Czechoslovak Mathematical Journal

Toshihide Futamura; Tetsu Shimomura

Generalized fractional integral operators on weak Choquet spaces over quasi-metric measure spaces

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 3, 905–913

Persistent URL: http://dml.cz/dmlcz/152588

Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 74 (149) (2024), 905–913

GENERALIZED FRACTIONAL INTEGRAL OPERATORS ON WEAK CHOQUET SPACES OVER QUASI-METRIC MEASURE SPACES

Toshihide Futamura, Nagoya, Tetsu Shimomura, Higashi-Hiroshima

Received April 1, 2024. Published online July 18, 2024.

Abstract. We prove the boundedness of the generalized fractional maximal operator M_α and the generalized fractional integral operator I_α on weak Choquet spaces with respect to Hausdorff content over quasi-metric measure spaces.

Keywords: fractional integral operator; quasi-metric measure space; Hausdorff content; weak Choquet space; Ahlfors regular

MSC 2020: 42B25, 28A12, 46E30

1. Introduction

For $0 \leqslant \alpha < N$ and a locally integrable function f on $\mathbb{R}^N$, the fractional maximal function $\mathcal{M}_\alpha f$ is defined by

$$\mathcal{M}_\alpha f(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, \mathrm{d}y.$$

When $\alpha = 0$, $\mathcal{M}_\alpha$ is the Hardy-Littlewood maximal operator $\mathcal{M}$. Adams in [1] defined a Choquet space with respect to Hausdorff capacity and studied the boundedness of $\mathcal{M}_\alpha$ from L^p with respect to Hausdorff capacity to Choquet spaces, see [2], Theorem 7. For the boundedness of $\mathcal{M}$ on Choquet spaces with respect to Hausdorff content, see [10]. Watanabe in [12], [13] extended Theorem 7 of [2] to quasi-metric measure spaces.

For $0 < \alpha < N$, we define the Riesz potential of order α for a locally integrable function f on $\mathbb{R}^N$ by

$$U_\alpha f(x) = \int_{\mathbb{R}^N} |x-y|^{\alpha-N} f(y) \, \mathrm{d}y.$$

The operator U_α is also called the *fractional integral operator*.

DOI: 10.21136/CMJ.2024.0133-24

© Institute of Mathematics, Czech Academy of Sciences 2024.

Recently, in [5], the boundedness of $\mathcal{M}_\alpha$ and U_α on weak Choquet spaces with respect to Hausdorff content in the Euclidean setting has been studied.

In the present note, we extend [5] from the Euclidean case to a quasi-metric measure setting. Let $X = (X, d, \mu)$ be a quasi-metric measure space with a quasi-metric d and a nonnegative complete Borel regular outer measure μ which is finite in every bounded set; a function $d\colon\ X \times X \to [0, \infty)$ is called a *quasi-metric* if the following conditions are satisfied for all $x, y, z \in X$:

(i) $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$,

(iii) there exists $K_1 \geqslant 1$ such that

$$d(x, y) \leqslant K_1(d(x, z) + d(z, y)).$$

By $B(x, r) = \{y \in X\colon\ d(x, y) < r\}$, we denote a ball in X. We say that a measure μ is Ahlfors Q-regular if there exist constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 r^Q \leqslant \mu(B(x, r)) \leqslant c_2 r^Q \tag{1.1}$$

for all $x \in X$ and $0 < r < d_X := \operatorname{diam} X$, see e.g. [3], [7]. Note that Ahlfors Q-regular measure μ satisfies that $\mu(\{x\}) = 0$ for $x \in X$ and $0 < \mu(B(x, r)) < \infty$ for $x \in X$ and $r > 0$. In the present note we assume that μ is Ahlfors Q-regular.

Let $0 < d \leqslant Q$ and let E be a set in X. Then we define the Hausdorff content $H^d_\infty(E)$ by

$$H^d_\infty(E) = \inf\Big\{\sum_{i=1}^\infty r_i^d\colon\ E \subset \bigcup_{i=1}^\infty B(x_i, r_i)\Big\}.$$

We refer to [1], [2] for the usual Hausdorff content.

For a nonnegative function f, the integral of f with respect to H^d_∞ is taken in the Choquet sense:

$$\int_X f(y)\,\mathrm{d}H^d_\infty(y) = \int_0^\infty H^d_\infty(\{x \in X\colon\ f(x) > t\})\,\mathrm{d}t.$$

For $0 < p < \infty$, let the Choquet space $L^p(H^d_\infty)$ be the set of all functions such that

$$\|f\|_{L^p(H^d_\infty)} = \Big(\int_X |f(y)|^p\,\mathrm{d}H^d_\infty(y)\Big)^{1/p} < \infty,$$

and let the weak Choquet space $wL^p(H^d_\infty)$ be the set of all functions such that

$$\|f\|_{wL^p(H^d_\infty)} = \sup_{t>0} tH^d_\infty(\{x \in X\colon\ |f(x)| > t\})^{1/p} < \infty.$$

Note that

$$tH^d_\infty(\{x \in X\colon\ |f(x)| > t\})^{1/p} \leqslant \|f\|_{L^p(H^d_\infty)} \quad \forall\, t > 0. \tag{1.2}$$

For a locally integrable function f on X, the fractional maximal function $M_\alpha f$ is defined by

$$M_\alpha f(x) = \sup_{r>0} \mu(B(x,r))^{(\alpha-Q)/Q} \int_{B(x,r)} |f(y)| \, \mathrm{d}\mu(y).$$

For $\alpha > 0$, we define the generalized Riesz potential of order α for a locally integrable function f on X by

$$I_\alpha f(x) = \int_X \frac{d(x,y)^\alpha f(y)}{\mu(B(x, d(x,y)))} \, \mathrm{d}\mu(y),$$

see e.g., [4], [8], [9], [11]. The operator I_α is also called the *generalized fractional integral operator*. Observe that this naturally extends the Riesz potential operator $U_\alpha f(x)$ when (X, d, μ) is the N-dimensional Euclidean space and $\mu = \mathrm{d}x$.

Our aim in this note is to establish the following theorem as an extension of Theorem 1.1 in [5].

Theorem 1.1.

(i) *Let $0 < d \leqslant Q$ and $0 \leqslant \alpha < Q$. Suppose that $d/Q \leqslant r < p < d/\alpha$ and*

$$\frac{d - \alpha r}{q} = \frac{d - \alpha p}{p}.$$

Then there exists a constant C such that

$$\|M_\alpha f\|_{wL^q(H^{d-\alpha r}_\infty)} \leqslant C \|f\|_{wL^p(H^d_\infty)}.$$

(ii) *Let $0 < d < Q$ and $0 < \alpha < Q$. Suppose that $d/Q < r < p < d/\alpha$ and*

$$\frac{d - \alpha r}{q} = \frac{d - \alpha p}{p}. \tag{1.3}$$

Then there exists a constant C such that

$$\|I_\alpha f\|_{wL^q(H^{d-\alpha r}_\infty)} \leqslant C \|f\|_{wL^p(H^d_\infty)}.$$

(iii) *Let $0 < d < Q$ and $0 < \alpha < Q$. Suppose that $d/Q < r < p < d/\alpha$ and*

$$\frac{d - \alpha r}{q} = \frac{d - \alpha p}{p}.$$

Then there exists a constant C such that

$$\|I_\alpha f\|_{L^q(H^{d-\alpha r}_\infty)} \leqslant C \|f\|_{L^p(H^d_\infty)}.$$

Our strategy is to give an estimate of $I_\alpha f$ by the use of the fractional maximal function. The key lemma for our proof is Lemma 2.8, which is proved by Hedberg's method, see [6].

Throughout this paper, let C denote various constants independent of the variables in question and $C(a, b, \ldots)$ be a constant that depends on $a, b, \ldots$

2. Lemmas

By (1.2) and [4], Theorem 14.11, we obtain the following lemma. See also [5], Propositions 2.3 and 2.5.

Lemma 2.1. *For $0 < q < p$ and every mesurable set A satisfying $H^d_\infty(A) < \infty$,*

$$H^d_\infty(A)^{-1/q+1/p}\|f\chi_A\|_{L^q(H^d_\infty)} \leqslant 2^{1/q}\Big(\frac{q}{p-q}\Big)^{1/p}\|f\|_{wL^p(H^d_\infty)}. \tag{2.1}$$

For $0 < q \leqslant p < \infty$, we put

$$\|f\|_{M^p_q(H^d_\infty)} = \sup_{r>0, x\in X} \mu(B(x,r))^{(d/p-d/q)/Q}\bigg(\int_{B(x,r)} |f(y)|^q \,\mathrm{d}H^d_\infty(y)\bigg)^{1/q}.$$

Note that if $0 < p = q < \infty$, then $\|f\|_{M^p_p(H^d_\infty)} = \|f\|_{L^p(H^d_\infty)}$. By (1.1) and Hölder's inequality, we have that if $0 < q \leqslant q_1 \leqslant p$, then

$$\|f\|_{M^p_q(H^d_\infty)} \leqslant C\|f\|_{M^p_{q_1}(H^d_\infty)} \leqslant C\|f\|_{L^p(H^d_\infty)}. \tag{2.2}$$

We know the following lemmas from [12].

Lemma 2.2 ([12], Theorem (i))**.** *Let $d/Q < p < d/\alpha$. Then*

$$\|M_\alpha f\|_{L^p(H^{d-\alpha p}_\infty)} \leqslant C\|f\|_{L^p(H^d_\infty)}.$$

Lemma 2.3 ([12], Theorem (ii))**.** *Let $0 < d \leqslant Q$. Then*

$$\|M_\alpha f\|_{wL^{d/Q}(H^{d-\alpha d/Q}_\infty)} \leqslant C\|f\|_{L^{d/Q}(H^d_\infty)}.$$

Lemma 2.4 ([5], Lemma 2.1)**.** *Let $0 < d \leqslant Q$ and $0 < q \leqslant p < \infty$. Then for $1 \leqslant \theta \leqslant Q/d$,*

$$\|f\|_{L^{\theta p}(H^{\theta d}_\infty)} \leqslant \theta^{1/\theta p}\|f\|_{L^p(H^d_\infty)} \quad \text{and} \quad \|f\|_{M^{\theta p}_{\theta q}(H^{\theta d}_\infty)} \leqslant \theta^{1/\theta q}\|f\|_{M^p_q(H^d_\infty)}.$$

Using (1.1) and Lemma 2.4 with $p = d/Q$ and $\theta = Q/d$, we obtain the next lemma.

Lemma 2.5 (cf. [12], Lemma 2.5)**.** *Let $0 < d \leqslant Q$. Then*

$$\|f\|_{L^1(\mu)} \leqslant \frac{c_2 Q}{d}\|f\|_{L^{d/Q}(H^d_\infty)}.$$

Lemma 2.6. *Let $0 \leqslant \beta < \alpha < Q$ and let f be a nonnegative measurable function on X. Then there exists a constant $C > 0$ such that*

$$\int_{B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) \leqslant C\frac{r^{\alpha-\beta}}{\alpha-\beta}M_\beta f(x)$$

for all $x \in X$.

P r o o f. By (1.1), we have

$$\begin{aligned}
\int_{B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) &= \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}r)\setminus B(x,2^{-j}r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) \\
&\leqslant \sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}r)\setminus B(x,2^{-j}r)} \frac{(2^{-j+1}r)^\alpha f(y)}{\mu(B(x,2^{-j}r))}\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{j=1}^{\infty} \int_{B(x,2^{-j+1}r)\setminus B(x,2^{-j}r)} \frac{(2^{-j+1}r)^\alpha f(y)}{\mu(B(x,2^{-j+1}r))}\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{j=1}^{\infty} (2^{-j+1}r)^{\alpha-\beta} M_\beta f(x) \leqslant C\frac{r^{\alpha-\beta}}{\alpha-\beta}M_\beta f(x)
\end{aligned}$$

for $x \in X$, since $2^{\alpha-\beta} - 1 \geqslant C(\alpha-\beta)$, which completes the proof. □

Lemma 2.7. *Let $0 < d < Q$, $0 < \alpha < Q$ and let f be a nonnegative measurable function on X. Suppose that $d/Q \leqslant p < d/\alpha$. Then there exists a constant $C > 0$ such that*

$$\int_{X\setminus B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) \leqslant C\frac{r^{\alpha-d/p}}{d/p-\alpha}\|f\|_{M^p_{d/Q}(H^d_\infty)}$$

for all $x \in X$.

P r o o f. By (1.1), we have

$$\begin{aligned}
\int_{X\setminus B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) &= \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}r)\setminus B(x,2^{j}r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) \\
&\leqslant \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}r)\setminus B(x,2^{j}r)} \frac{(2^{j+1}r)^\alpha f(y)}{\mu(B(x,2^{j}r))}\,\mathrm{d}\mu(y) \\
&\leqslant C\sum_{j=0}^{\infty} \int_{B(x,2^{j+1}r)\setminus B(x,2^{j}r)} \frac{(2^{j+1}r)^\alpha f(y)}{\mu(B(x,2^{j+1}r))}\,\mathrm{d}\mu(y)
\end{aligned}$$

for $x \in X$. By Lemma 2.5 and (1.1), we have

$$\begin{aligned}
&\int_{X\setminus B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y)\\
&\quad\leqslant C\sum_{j=0}^{\infty}(2^{j+1}r)^\alpha \mu(B(x,2^{j+1}r))^{-1+(Q-d/p)/Q}\mu(B(x,2^{j+1}r))^{(d/p-Q)/Q}\\
&\qquad\times\left(\int_{B(x,2^{j+1}r)} f(y)^{d/Q}\,dH^d_\infty(y)\right)^{Q/d}\\
&\quad\leqslant C\sum_{j=0}^{\infty}(2^{j+1}r)^{\alpha-d/p}\|f\|_{M^p_{d/Q}(H^d_\infty)} \leqslant C\frac{r^{\alpha-d/p}}{d/p-\alpha}\|f\|_{M^p_{d/Q}(H^d_\infty)}
\end{aligned}$$

for $x \in X$, since $2^{d/p-\alpha}-1 \geqslant C(d/p-\alpha)$. Thus, the lemma is proved. □

We give an estimate of $I_\alpha f$ by use of $M_\beta f$, which is an extension of Lemma 2.7 in [5].

Lemma 2.8. *Let $0 < d < Q$, $0 \leqslant \beta < \alpha < Q$ and $d/Q \leqslant p < q < \infty$ with*

$$\frac{d-\beta p}{q} = \frac{d-\alpha p}{p}.$$

Then there exists a constant $C > 0$ such that the inequality

$$I_\alpha f(x) \leqslant C\Big(\frac{d}{p}-\alpha\Big)^{p/q-1}(\alpha-\beta)^{-p/q}\|f\|^{1-p/q}_{M^p_{d/Q}(H^d_\infty)}(M_\beta f(x))^{p/q}$$

holds for all $x \in X$.

P r o o f. We note that $d-\alpha p = (q/p-1)^{-1}p(\alpha-\beta) > 0$ by our assumption. By Lemmas 2.6 and 2.7, we obtain

$$\begin{aligned}
I_\alpha f(x) &= \int_{B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y) + \int_{X\setminus B(x,r)} \frac{d(x,y)^\alpha f(y)}{\mu(B(x,d(x,y)))}\,\mathrm{d}\mu(y)\\
&\leqslant C\frac{r^{\alpha-\beta}}{\alpha-\beta}M_\beta f(x) + C\frac{r^{\alpha-d/p}}{d/p-\alpha}\|f\|_{M^p_{d/Q}(H^d_\infty)}.
\end{aligned}$$

Take

$$r = \Big\{(\alpha-\beta)\Big(\frac{d}{p}-\alpha\Big)^{-1}\|f\|_{M^p_{d/Q}(H^d_\infty)}(M_\beta f(x))^{-1}\Big\}^{1/(d/p-\beta)}.$$

Since

$$\frac{p}{q} = \frac{d-\alpha p}{d-\beta p},$$

we find

$$I_\alpha f(x) \leqslant C\Big(\frac{d}{p}-\alpha\Big)^{p/q-1}(\alpha-\beta)^{-p/q}\|f\|^{1-p/q}_{M^p_{d/Q}(H^d_\infty)}(M_\beta f(x))^{p/q},$$

as required. □

3. Proof of Theorem 1.1

We can prove Theorem 1.1 like in the proof of Theorem 1.1 in [5]. For the reader's convenience, we give a proof of Theorem 1.1.

P r o o f of Theorem 1.1 (i). Put $G_t = \{x \in X\colon\ M_\alpha f(x) > t\}$ for $t > 0$. Then we have by (1.2) and Lemma 2.2

$$
\begin{aligned}
(3.1)\qquad tH_\infty^{d-\alpha r}(G_t)^{1/q} &= H_\infty^{d-\alpha r}(G_t)^{1/q-1/r} \cdot tH_\infty^{d-\alpha r}(G_t)^{1/r} \\
&\leqslant H_\infty^{d-\alpha r}(G_t)^{1/q-1/r} \|(M_\alpha f)\chi_{G_t}\|_{L^r(H_\infty^{d-\alpha r})} \\
&\leqslant CH_\infty^{d-\alpha r}(G_t)^{1/q-1/r} \|f\chi_{G_t}\|_{L^r(H_\infty^d)},
\end{aligned}
$$

when $r > d/Q$. Further (3.1) holds when $r = d/Q$ by Lemma 2.3.

Now $1/q - 1/r < 0$ and $d^{-1}(d - \alpha r)(1/q - 1/r) = 1/p - 1/r$. Then we have

$$
H_\infty^{d-\alpha r}(G_t)^{1/q-1/r} \leqslant H_\infty^d(G_t)^{d^{-1}(d-\alpha r)(1/q-1/r)} = H_\infty^d(G_t)^{1/p-1/r}.
$$

Hence, we obtain by (2.1)

$$
tH_\infty^{d-\alpha r}(G_t)^{1/q} \leqslant CH_\infty^d(G_t)^{1/p-1/r} \|f\chi_{G_t}\|_{L^r(H_\infty^d)} \leqslant C\|f\|_{wL^p(H_\infty^d)}
$$

as required. □

P r o o f of Theorem 1.1 (ii). First of all, we take a number θ such that $1 < \theta < \min\{Q/d, (d - \alpha r)/(d - \alpha p)\}$. Set $\beta = d/p - (d - \alpha r)/(\theta p)$, $d_1 = \theta d$, $p_1 = \theta p$ and $r_1 = \alpha r/\beta$. Then $0 < d_1 < Q$, $0 < \beta < \alpha$, $d_1/Q < p_1 < q$, $d/Q < r < r_1 < p$ and

$$
\frac{d_1 - \beta p_1}{q} = \frac{d - \alpha r}{q} = \frac{d - \alpha p}{p} = \frac{d_1 - \alpha p_1}{p_1}.
$$

Since

$$
\begin{aligned}
&\|f\|_{M_{d/Q}^p(H_\infty^d)} \\
&\quad = \sup_{r>0,\ x\in X} \mu(B(x,r))^{(d/p-Q)/Q} \|f\chi_{B(x,r)}\|_{L^{d/Q}(H_\infty^d)} \\
&\quad \leqslant C \sup_{r>0, x\in X} \mu(B(x,r))^{(d/p-Q)/Q} H_\infty^d(B(x,r))^{Q/d-1/p} \|f\chi_{B(x,r)}\|_{wL^p(H_\infty^d)} \\
&\quad \leqslant C \sup_{r>0} \bigl(c_1 r^Q\bigr)^{(d/p-Q)/Q} (r^d)^{Q/d-1/p} \|f\|_{wL^p(H_\infty^d)} = C\|f\|_{wL^p(H_\infty^d)}
\end{aligned}
$$

by (2.1), we obtain by Lemmas 2.8 and 2.4

$$
\begin{aligned}
|I_\alpha f(x)| &\leqslant C\|f\|^{1-p_1/q}_{M^{p_1}_{d_1/Q}(H^{d_1}_\infty)} (M_\beta f(x))^{p_1/q} \leqslant C\|f\|^{1-p_1/q}_{M^p_{d/Q}(H^d_\infty)} (M_\beta f(x))^{p_1/q} \\
&\leqslant C\|f\|^{1-p_1/q}_{wL^p(H^d_\infty)} (M_\beta f(x))^{p_1/q}.
\end{aligned}
$$

Therefore, we have

$$\|I_\alpha f\|_{wL^q(H^{d-\alpha r}_\infty)} \leqslant C\|f\|^{1-p_1/q}_{wL^p(H^d_\infty)}\|(M_\beta f)^{p_1/q}\|_{wL^q(H^{d-\alpha r}_\infty)} = C\|f\|^{1-p_1/q}_{wL^p(H^d_\infty)}\|M_\beta f\|^{p_1/q}_{wL^{p_1}(H^{d-\beta r_1}_\infty)}. \tag{3.2}$$

Since $(d-\beta r_1)/p_1 = (d-\beta p)/p$, we have by (i)

$$\|M_\beta f\|_{wL^{p_1}(H^{d-\beta r_1}_\infty)} \leqslant C\|f\|_{wL^p(H^d_\infty)}. \tag{3.3}$$

By (3.2) and (3.3), (ii) is proved. □

P r o o f of Theorem 1.1 (iii). Set $\beta = \alpha r/p$. Then we have $0 < \beta < \alpha$, $p < d/\beta$ and

$$\frac{d-\beta p}{q} = \frac{d-\alpha p}{p}.$$

It follows from Lemmas 2.8, 2.2, and (2.2) that

$$\begin{aligned}\|I_\alpha f\|_{L^q(H^{d-\beta p}_\infty)} &\leqslant C\|f\|^{1-p/q}_{M^p_{d/Q}(H^d_\infty)}\|M_\beta f\|^{p/q}_{L^p(H^{d-\beta p}_\infty)}\\ &\leqslant C\|f\|^{1-p/q}_{L^p(H^d_\infty)}\|f\|^{p/q}_{L^p(H^d_\infty)} = C\|f\|_{L^p(H^d_\infty)},\end{aligned}$$

as required. □

Acknowledgement. We would like to express our deep thanks to the referee for carefully reading the manuscript and giving useful comments.

References

[1] *D. R. Adams*: A note on Choquet integrals with respect to Hausdorff capacity. Function Spaces and Applications. Lecture Notes in Mathematics 1302. Springer, Berlin, 1988, pp. 115–124. zbl MR doi
[2] *D. R. Adams*: Choquet integrals in potential theory. Publ. Mat., Barc. *42* (1998), 3–66. zbl MR doi
[3] *A. Björn, J. Björn*: Nonlinear Potential Theory on Metric Spaces. EMS Tracts in Mathematics 17. EMS Press, Zürich, 2011. zbl MR doi
[4] *P. Hajłasz, P. Koskela*: Sobolev met Poincaré. Mem. Am. Math. Soc. *145* (2000), 101 pages. zbl MR doi
[5] *N. Hatano, R. Kawasumi, H. Saito, H. Tanaka*: Choquet integrals, Hausdorff content and fractional operators. To appear in Bull. Aust. Math. Soc. doi
[6] *L. I. Hedberg*: On certain convolution inequalities. Proc. Am. Math. Soc. *36* (1972), 505–510. zbl MR doi
[7] *J. Heinonen*: Lectures on Analysis on Metric Spaces. Universitext. Springer, New York, 2001. zbl MR doi
[8] *A. Kairema*: Two-weight norm inequalities for potential type and maximal operators in a metric space. Publ. Mat., Barc. *57* (2013), 3–56. zbl MR doi

[9] *Y. Mizuta, T. Shimomura, T. Sobukawa*: Sobolev's inequality for Riesz potentials of functions in non-doubling Morrey spaces. Osaka J. Math. *46* (2009), 255–271. zbl MR

[10] *J. Orobitg, J. Verdera*: Choquet integrals, Hausdorff content and the Hardy-Littlewood maximal operator. Bull. Lond. Math. Soc. *30* (1998), 145–150. zbl MR doi

[11] *Y. Sawano, T. Shimomura*: Fractional maximal operator on Musielak-Orlicz spaces over unbounded quasi-metric measure spaces. Result. Math. *76* (2021), Article ID 188, 22 pages. zbl MR doi

[12] *H. Watanabe*: Estimates of fractional maximal functions in a quasi-metric space. Nat. Sci. Rep. Ochanomizu Univ. *56* (2006), 21–31. zbl MR

[13] *H. Watanabe*: Estimates of maximal functions by Hausdorff contents in a metric space. Potential Theory in Matsue. Advanced Studies in Pure Mathematics 44. Mathematical Society of Japan, Tokyo, 2006, pp. 377–389. zbl MR doi

Authors' addresses: Toshihide Futamura, Department of Mathematics, Daido University, 10-3 Takiharucho, Minami-ku, Nagoya 457-8530, Japan, e-mail: `futamura@daido-it.ac.jp`; Tetsu Shimomura (corresponding author), Department of Mathematics, Graduate School of Humanities and Social Sciences, Hiroshima University, 1-1-1 Kagamiyama, Higashi-Hiroshima 739-8524, Japan, e-mail: `tshimo@hiroshima-u.ac.jp`.