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A REMARK ON A DIOPHANTINE EQUATION OF S. S. PILLAI

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Abstract. S. S. Pillai proved that for a fixed positive integer a, the exponential Diophantine equation $x^y - y^x = a$, $\min(x, y) > 1$, has only finitely many solutions in integers x and y. We prove that when a is of the form $2z^2$, the above equation has no solution in integers x and y with $\gcd(x, y) = 1$.

Keywords: Pillai's Diophantine equation; Lehmer sequence; primitive divisor

MSC 2020: 11D61, 11D72

1. Introduction

Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of positive integers and all integers, respectively. In 1932, Pillai in [9] proved that for $a \to \infty$, the number of solutions (x, y) of the Diophantine inequalities

$$0 < x^y - y^x \leqslant a, \quad x, y \in \mathbb{N},$$

is asymptotically equal to

$$\frac{(\log a)^2}{2(\log\log a)^2}.$$

In the same paper, he used the above result to prove that for any $\delta > 0$ and for a sufficiently large in terms of δ, the number of solutions (x, y) to the exponential Diophantine equation

$$x^y - y^x = a, \quad x, y \in \mathbb{N}, \quad \min(x, y) > 1, \tag{1.1}$$

is at most

$$(1 + \delta)\frac{\log a}{\log\log a}.$$

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Thanks to Professor Michel Waldschmidt for the beautiful memoir (see [12]), which have helped us to reach out to the above results of Pillai.

It follows from the works of Luca and Mignotte (see [8]), and Le (see [7]) that (1.1) has no solution (x, y) with $\gcd(x, y) = 1$ when $a > 1$ is a square. On the other hand, $(x, y) = (3, 2)$ is the only solution of (1.1) when $a = 1$. They precisely studied the Diophantine equation

$$x^y - y^x = z^2, \quad x, y, z \in \mathbb{N}, \quad \min(x, y) > 1, \quad \gcd(x, y) = 1. \tag{1.2}$$

Luca and Mignotte in [8] used a combination of lower bounds of linear forms in p-adic and Archimedian logarithms to prove that $(x, y, x) = (3, 2, 1)$ is the only solution of (1.2) with xy even. In contrast, Le in [7] used the existence of primitive divisors of Lucas numbers to prove that (1.2) has no solution when xy is odd.

In this paper, we extend the argument of Le (see [7]) to use Lehmer numbers and as a result, we consider (1.1) when a is of the form $2z^2$ for a positive integer z, that is, the following equation:

$$x^y - y^x = 2z^2, \quad x, y, z \in \mathbb{N}. \tag{1.3}$$

Clearly, $x = 1$ is not possible. On the other hand, $y = 1$ gives the following family of parametric solutions:

$$(x, y, z) = (2t^2 + 1, 1, t), \quad t \in \mathbb{N}.$$

Therefore, we consider equation (1.3) with $\min(x, y) > 1$ and $\gcd(x, y) = 1$. More precisely, we consider the Diophantine equation

$$x^y - y^x = 2z^2, \quad x, y, z \in \mathbb{N}, \quad \min(x, y) > 1, \quad \gcd(x, y) = 1. \tag{1.4}$$

Theorem 1.1. *Equation* (1.4) *has no solutions* (x, y, z).

We use a factorization argument, upper bounds on the class numbers of binary quadratic forms and a classical result due to Bilu, Hanrot and Voutier (see [1]) on the primitive divisors of Lehmer numbers.

2. Technical lemmas

Let D_1 and D_2 be coprime positive integers, $D = D_1 D_2$ and $h(-4D)$ denote the class number of positive binary quadratic form of discriminant $-4D$. The following classical lemma is a particular case of [13], Corollary 3.1. The remark in [13], page 314 allow us to derive this lemma from [13], Corollary 3.1.

Lemma 2.1. *Let $k > 1$ be an integer with $\gcd(k, D) = 1$. If $D_1 \neq 1$ and $D \neq 3$, then every solution (X, Y, Z) of the equation*

$$D_1X^2 + D_2Y^2 = k^Z, \quad X, Y, Z \in \mathbb{Z}, \quad \gcd(X, Y) = 1, \quad Z > 0, \tag{2.1}$$

can be expressed as

$$X\sqrt{-D_1} + Y\sqrt{D_2} = \lambda_1(\lambda_2 X_1\sqrt{-D_1} + Y_1\sqrt{D_2})^t, \quad Z = Z_1 t, \quad \lambda_1, \lambda_2 \in \{1, -1\},$$

where $t \geqslant 1$ is an integer and X_1, Y_1, Z_1 are positive integers satisfying

$$D_1X_1^2 + D_2Y_1^2 = k^{Z_1}, \quad \gcd(X_1, Y_1) = 1, \quad h(-4D) \equiv 0 \pmod{2Z_1}. \tag{2.2}$$

The lemma below sums up the basic statements of the theory of binary quadratic forms. Combining Theorems 12.10.1, 12.14.3 and 11.4.3 of Hua [6], one can get the proof of this lemma.

Lemma 2.2. *Let D be a positive integer. Then*

$$h(-4D) < \frac{4\sqrt{D}}{\pi}\log(2\mathrm{e}\sqrt{D}).$$

Let α and β be algebraic integers. Then (α, β) is a *Lehmer pair* if $(\alpha+\beta)^2$ and $\alpha\beta$ are nonzero coprime rational integers, and α/β is not a root of unity. Given a Lehmer pair (α, β), one defines the corresponding sequence of *Lehmer numbers* by

$$\mathcal{L}_l(\alpha, \beta) = \begin{cases} \dfrac{\alpha^l - \beta^l}{\alpha - \beta} & \text{if } l \text{ is odd,} \\ \dfrac{\alpha^l - \beta^l}{\alpha^2 - \beta^2} & \text{if } l \text{ is even.} \end{cases}$$

All Lehmer numbers are nonzero rational integers. Two Lehmer pairs (α_1, β_1) and (α_2, β_2) are *equivalent* if $\alpha_1/\alpha_2 = \beta_1/\beta_2 \in \{\pm 1, \pm\sqrt{-1}\}$. For such pairs, $\mathcal{L}_l(\alpha_1, \beta_1) = \pm\mathcal{L}_l(\alpha_2, \beta_2)$ for any positive integer l. A prime number p is said to be a *primitive* divisor of $\mathcal{L}_l(\alpha, \beta)$ if $p \mid \mathcal{L}_l(\alpha, \beta)$ and $p \nmid (\alpha^2 - \beta^2)^2\mathcal{L}_1(\alpha, \beta)\mathcal{L}_2(\alpha, \beta)\ldots\mathcal{L}_{l-1}(\alpha, \beta)$.

Given two algebraic integers α and β, we define $a := (\alpha + \beta)^2$ and $b := a - 4\alpha\beta$. Then $\alpha = \frac{1}{2}(\sqrt{a} + \lambda\sqrt{b})$ and $\beta = \frac{1}{2}(\sqrt{a} - \lambda\sqrt{b})$ with $\lambda \in \{1, -1\}$. This pair (a, b) is called the *parameters* of the Lehmer pair (α, β).

Theorem A ([1], Theorem 1.4)**.** *For every integer $l > 30$, the Lehmer numbers $\mathcal{L}_l(\alpha, \beta)$ have primitive divisors.*

The following result is a particular case of [11], Theorem 1 (ii) and [1], Theorem 1.3.

Theorem B. *Let l be an odd integer such that $5 \leqslant l \leqslant 30$. If the Lehmer numbers $\mathcal{L}_l(\alpha, \beta)$ have no primitive divisor, then up to equivalence, the parameters (a, b) of the corresponding Lehmer pair (α, β) are given below.*

(i) *For $l = 5$, $(a, b) = \begin{cases} (F_{k-2\varepsilon}, F_{k-2\varepsilon} - 4F_k) \text{ with } k \geqslant 3, \\ (L_{k-2\varepsilon}, L_{k-2\varepsilon} - 4L_k) \text{ with } k \neq 1; \end{cases}$*

(ii) *for $l = 7$, $(a, b) = (1, -7), (1, -19), (3, -5), (5, -7), (13, -3), (14, -22)$;*

(iii) *for $l = 9$, $(a, b) = (5, -3), (7, -1), (7, -5)$;*

(iv) *for $l = 13$, $(a, b) = (1, -7)$;*

(v) *for $l = 15$, $(a, b) = (7, -1), (10, -2)$;*

where k is a nonnegative integer, $\varepsilon = \pm 1$, and F_k (or L_k) denotes the kth Fibonacci (or Lucas) number.

We recall the following two results.

Lemma 2.3 ([5], Lemma 2.1)**.** *Let F_k and L_k be as in Theorem* B*. Then for $\varepsilon = \pm 1$,*

(i) $4F_k - F_{k-2\varepsilon} = L_{k+\varepsilon}$;

(ii) $4L_k - L_{k-2\varepsilon} = 5F_{k+\varepsilon}$.

The first part of the following theorem is basically Theorem 2 of [4], while the second part follows from Theorem 3 of [10], .

Theorem C. *Let F_k and L_k be as in Theorem* B*. Then*

(i) *if $L_k = 2x^2$, then $k = 0, 6$;*

(ii) *the equation $F_k = 10x^2$ has no solution.*

3. Proof of Theorem 1.1

Since $\gcd(x, y) = 1$, we infer from (1.4) that both x and y are odd. Our assumptions in (1.4) that $\min(x, y) > 1$ and $\gcd(x, y) = 1$ together imply that $y \geqslant 5$ and $x \geqslant 3$.

We now rewrite (1.4) as follows:

$$2z^2 + y(y^{(x-1)/2})^2 = x^y.$$

This shows that $(X, Y, Z) = (z, y^{(x-1)/2}, y)$ is a solution of (2.1) for $(D_1, D_2, k) = (2, y, x)$. Thus, by Lemma 2.1, we get

$$z\sqrt{-2} + y^{(x-1)/2}\sqrt{y} = \lambda_1(\lambda_2 X_1\sqrt{-2} + Y_1\sqrt{y})^t, \quad y = Z_1 t,\ \lambda_1, \lambda_2 \in \{1, -1\}, \tag{3.1}$$

where $t \geqslant 1$ is an integer and X_1, Y_1, Z_1 are positive integers satisfying

$$2X_1^2 + yY_1^2 = x^{Z_1}, \quad \gcd(X_1, Y_1) = 1, \quad h(-8y) \equiv 0 \pmod{2Z_1}. \tag{3.2}$$

We now define $\alpha := \lambda_2 X_1\sqrt{-2} + Y_1\sqrt{y}$ and $\beta := \lambda_2 X_1\sqrt{-2} - Y_1\sqrt{y}$. Then $(\alpha+\beta)^2 = -8X_1^2$ and $\alpha\beta = -(2X_1^2 + yY_1^2) = -x^{Z_1}$. Since $\gcd(X_1, Y_1) = 1$ and $\gcd(x, y) = 1$, we have $\gcd(X_1, x) = 1$. As x is odd, $\gcd((\alpha+\beta)^2, \alpha\beta) = 1$.

Further,

$$\frac{(\alpha+\beta)^2}{\alpha\beta} = \frac{8X_1^2}{x^{Z_1}}$$

implies that

$$x^{Z_1}\left(\frac{\alpha}{\beta}\right)^2 + 2(x^{Z_1} - 4X_1^2)\frac{\alpha}{\beta} + x^{Z_1} = 0.$$

Now $\gcd(x^{Z_1}, 2(x^{Z_1} - 4X_1^2)) = \gcd(x^{Z-1}, 8X_1^2)$ and since x is odd, it is equal to $\gcd(x^{Z_1}, X_1^2)$, which is 1 since we saw above that $\gcd(x, X_1) = 1$. Moreover, it is easy to verify that α/β is a root of none of the quadratic polynomials $X^2 \pm 1$, $X^2 \pm X + 1$ and $(X \pm 1)^2$. Therefore, α/β is not a root of unity. This shows that (α, β) is a Lehmer pair with parameter $(-8X_1^2, 4yY_1^2)$. Assume that $\mathcal{L}_n(\alpha, \beta)$, $n = 0, 1, 2, \ldots$, denotes the corresponding Lehmer numbers. As we saw in the above that $Z = y = Z_1 t$ and we assume that y is odd, t must be odd. Therefore,

$$|\mathcal{L}_t(\alpha, \beta)| = \Big|\frac{\alpha^t - \beta^t}{\alpha - \beta}\Big|.$$

Employing (3.1), we get

$$|\mathcal{L}_t(\alpha, \beta)| = \frac{y^{(x-1)/2}}{Y_1}.$$

It follows from the above equation that the prime divisors of $|\mathcal{L}_t(\alpha, \beta)|$ are prime divisors of y. Since $(\alpha^2 - \beta^2)^2 = -32X_1^2Y_1^2y$, the prime divisors of $|\mathcal{L}_t(\alpha, \beta)|$ also divide $(\alpha-\beta)^2$. By the definition of primitive divisors, this means that $\mathcal{L}_t(\alpha, \beta)$ has no primitive divisors. Thus, Theorem A confirms that $t \leqslant 30$. We already showed that t is odd. Hence, by Theorem B, we obtain $t \in \{1, 3, 5, 7, 9, 13, 15\}$. Since $(-8X_1^2, 4yY_1^2)$ are the parameters corresponding to the Lehmer numbers $\mathcal{L}_t(\alpha, \beta)$, (ii)–(v) of Theorem B confirms that $t \leqslant 5$. Further, (i) of Theorem B gives us

$$(-8X_1^2, 4yY_1^2) = (F_{k-2\varepsilon} - 4F_k, F_{k-2\varepsilon}) \quad \text{with } k \geqslant 3,$$

and

$$(-8X_1^2, 4yY_1^2) = (L_{k-2\varepsilon} - 4L_k, L_{k-2\varepsilon}) \quad \text{with } k \neq 1;$$

with $\varepsilon = \pm 1$. Using Lemma 2.3 in the above equations, we get

$$(8X_1^2, 4yY_1^2) = (L_{k+\varepsilon}, F_{k-2\varepsilon}) \quad \text{with } k \geqslant 3, \tag{3.3}$$

and

$$(8X_1^2, 4yY_1^2) = (5F_{k+\varepsilon}, L_{k-2\varepsilon}) \quad \text{with } k \neq 1. \tag{3.4}$$

We see from (3.3) that $L_{k+\varepsilon} = 2(2X_1)^2$ and that using Theorem C, it is not possible. Further, (3.4) gives us $5F_{k+\varepsilon} = 8X_1^2$. If we put $X_1 = 5X_2$, then we have $F_{k+\varepsilon} = 10(2X_2)^2$. However, it not possible by Theorem C. Therefore, we arrive at $t \in \{1, 3\}$.

We first deal with the case $t = 1$. In this case, by (3.1) and (3.2), we get

$$h(-8y) \equiv 0 \pmod{2y}. \tag{3.5}$$

This holds only when $2y \leqslant h(-8y)$, and therefore by Lemma 2.2, we arrive at

$$y \leqslant \frac{2\sqrt{2y}}{\pi} \log(2\mathrm{e}\sqrt{2y}).$$

We use MAGMA (see [2]) to compute that $y \leqslant 7$. However, we already saw that $y \geqslant 5$, and thus the only possibilities are $y = 5, 7$. It is easy to calculate (using [3], Algorithm 5.3.5, page 228) that $h(-8y) = 2, 4$ when $y = 5, 7$, respectively, and thus (3.5) does not hold.

We now consider the case $t = 3$. Utilizing (3.1) and (3.2), we get

$$h(-8y) \equiv 0 \pmod{(\tfrac{2}{3}y)}. \tag{3.6}$$

This implies that $2y \leqslant 3h(-8y)$, and hence using Lemma 2.2, we obtain

$$y \leqslant \frac{6}{\pi}\sqrt{2y} \log(2\mathrm{e}\sqrt{2y}).$$

Using MAGMA (see [2]), we see that the above inequality holds only when $y < 150$. We verify by MAGMA that (3.6) holds only when $y = 2$, which contradicts the fact that $y \geqslant 5$. This completes the proof of Theorem 1.1.

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