 5.5, one gets $\phi(I_0) = 0$, $\phi(I_1) = 0$, $\phi(I_2) = 0$, $\phi(I_3) = 0$ and $\phi(I_4) = 0$. Hence, ϕ induces a ring epimorphism $\bar{\phi}\colon\ \mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle \to G_0(D(\Bbbk Q_8))$ such that $\phi = \bar{\phi}\pi$, where $\pi\colon\ \mathbb{Z}[H] \to \mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle$ is the canonical projection. Let $\bar{x} := \pi(x)$ for any $x \in \mathbb{Z}[H]$. It follows from the proof of Theorem 5.1 that $\{\bar{1}, \bar{x}, \overline{y}, \overline{z}, \overline{xy}, \overline{w}, \overline{xw}, \overline{yw}, \overline{zw}, \overline{xyw}\}$ is a $\mathbb{Z}$-basis of $\mathbb{Z}[x, y, z, w]/\langle I_0\rangle$. Note that $\mathbb{Z}[x, y, z, w]/\langle I_0\rangle \subset \mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle$.

Moreover, by the definition of I_1, I_2, I_3 and I_4, one gets $\mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle$ is generated as a $\mathbb{Z}$-module, by $F' := \{\bar{1}, \bar{x}, \overline{y}, \overline{z}, \overline{xy}, \overline{w}, \overline{xw}, \overline{yw}, \overline{zw}, \overline{xyw}, \overline{z_0}, \overline{z_1}, \overline{yz_0}, \overline{yz_1}, \overline{t_0}, \overline{t_1}, \overline{xt_0}, \overline{xt_1}, \overline{p_0}, \overline{p_1}, \overline{xp_1}, \overline{xp_1}\}$. By $\phi = \bar{\phi}\pi$ and Theorem 3.3, F' is a linearly independent set over $\mathbb{Z}$. It follows that the set F' is a $\mathbb{Z}$-basis of $\mathbb{Z}[H]/\langle I_0, I_1, I_2, I_3, I_4\rangle$. Consequently, $\bar{\phi}$ is a $\mathbb{Z}$-module isomorphism, and so it is a ring isomorphism. □

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