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REGULARIZING EFFECT OF THE INTERPLAY BETWEEN COEFFICIENTS IN SOME NONCOERCIVE INTEGRAL FUNCTIONALS

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Abstract. We are interested in regularizing effect of the interplay between the coefficient of zero order term and the datum in some noncoercive integral functionals of the type

$$\mathcal{J}(v) = \int_\Omega j(x, v, \nabla v)\,\mathrm{d}x + \int_\Omega a(x)|v|^2\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x, \quad v \in W_0^{1,2}(\Omega),$$

where $\Omega \subset \mathbb{R}^N$, j is a Carathéodory function such that $\xi \mapsto j(x, s, \xi)$ is convex, and there exist constants $0 \leqslant \tau < 1$ and $M > 0$ such that

$$\frac{|\xi|^2}{(1+|s|)^\tau} \leqslant j(x, s, \xi) \leqslant M|\xi|^2$$

for almost all $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$. We show that, even if $0 < a(x)$ and $f(x)$ only belong to $L^1(\Omega)$, the interplay

$$|f(x)| \leqslant 2Qa(x)$$

implies the existence of a minimizer $u \in W_0^{1,2}(\Omega)$ which belongs to $L^\infty(\Omega)$.

Keywords: regularizing effect; interplay; minimizer; noncoercive integral functional

MSC 2020: 49J45

1. Introduction

In this note we are interested in noncoercive integral functionals of the type

$$\mathcal{J}(v) = \int_\Omega j(x, v(x), \nabla v(x))\,\mathrm{d}x + \int_\Omega a(x)|v(x)|^2\,\mathrm{d}x - \int_\Omega f(x)v(x)\,\mathrm{d}x, \tag{1.1}$$

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where Ω is a bounded open subset of $\mathbb{R}^N$, $N > 2$, $j\colon \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function (that is, $x \mapsto j(x, s, \xi)$ is measurable for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $(s, \xi) \mapsto j(x, s, \xi)$ is continuous for almost all $x \in \Omega$) such that for almost every $x \in \Omega$ and for every $s \in \mathbb{R}$,

$$\xi \mapsto j(x, s, \xi) \text{ is convex,} \tag{1.2}$$

and there exist constants $0 \leqslant \tau < 1$ and $M > 0$ such that

$$\frac{|\xi|^2}{(1 + |s|)^\tau} \leqslant j(x, s, \xi) \leqslant M|\xi|^2 \tag{1.3}$$

for almost all $x \in \Omega$, all $s \in \mathbb{R}$ and all $\xi \in \mathbb{R}^N$; for the coefficient $a(x)$ of the zero order term and the datum $f(x)$ in (1.1), we assume

$$0 < a(x) \in L^1(\Omega), \tag{1.4}$$

and there exists a constant $Q > 0$ such that

$$|f(x)| \leqslant 2Qa(x) \quad \text{for a.e. } x \in \Omega. \tag{1.5}$$

We remark that (1.5) together with (1.4) implies $f(x) \in L^1(\Omega)$.

We notice that, although the functional $\mathcal{J}(v)$ may not be well defined for $v \in W_0^{1,2}(\Omega)$, it is bounded from below. In fact, thanks to the interplay (1.5) we have

$$\begin{aligned}\int_\Omega (a(x)|v|^2 - fv)\,\mathrm{d}v &\geqslant \int_\Omega (a(x)|v|^2 - 2Qa(x)|v|)\,\mathrm{d}x \\ &= \int_\Omega a(x)[|v|^2 - 2Q|v|]\,\mathrm{d}x \geqslant -Q^2 \int_\Omega a(x)\,\mathrm{d}x.\end{aligned}$$

A function $u \in W_0^{1,2}(\Omega)$ is called a *minimizer* of (1.1) if

$$\mathcal{J}(u) = \inf\{\mathcal{J}(v)\colon\ v \in W_0^{1,2}(\Omega)\}.$$

Before going on, let us first recall some definitions and known results.

Let X be a Banach space and $\|x\|$ denote the norm of x in X. Let $\mathcal{J}\colon\ X \to \mathbb{R}$ be a functional.

(i) $\mathcal{J}$ is weakly lower semicontinuous if $\liminf_{n\to\infty} \mathcal{J}(x_n) \geqslant \mathcal{J}(x)$ for every sequence x_n weakly converging to x;

(ii) $\mathcal{J}$ is coercive if $\lim_{\|x\|\to\infty} \mathcal{J}(x) = \infty$.

The following Weierstrass theorem can be found, for example, in [3], Theorem 9.1.

Theorem 1.1. *Let X be a Banach reflexive space. Let $\mathcal{J}\colon\ X \to \mathbb{R}$ be coercive, bounded from below and weakly lower semicontinuous. Then $\mathcal{J}$ has a minimizer.*

The following lemma comes from Theorem 4.5 of [8]. We write it in the present form suitable for our purposes. As it is usual, we shall use an arrow "$\to$" to denote the strong convergence, and a half-arrow "$\rightharpoonup$" to mark the weak convergence.

Lemma 1.1. *Let Ω be an open set in $\mathbb{R}^N$, and let $F(x,u,z)$ be a function defined in $\Omega\times\mathbb{R}\times\mathbb{R}^N$ and such that*

(i) *F is a Carathéodory function, that is, measurable in x for every $(u,z)\in\mathbb{R}\times\mathbb{R}^N$ and continuous in (u,z) for almost every $x\in\Omega$;*

(ii) *$F(x,u,z)$ is convex in z for almost every $x\in\Omega$ and for every $u\in\mathbb{R}$;*

(iii) *$F\geqslant 0$.*

Let $u_h,u\in L^1(\Omega,\mathbb{R})$, $z_h,z\in L^1(\Omega,\mathbb{R}^N)$, and assume that $u_h\to u$ and $z_h\rightharpoonup z$ in $L^1_{\mathrm{loc}}(\Omega)$. Then

$$\int_\Omega F(x,u,z)\,\mathrm{d}x\leqslant\liminf_{h\to\infty}\int_\Omega F(x,u_h,z_h)\,\mathrm{d}x.$$

Let us come back to the integral functional (1.1). The following example illustrates that it is not coercive. In fact, let us consider the special case $a(x)=f(x)=0$ a.e. in Ω. The sequence

$$v_m(x)=|x|^{m(2-N)/((m+1)(2-\tau))}-1,\quad m=1,2,\dots,$$

defined in $B_1(0)$, the unit ball centered at 0 in $\mathbb{R}^N$, satisfies for every $m\geqslant(2-\tau)/\tau$,

$$\begin{aligned}\int_{B_1(0)}&|\nabla v_m|^2\,\mathrm{d}x\\&=\Big(\frac{m(N-2)}{(m+1)(2-\tau)}\Big)^2\int_{B_1(0)}|x|^{-2(Nm+2-(m+1)\tau)/((m+1)(2-\tau))}\,\mathrm{d}x=\infty,\end{aligned}$$

so

$$\|v_m\|_{W^{1,2}_0(\Omega)}=\infty\quad\text{for every } m\geqslant(2-\tau)/\tau. \tag{1.6}$$

In the meantime, for all $m=1,2,\dots$,

$$\int_{B_1(0)}\frac{|\nabla v_m|^2}{(1+|v_m|)^\tau}\,\mathrm{d}x=\Big(\frac{m(N-2)}{(m+1)(2-\tau)}\Big)^2\int_{B_1(0)}|x|^{-(Nm+2)/(m+1)}\,\mathrm{d}x<\infty. \tag{1.7}$$

Therefore, (1.6) together with (1.7) and (1.3) implies

$$\begin{aligned}\frac{1}{\|v_m\|_{W^{1,2}_0(\Omega)}}&\int_{B_1(0)}\frac{|\nabla v_m|^2}{(1+|v_m|)^\tau}\,\mathrm{d}x\\&\leqslant\frac{1}{\|v_m\|_{W^{1,2}_0(\Omega)}}\int_{B_1(0)}j(x,v,\nabla v)\,\mathrm{d}x=0\quad\text{as } m\to\infty,\end{aligned}$$

which shows that the integral $\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x$ is not coercive on $W^{1,2}_0(\Omega)$.

In this note we pay our attention to regularizing effect of the interplay between the coefficients $f(x)$ and $a(x)$, see (1.5). This hypothesis or variants of it has been used for different problems. In [1], the authors proved that, even if f only belongs to $L^1(\Omega)$, the assumption (1.5) implies the existence of a bounded solution of boundary value problems, whose simplest example is the linear problem

$$\begin{cases} -\mathrm{div}(M(x)\nabla u) + a(x)u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $M(x)$ is a bounded elliptic matrix; in [2] the authors continued the study of L^q interaction between the coefficients $a(x)$ and the datum $f(x)$, that is, Q is a function in $L^q(\Omega)$. There have been some other remarkable advances in this field in recent years, we refer to [4] for the existence and regularity of minima for integral functionals noncoercive in the energy space, [7] for the case of singular drift terms, [9], [10] for the parabolic problem, and [12] for non-uniformly elliptic equations. Some recent developments dealing with the second order regularity of the solutions, also for non-uniformly elliptic problems, can be found in [5], [6], [11].

2. Existence and regularity of a minimizer

In this section we show that the interplay (1.5) between $a(x)$ and $f(x)$ has regularizing effect on the existence and regularity of a minimizer $u \in W_0^{1,2}(\Omega)$ of (1.1), even if $a(x)$ and $f(x)$ are only assumed to belong to $L^1(\Omega)$.

Theorem 2.1. *Suppose* (1.2)–(1.5). *Then there exists a minimizer* $u \in W_0^{1,2}(\Omega)$ *of* $\mathcal{J}$ *which belongs to* $L^\infty(\Omega)$.

Due to the lack of coerciveness of the integral functional (1.1) on $W_0^{1,2}(\Omega)$, it is not easy to derive directly that there exists a minimizer $u \in W_0^{1,2}(\Omega)$ of (1.1), see Theorem 1.1 above. We show instead that, in a suitable larger space $W_0^{1,\delta}(\Omega)$ with a suitable $\delta \in (1,2)$, the minimizer of (1.1) exists, then we show that this minimizer lies in the space $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$. To this aim, we first prove the following lemma.

Lemma 2.1. *Suppose* (1.2)–(1.5). *Then there exists a minimizer* $u \in W_0^{1,\delta}(\Omega)$, $\delta = N(2-\tau)/(N-\tau)$, *of* $\mathcal{J}$. *That is, there exists* $u \in W_0^{1,\delta}(\Omega)$ *such that*

$$\mathcal{J}(u) = \inf\{\mathcal{J}(v),\, v \in W_0^{1,\delta}(\Omega)\}.$$

Note that $1 < \delta = N(2-\tau)/(N-\tau) < 2$.

P r o o f. Take a Banach reflexive space $X = W_0^{1,\delta}(\Omega)$. We prove that $\mathcal{J}\colon W_0^{1,\delta}(\Omega) \to \mathbb{R}$ is

(1) coercive;

(2) bounded from below;

(3) weakly lower semicontinuous.

Then the result follows from the Weierstrass theorem.

We begin with the coerciveness and boundedness of $\mathcal{J}$. In virtue of (1.3) and the Hölder inequality for any $v \in W_0^{1,\delta}(\Omega)$,

$$\begin{aligned}\int_\Omega |\nabla v|^\delta \,\mathrm{d}x &= \int_\Omega \frac{|\nabla v|^\delta}{(1+|v|)^{\delta\tau/2}}(1+|v|)^{\delta\tau/2}\,\mathrm{d}x\\ &\leqslant \left(\int_\Omega \frac{|\nabla v|^2}{(1+|v|)^\tau}\,\mathrm{d}x\right)^{\delta/2}\left(\int_\Omega (1+|v|)^{\delta\tau/(2-\delta)}\,\mathrm{d}x\right)^{1-\delta/2}\\ &\leqslant \left(\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x\right)^{\delta/2}\left(1+\int_\Omega |v|^{\delta\tau/(2-\delta)}\,\mathrm{d}x\right)^{1-\delta/2}.\end{aligned}$$

Since $\delta = N(2-\tau)/(N-\tau)$, then $\delta\tau/(2-\delta) = \delta^*$. The Sobolev embedding theorem yields

$$\begin{aligned}\int_\Omega |\nabla v|^\delta \,\mathrm{d}x &\leqslant \left(\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x\right)^{\delta/2}\left(1+\int_\Omega |v|^{\delta^*}\,\mathrm{d}x\right)^{1-\delta/2}\\ &\leqslant \left(\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x\right)^{\delta/2}\left(1+c\left(\int_\Omega |\nabla v|^\delta\,\mathrm{d}x\right)^{\delta^*/\delta}\right)^{1-\delta/2},\end{aligned}$$

where c is a constant depending upon N and δ. It is no loss of generality to assume that $\|v\|_{W_0^{1,\delta}(\Omega)} \geqslant 1$, thus

$$\int_\Omega |\nabla v|^\delta \,\mathrm{d}x \leqslant (1+c)\left(\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x\right)^{\delta/2}\left(\int_\Omega |\nabla v|^\delta\,\mathrm{d}x\right)^{(1-\delta/2)\delta^*/\delta},$$

so that

$$\begin{aligned}\int_\Omega j(x,v,\nabla v)\,\mathrm{d}x &\geqslant (1+c)^{-2/\delta}\left(\int_\Omega |\nabla v|^\delta\,\mathrm{d}x\right)^{[1-(1-\delta/2)\delta^*/\delta]2/\delta}\\ &= (1+c)^{-2/\delta}\left(\int_\Omega |\nabla v|^\delta\,\mathrm{d}x\right)^{(N-2)/(N-\delta)}\\ &= (1+c)^{-2/\delta}\|v\|_{W_0^{1,\delta}(\Omega)}^{(N-2)\delta/(N-\delta)}.\end{aligned} \tag{2.1}$$

The assumption (1.5) implies

$$
\begin{aligned}
(2.2)\qquad &\int_\Omega a(x)|v|^2\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x \\
&\geqslant \int_\Omega a(x)|v|^2\,\mathrm{d}x - \int_\Omega |f||v|\,\mathrm{d}x \geqslant \int_\Omega a(x)|v|^2\,\mathrm{d}x - 2\int_\Omega Qa(x)|v|\,\mathrm{d}x \\
&\geqslant \int_\Omega a(x)|v|^2\,\mathrm{d}x - Q^2\int_\Omega a(x)\,\mathrm{d}x - \int_\Omega a(x)|v|^2\,\mathrm{d}x = -Q^2\int_\Omega a(x)\,\mathrm{d}x.
\end{aligned}
$$

Combining (2.1) and (2.2) one gets

$$
(2.3)\qquad \mathcal{J}(v) \geqslant (1+c)^{-2/\delta}\|v\|^{(N-2)\delta/(N-\delta)}_{W_0^{1,\delta}(\Omega)} - Q^2\int_\Omega a(x)\,\mathrm{d}x,
$$

which implies

$$
\lim_{\|v\|_{W_0^{1,\delta}(\Omega)}\to\infty} \mathcal{J}(v) = \infty,
$$

that is, $\mathcal{J}$ is coercive on $W_0^{1,\delta}(\Omega)$. The estimate (2.3) implies also that $\mathcal{J}(v)$ is bounded from below since for any $v \in W_0^{1,\delta}(\Omega)$,

$$
\mathcal{J}(v) \geqslant -Q^2\int_\Omega a(x)\,\mathrm{d}x > -\infty.
$$

We next use Lemma 1.1 to prove that $\mathcal{J}$ is weakly lower semicontinuous. Define $F(x,u,z)\colon\ \Omega\times\mathbb{R}\times\mathbb{R}^N$ by

$$
F(x,u,z) = j(x,u,z) + a(x)|u|^2 - fu + Q^2a(x),
$$

then we need to show that

(i) $F(x,s,u)$ is a Carathéodory function;

(ii) $F(x,s,z)$ is convex in z for almost every $x\in\Omega$ and for every $u\in\mathbb{R}$;

(iii) $F \geqslant 0$.

The statements (i) and (ii) are easy due to the above assumptions. We now prove (iii):

$$
\begin{aligned}
F(x,u,z) &= j(x,u,z) + a(x)|u|^2 - fu + Q^2a(x) \\
&\geqslant a(x)|u|^2 - |f||u| + Q^2a(x) \\
&\geqslant a(x)|u|^2 - 2Qa(x)|u| + Q^2a(x) \\
&\geqslant a(x)|u|^2 - Q^2a(x) - a(x)|u|^2 + Q^2a(x) = 0,
\end{aligned}
$$

where we used again (1.5) and the fact that by (1.3), $j \geqslant 0$.

For every sequence $\{u_h\}$ weakly converging to u in $W_0^{1,\delta}(\Omega)$, let $z_h = \nabla u_h$, $z = \nabla u$. The Rellich theorem guarantees that u_h strongly converges to u in $L^1(\Omega)$, and it is obvious that z_h converges to z weakly in $L^1(\Omega)$. We use Lemma 1.1 and have

$$\int_\Omega F(x,u,z)\,\mathrm{d}x \leqslant \liminf_{h\to\infty} \int_\Omega F(x,u_h,z_h)\,\mathrm{d}x.$$

Deleting $Q^2 \int_\Omega a(x)\,\mathrm{d}x$ from both sides we derive

$$\mathcal{J}(u) \leqslant \liminf_{h\to\infty} \mathcal{J}(u_h),$$

which means that $\mathcal{J}$ is weakly lower semicontinuous. It completes the proof of Lemma 2.1. □

We next prove that the minimizer $u \in W_0^{1,\delta}(\Omega)$ of (1.1) belongs to $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

Lemma 2.2. *Under the same assumptions as in Theorem* 2.1, *any minimizer* u *of* $\mathcal{J}$ *on* $W_0^{1,\delta}(\Omega)$ *belongs to* $W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.

P r o o f. Let T_Q be the truncation operator at level $Q > 0$, i.e.,

$$T_Q(s) = \max\{-Q, \min\{s, Q\}\} = \begin{cases} s, & |s| \leqslant Q, \\ Q \cdot \operatorname{sign}(s), & |s| > Q. \end{cases} \tag{2.4}$$

Let u be a minimizer of $\mathcal{J}$ on $W_0^{1,\delta}(\Omega)$, then $\mathcal{J}(u) \leqslant \mathcal{J}(T_Q(u))$. Put $A_Q = \{x \in \Omega\colon |u| > Q\}$. Note that $T_Q(u) = u$ for all x such that $|u| \leqslant Q$ and $j(x,s,0) = 0$ by (1.3). We then get

$$\begin{aligned} \int_{A_Q} j(x,u,\nabla u)\,\mathrm{d}x + \int_{A_Q} a(x)|u|^2\,\mathrm{d}x - \int_{A_Q} fu\,\mathrm{d}x \\ \leqslant \int_{A_Q} a(x)|T_Q(u)|^2\,\mathrm{d}x - \int_{A_Q} fT_Q(u)\,\mathrm{d}x, \end{aligned}$$

thus by (1.3),

$$\begin{aligned} \int_{A_Q} \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_{A_Q} a(x)[|u|^2 - |T_Q(u)|^2]\,\mathrm{d}x \\ \leqslant \int_{A_Q} f(u - T_Q(u))\,\mathrm{d}x \leqslant \int_{A_Q} |f||u - T_Q(u)|\,\mathrm{d}x. \end{aligned}$$

Thanks to the interplay (1.5), we have

$$\int_{A_Q} \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_{A_Q} a(x)[|u|^2 - |T_Q(u)|^2]\,\mathrm{d}x \leqslant \int_{A_Q} 2Qa(x)|u - T_Q(u)|\,\mathrm{d}x,$$

that is,

$$\int_{A_Q} \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_{A_Q} a(x)[|u|^2 - |T_Q(u)|^2 - 2Q|u - T_Q(u)|]\,\mathrm{d}x \leqslant 0.$$

The first term above is nonnegative, dropping it one obtains

$$\int_{A_Q} a(x)[|u|^2 - |T_Q(u)|^2 - 2Q|u - T_Q(u)|]\,\mathrm{d}x \leqslant 0.$$

It is not difficult to prove that for $x \in A_Q$,

$$|u|^2 - |T_Q(u)|^2 - 2Q|u - T_Q(u)| = |u - T_Q(u)|^2,$$

thus,

$$\int_{A_Q} a(x)|u - T_Q(u)|^2\,\mathrm{d}x \leqslant 0.$$

The above inequality together with the fact $a(x) > 0$ implies $|u| \leqslant Q$ a.e. in Ω, that is, $u \in L^\infty(\Omega)$.

Next we prove $u \in W_0^{1,2}(\Omega)$. Let u be a minimizer of $\mathcal{J}$ on $W_0^{1,\delta}(\Omega)$, then $\mathcal{J}(u) \leqslant \mathcal{J}(0) = 0$, that is,

$$\int_\Omega j(x, u, \nabla u)\,\mathrm{d}x + \int_\Omega a(x)|u|^2\,\mathrm{d}x - \int_\Omega fu\,\mathrm{d}x \leqslant 0.$$

Thus by (1.3),

$$\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_\Omega a(x)|u|^2\,\mathrm{d}x - \int_\Omega |f||u|\,\mathrm{d}x \leqslant 0,$$

and thanks to the interplay (1.5), we have

$$\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_\Omega a(x)|u|^2\,\mathrm{d}x - \int_\Omega 2Qa(x)|u|\,\mathrm{d}x \leqslant 0,$$

therefore,

$$\int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_\Omega a(x)|u|(|u| - 2Q)\,\mathrm{d}x \leqslant 0. \tag{2.5}$$

We use the boundedness of u and (2.5) in order to get

$$\begin{aligned}\frac{1}{(1+Q)^\tau}\int_\Omega |\nabla u|^2\,\mathrm{d}x &\leqslant \int_\Omega \frac{|\nabla u|^2}{(1+|u|)^\tau}\,\mathrm{d}x + \int_{\{|u|\geqslant 2Q\}} a(x)|u|(|u| - 2Q)\,\mathrm{d}x \\ &\leqslant \int_{\{|u|<2Q\}} a(x)|u|(2Q - |u|)\,\mathrm{d}x \leqslant 2Q\int_{\{|u|<2Q\}} a(x)|u|\,\mathrm{d}x \\ &\leqslant 2Q^2\int_\Omega a(x)\,\mathrm{d}x < \infty,\end{aligned}$$

from which we derive $u \in W_0^{1,2}(\Omega)$ and

$$\|u\|_{W_0^{1,2}(\Omega)} \leqslant (2Q^2(1+Q)^\tau \|a(x)\|_{L^1(\Omega)})^{1/2}.$$

□

P r o o f of Theorem 2.1. Lemma 2.1 says that the integral functional (1.1) has a minimizer u on the space $W_0^{1,\delta}(\Omega)$. Lemma 2.2 tells us that this minimizer belongs to $W_0^{1,2}(\Omega)$, thus u is in fact a minimizer for (1.1) on the space $W_0^{1,2}(\Omega)$ since

$$\mathcal{J}(u) = \inf\{\mathcal{J}(v), v \in W_0^{1,\delta}(\Omega)\} \leqslant \inf\{\mathcal{J}(v), v \in W_0^{1,2}(\Omega)\}.$$

□

3. An alternative proof

This section gives an alternative proof of Theorem 2.1 by the truncation argument, which has been used by Boccardo and his coauthors for many times. Consider, for v in $W_0^{1,2}(\Omega)$ and $n \in \mathbb{N}$, the sequence of functionals

$$\mathcal{J}_n(v) = \int_\Omega j(x, T_n(v), \nabla v)\,\mathrm{d}x + \int_\Omega a(x)|v|^2\,\mathrm{d}x - \int_\Omega fv\,\mathrm{d}x, \tag{3.1}$$

where $T_n(s)$ is the truncation of s at level n, see (2.4).

We show that, every integral functional (3.1) is coercive, bounded from below, and weakly lower semicontinuous on $W_0^{1,2}(\Omega)$. In fact, by (1.3),

$$j(x, T_n(v), \nabla v) \geqslant \frac{|\nabla v|^2}{(1 + |T_n(v)|)^\tau} \geqslant \frac{|\nabla v|^2}{(1+n)^\tau}. \tag{3.2}$$

Substituting (3.2) and (2.2) into (3.1), one gets

$$\mathcal{J}_n(v) \geqslant \frac{1}{(1+n)^\tau}\int_\Omega |\nabla v|^2\,\mathrm{d}x - Q^2\int_\Omega a(x)\,\mathrm{d}x.$$

Then $\mathcal{J}_n$ turns out to be coercive and bounded from below. As in Section 2, one can show that $\mathcal{J}_n$ is weakly lower semicontinuous on $W_0^{1,2}(\Omega)$.

The Weierstrass theorem guarantees the existence of a minimizer $u_n \in W_0^{1,2}(\Omega)$:

$$\mathcal{J}_n(u_n) = \inf\{\mathcal{J}_n(v)\colon\ v \in W_0^{1,2}(\Omega)\}. \tag{3.3}$$

We next prove that the sequence $\{u_n\}$ obtained above is uniformly bounded in $L^\infty(\Omega)$. Choosing $v = T_Q(u_n)$ in (3.3), we get

$$\begin{aligned}
\int_{A_{Q,n}} j(x, T_n(u_n), \nabla T_n(u_n))\,\mathrm{d}x + \int_{A_{Q,n}} a(x)|u_n|^2\,\mathrm{d}x - \int_{A_{Q,n}} fu_n\,\mathrm{d}x \\
&\leqslant \int_{A_{Q,n}} j(x, T_n(T_Q(u_n)), \nabla T_Q(u_n))\,\mathrm{d}x \\
&\quad + \int_{A_{Q,n}} a(x)|T_Q(u_n)|^2\,\mathrm{d}x - \int_{A_{Q,n}} fT_Q(u_n)\,\mathrm{d}x \\
&\leqslant \int_{A_{Q,n}} Q^2 a(x)\,\mathrm{d}x - \int_{A_{Q,n}} fT_Q(u_n)\,\mathrm{d}x,
\end{aligned}$$

where $A_{Q,n} = \{x \in \Omega : |u_n(x)| > Q\}$. The above inequality implies, by dropping a nonnegative term, that

$$\int_{A_{Q,n}} a(x)(|u_n|^2 - Q^2)\,\mathrm{d}x \leqslant \int_{A_{Q,n}} fG_Q(u_n)\,\mathrm{d}x \leqslant \int_{A_{Q,n}} |f||G_Q(u_n)|\,\mathrm{d}x \leqslant \int_{A_{Q,n}} 2Qa(x)|G_Q(u_n)|\,\mathrm{d}x, \tag{3.4}$$

where $G_Q(u_n) = u_n - T_Q(u_n)$ and we used (1.5) again. The left hand side of (3.4) can be estimated as

$$\int_{A_{Q,n}} a(x)(|u_n|^2 - Q^2)\,\mathrm{d}x = \int_{A_{Q,n}} a(x)(|u_n| + Q)(|u_n| - Q)\,\mathrm{d}x \geqslant \int_{A_{Q,n}} a(x)(|G_Q(u_n)| + 2Q)|G_Q(u_n)|\,\mathrm{d}x. \tag{3.5}$$

Then (3.4) and (3.5) merge by deleting the common term

$$\int_{A_{Q,n}} a(x)|G_Q(u_n)|^2\,\mathrm{d}x \leqslant 0,$$

which yields $|u_n| \leqslant Q$ a.e. Ω.

The existence of a minimizer in $W_0^{1,2}(\Omega)$ is now easy. Let us choose $n \geqslant Q$, then the facts that $u_n \in W_0^{1,2}(\Omega)$ is a minimizer of $\mathcal{J}_n$ and $\mathcal{J}_n(u_n) = \mathcal{J}(u_n)$ for $n \geqslant Q$ imply that u_n is a minimizer of $\mathcal{J}$, as desired.

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