Czechoslovak Mathematical Journal

Zhengtian Qiu; Guiyun Chen; Jianjun Liu

A note on the Π-property of some subgroups of finite groups

Czechoslovak Mathematical Journal, Vol. 74 (2024), No. 3, 943–953

Persistent URL: http://dml.cz/dmlcz/152591

Terms of use:

© Institute of Mathematics AS CR, 2024

Institute of Mathematics of the Czech Academy of Sciences provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.



This document has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://dml.cz

Czechoslovak Mathematical Journal, 74 (149) (2024), 943–953

A NOTE ON THE Π-PROPERTY OF SOME SUBGROUPS OF FINITE GROUPS

Zhengtian Qiu, Guiyun Chen, Jianjun Liu, Chongqing

Received June 7, 2024. Published online September 16, 2024.

Abstract. Let H be a subgroup of a finite group G. We say that H satisfies the Π-property in G if for any chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. We obtain some criteria for the p-supersolubility or p-nilpotency of a finite group and extend some known results by concerning some subgroups that satisfy the Π-property.

Keywords: finite group; p-supersoluble group; p-nilpotent group; the Π-property

MSC 2020: 20D10, 20D20

1. Introduction

All groups considered in this paper are finite. We use conventional notions as in [9]. Throughout the paper, G always denotes a finite group, p denotes a fixed prime, π denotes a set of primes and $\pi(G)$ denotes the set of all primes dividing $|G|$. An integer n is called a π-number if all prime divisors of n belong to π.

Suppose that P is a p-group for a prime p. Let $\mathcal{M}(P)$ be the set of all maximal subgroups of P. Let d be the smallest generator number of P, i.e., $p^d = |P/\Phi(P)|$, where $\Phi(P)$ is the Frattini subgroup of P. According to [12], $\mathcal{M}_d(P) = \{P_1, \dots, P_d\}$ is a subset of $\mathcal{M}(P)$ such that $\bigcap_{i=1}^{d} P_i = \Phi(P)$. Notice that the subset $\mathcal{M}_d(P)$ is not unique for a fixed p-group P in general. Li and He in [12] showed that $|\mathcal{M}(P)| \gg |\mathcal{M}_d(P)|$ because

$$\lim_{d\to\infty} \frac{|\mathcal{M}(P)|}{|\mathcal{M}_d(P)|} = \lim_{d\to\infty} \frac{(p^d-1)/(p-1)}{d} = \infty.$$

This work is supported by the National Natural Science Foundation of China (Grant Nos. 12071376 and 12471019).

DOI: 10.21136/CMJ.2024.0226-24

© Institute of Mathematics, Czech Academy of Sciences 2024.

In [11], Li introduced the concept of the Π-property of subgroups of finite groups. Let H be a subgroup of a group G. We say that H satisfies the Π-property in G if for any chief factor L/K of G, $|G/K : N_{G/K}(HK/K \cap L/K)|$ is a $\pi(HK/K \cap L/K)$-number. There are many examples of embedding properties of subgroups implying the possession of the Π-property, see Section 4.

In this note, we obtain some criteria for the p-supersolubility or p-nilpotency of a group G by assuming that some p-subgroups of N satisfy the Π-property in G, where N is a normal subgroup of G such that G/N is p-supersoluble. More precisely, we prove the following results:

Theorem A. *Let G be a p-soluble group, and let P be a Sylow p-subgroup of G, where p is a prime dividing the order of G. Assume that N is a normal subgroup of G such that G/N is p-supersoluble. Then G is p-supersoluble if and only if $P_i \cap N$ satisfies the* Π*-property in G for every member P_i of a fixed $\mathcal{M}_d(P)$.*

Theorem B. *Let P be a Sylow p-subgroup of a group G for a prime $p \in \pi(G)$. Assume that N is a normal subgroup of G such that G/N is p-supersoluble. Then G is p-nilpotent if and only if $N_G(P)$ is p-nilpotent and $P_i \cap N$ satisfies the* Π*-property in G for every member P_i of a fixed $\mathcal{M}_d(P)$.*

Theorem C. *Let p be a prime dividing the order of a group G with $(|G|, p-1) = 1$ and $P \in \mathrm{Syl}_p(G)$. Assume that N is a normal subgroup of G such that G/N is p-nilpotent. Then G is p-nilpotent if and only if $P_i \cap N$ satisfies the* Π*-property in G for every member P_i of a fixed $\mathcal{M}_d(P)$.*

The conditions that G is p-soluble in Theorem A, $N_G(P)$ is p-nilpotent in Theorem B, and $(|G|, p-1) = 1$ in Theorem C, cannot be removed. Let us see the following example.

Example 1.1. Let $A = \langle a \colon a^p = 1 \rangle$ be a cyclic group of order p, where p is a prime. Let B be a non-abelian simple group such that the Sylow p-subgroups of B have order p. Let b be an element of B of order p. Set $G = A \times B$. Let P be a Sylow p-subgroup of G such that $a, b \in P$. Write $P_1 = A$ and $P_2 = \langle ab \rangle$. It is easy to see that $P_i \cap B$ satisfies the Π-property in G for any $i = 1, 2$. However, G is not p-supersoluble or p-nilpotent.

2. Preliminaries

In this section, we collect some results which are needed in the proofs.

Lemma 2.1 ([11], Proposition 2.1 (1)). *Let H be a subgroup of a group G and N a normal subgroup of a group G. If H satisfies the* Π*-property in G, then HN/N satisfies the* Π*-property in G/N.*

Lemma 2.2. *Let p be a prime dividing the order of a group G and P_1 be a p-subgroup of G. Let $L \trianglelefteq G$ and N be a normal p'-subgroup of G. Then $P_1N/N \cap LN/N = (P_1 \cap L)N/N$.*

Lemma 2.3 ([9], Kapitel I, Hauptsatz 17.4). *Suppose that N is an abelian normal subgroup of a group G and $N \leqslant M \leqslant G$ such that $(|N|, |G : M|) = 1$. If N is complemented in M, then N is complemented in G.*

Lemma 2.4. *Let H be a p-subgroup of a group G for a prime $p \in \pi(G)$. If G is p-supersoluble, then H satisfies the* Π*-property in G.*

P r o o f. By [20], Lemma 2.3, the conclusion holds. □

Lemma 2.5 ([15], Lemma 2.1 (6)). *Let p be a prime dividing the order of a group G, H be a p-subgroup of G and $N \trianglelefteq G$. If H satisfies the* Π*-property in G, then $H \cap N$ satisfies the* Π*-property in G.*

3. Proofs

P r o o f of Theorem A. By Lemma 2.4, we only need to prove the sufficiency. Suppose that G is not p-supersoluble and let G be a counterexample of minimal order. We divide the proof into the following steps.

Step 1. $O_{p'}(G) = 1$. Set $\overline{G} = G/O_{p'}(G)$. Clearly, $\overline{P}$ is a Sylow p-subgroup of $\overline{G}$, which is isomorphic to P. Thus, $\mathcal{M}_d(\overline{P}) = \{\overline{P_1}, \overline{P_2}, \ldots, \overline{P_d}\}$. By hypothesis, $P_i \cap N$ satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$. By Lemmas 2.1 and 2.2, $\overline{P_i \cap N} = \overline{P_i} \cap \overline{N}$ satisfies the Π-property in $\overline{G}$ for any $\overline{P_i} \in \mathcal{M}_d(\overline{P})$. Clearly, $\overline{G}/\overline{N} \cong G/NO_{p'}(G)$ is p-supersoluble and $\overline{G}$ is p-soluble. Thus, $\overline{G}$ satisfies the hypotheses of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-supersoluble by the minimal choice of G. Hence, G is p-supersoluble, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. In particular, $\Phi(O_p(G)) = 1$. Assume that $\Phi(P)_G > 1$, and we work to obtain a contradiction. Let V be a minimal normal subgroup of G contained

in $\Phi(P)_G$. Since every maximal subgroup of P contains $\Phi(P)$ and P/V has the same smallest generator number as P, we have $\mathcal{M}_d(P/V) = \{P_1/V, P_2/V, \ldots, P_d/V\}$. By Lemma 2.1 and Dedekind's lemma, $P_i/V \cap NV/V$ satisfies the Π-property in G/V for any $P_i \in \mathcal{M}_d(P)$. It is not difficult to see that G/V satisfies the hypotheses of the theorem. The minimal choice of G yields that G/V is p-supersoluble. Since $V \leqslant \Phi(P)$, it follows from [4], Chapter A, Theorem 9.2 (d) that $V \leqslant \Phi(G)$. Note that the class of all p-supersoluble groups is a saturated formation, and so G is p-supersoluble, a contradiction. Thus, $\Phi(P)_G = 1$. By [4], Chapter A, Theorem 9.2 (e), we have $\Phi(O_p(G)) = 1$.

Step 3. Every minimal normal subgroup of G contained in $O_p(G)$ is of order p. Since G is p-soluble and $O_{p'}(G) = 1$, we have $O_p(G) > 1$. Let K be a minimal normal subgroup of G contained in $O_p(G)$. By Step 2, there exists a member $P_j \in \mathcal{M}_d(P)$ such that $K \nleq P_j$. By hypothesis, $P_j \cap N$ satisfies the Π-property in G. For the G-chief factor $K/1$, we see that $|G : N_G(P_j \cap N \cap K)|$ is a p-number, and so $P_j \cap N \cap K \trianglelefteq G$. If $K \leqslant N$, then $P_j \cap K = 1$ by the minimality of K. Therefore, K has order p, as desired. Assume that $K \nleq N$. Since $O_{p'}(G) = 1$ and G/N is p-supersoluble, we deduce that $|K| = p$, as desired.

Step 4. The final contradiction. Since G is p-soluble, by Step 1, Step 2 and [5], Chapter 6, Theorem 3.2, we have $C_G(O_p(G)) = O_p(G)$. Assume that $O_p(G) \cap \Phi(G) > 1$. Let U be a minimal normal subgroup of G contained in $O_p(G) \cap \Phi(G)$. Then U has order p by Step 3. In view of Step 2, U is complemented in P. By Lemma 2.3, U is complemented in G, which is contrary to $U \leqslant \Phi(G)$. Therefore, $O_p(G) \cap \Phi(G) = 1$. By Step 3 and [19], Lemma 2.13, we have that $O_p(G) = U_1 \times U_2 \times \ldots \times U_s$, where U_i is a minimal normal subgroup of G of order p for any $i = 1, \ldots, s$. Note that $G/C_G(U_i)$ is cyclic of order dividing $p-1$ for any $i = 1, \ldots, s$. Hence, $G/\bigcap_{i=1}^{s} C_G(U_i) = G/C_G(O_p(G)) = G/O_p(G)$ is abelian. Since every chief factor of G below $O_p(G)$ is of order p, we see that G is p-supersoluble. This final contradiction completes the proof. □

P r o o f of Theorem B. By Lemma 2.4, we only need to prove the sufficiency. Suppose that G is not p-nilpotent and let G be a counterexample of minimal order.

Step 1. $O_{p'}(G) = 1$. Set $\overline{G} = G/O_{p'}(G)$. Arguing as in Step 1 of the proof of Theorem A, we know that $\overline{G}$ satisfies the hypotheses of the theorem. If $O_{p'}(G) > 1$, then $G/O_{p'}(G)$ is p-nilpotent by the minimal choice of G. Hence, G is p-nilpotent, a contradiction. Therefore, $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. In particular, $\Phi(O_p(G)) = 1$. Assume that $\Phi(P)_G > 1$. Let V be a minimal normal subgroup of G contained in $\Phi(P)_G$. With a similar argument as in Step 2 of the proof of Theorem A, we can obtain that G/V is p-nilpotent. Since

$V \leqslant \Phi(P)$, we have $V \leqslant \Phi(G)$. Note that the class of all p-nilpotent groups is a saturated formation, and hence, G is p-nilpotent, a contradiction. Thus, $\Phi(P)_G = 1$, and so $\Phi(O_p(G)) = 1$.

Step 3. Let K be a minimal normal subgroup of G such that $K \not\leqslant N$. Then $|K| = p$. Note that KN/N is a minimal normal subgroup of G/N. Since $O_{p'}(G) = 1$ and G/N is p-supersoluble, we deduce that $|K| = p$, as wanted.

Step 4. Every minimal normal subgroup of G is contained in $O_p(G)$. Assume that K is a minimal normal subgroup of G, which is not a p-group. We work to obtain a contradiction. Since $O_{p'}(G) = 1$, we have $p \mid |K|$. If $K \not\leqslant N$, then $|K| = p$ by Step 3, a contradiction. Therefore, $K \leqslant N$. By hypothesis, P_i satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$. Considering the G-chief factor $K/1$, we have that $|G : N_G(P_i \cap K)|$ is a p-number, and thus $P_i \cap K \trianglelefteq G$. Then either $P_i \cap K = 1$ or $P_i \cap K = K$. If $P_i \cap K = K$, then K is a p-group, a contradiction. Therefore, $P_i \cap K = 1$. Since $O_{p'}(G) = 1$, it follows that $|P \cap K| = p$. If $K = G$, then P is cyclic of order p. Since $N_G(P)$ is p-nilpotent, we deduce that $P \leqslant Z(N_G(P))$. By Burnside's theorem (see [5], Chapter 7, Theorem 4.3), it follows that G is p-nilpotent, a contradiction. Therefore, $K < G$. Set $L = PK$. Note that $|PK| = |P||K|/|P \cap K| = |P_i||K|$ and $P_i \cap K = 1$. Thus, $L = P_iK$. Observe that L/K is a p-group (also a p-supersoluble group), $P_i \cap K = 1$ satisfies the Π-property in G and $N_L(P)$ is p-nilpotent. Hence, L inherits the hypotheses of the theorem. If $L < G$, then L is p-nilpotent by the minimal choice of G. Thus, K is p-nilpotent. Since $O_{p'}(G) = 1$, we have $K \leqslant O_p(G)$, a contradiction. Hence, $P_iK = L = G$. Write $W = N_G(P \cap K)$. Clearly, $P \leqslant W$, $W = P_iK \cap W = P_i(K \cap W)$. Note that $W/(K \cap W)$ is a p-group, $P_i \cap (K \cap W) = P_i \cap K = 1$ satisfies the Π-property in G and $N_W(P)$ is p-nilpotent. Therefore, W satisfies the hypotheses of the theorem. If $W < G$, then $W = N_G(P \cap K)$ is p-nilpotent by the minimal choice of G, and so $N_K(P \cap K)$ is p-nilpotent. Since $|P \cap K| = p$, we have $P \cap K \leqslant Z(N_K(P \cap K))$. It follows that K is p-nilpotent. By Step 1, we know $K \leqslant O_p(G)$, a contradiction. If $W = G$, then $K = P \cap K$ by Step 1, a contradiction. Hence, Step 4 holds.

Step 5. Every minimal normal subgroup of G is of order p. Let K be a minimal normal subgroup of G. By Step 4, we have $K \leqslant O_p(G)$. If $K \not\leqslant N$, then $|K| = p$ by Step 3, as desired. Hence, we may assume that $K \leqslant N$. By hypothesis, P_i satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$. For the G-chief factor $K/1$, we know that $|G : N_G(P_i \cap K)|$ is a p-number, and so $P_i \cap K \trianglelefteq G$. The minimality of K implies that either $P_i \cap K = 1$ or $P_i \cap K = K$, where $i = 1, \ldots, d$. If $P_i \cap K = K$ for every $P_i \in \mathcal{M}_d(P)$, then $K \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P)$, which contradicts Step 2. So there exists a member $P_j \in \mathcal{M}_d(P)$ such that $P_j \cap K = 1$. Since $O_{p'}(G) = 1$, we have $|P \cap K| = p$, and thus $|K| = p$ by Step 4, as wanted.

Step 6. The final contradiction. Let R be a subgroup of G of minimal order such that $O_p(G)R = G$. Assume that $O_p(G) \cap R > 1$. By Step 2, we see that $O_p(G) \cap R \trianglelefteq G$. Let T be a minimal normal subgroup of G contained in $O_p(G) \cap R$. Then T has order p by Step 5. In view of Step 2, we see that T is complemented in P. By Lemma 2.3, T has a complement A in G. Hence, $R = R \cap TA = T(R \cap A)$, and so $G = O_p(G)R = O_p(G)(R \cap A)$. The minimal choice of R yields that $R \cap A = R$, i.e., $R \leqslant A$. As a consequence, $T \leqslant O_p(G) \cap R \leqslant A$. This is a contradiction. Therefore, $O_p(G) \cap R = 1$.

Assume that $O_p(G) \cap \Phi(G) > 1$. Let U be a minimal normal subgroup of G contained in $O_p(G) \cap \Phi(G)$. Then U has order p by Step 5. In view of Step 2, U is complemented in P. By Lemma 2.3, U is complemented in G, which contradicts the fact that $U \leqslant \Phi(G)$. Therefore, $O_p(G) \cap \Phi(G) = 1$. By Step 5 and [19], Lemma 2.13, $O_p(G) = U_1 \times U_2 \times \ldots \times U_s$, where $U_i \trianglelefteq G$ and $|U_i| = p$ for any $i = 1, \ldots, s$. Hence, $O_p(G) \leqslant Z(P)$. If $P \cap R = 1$, then $P = O_p(G)(P \cap R) = O_p(G)$, and thus, $G = N_G(P)$ is p-nilpotent, a contradiction. Therefore, $P \cap R > 1$. Let B be a minimal normal subgroup of G contained in $(P \cap R)^G$. By Step 4, we have $B \leqslant O_p(G)$. Since $O_p(G) \leqslant Z(P)$, we have $(P \cap R)^G = (P \cap R)^{O_p(G)R} = (P \cap R)^R \leqslant R$. Thus, $B \leqslant R$, which contradicts the fact that $O_p(G) \cap R = 1$. This final contradiction completes the proof. □

P r o o f of Theorem C. By Lemma 2.4, we only need to prove the sufficiency. Suppose that G is not p-nilpotent and let G be a counterexample of minimal order.

Step 1. $O_{p'}(G) = 1$. With a similar argument as in Step 1 of the proof of Theorem B, we can get that $O_{p'}(G) = 1$.

Step 2. $\Phi(P)_G = 1$. In particular, $\Phi(O_p(G)) = 1$. Arguing as in Step 2 of the proof of Theorem B, we see that $\Phi(P)_G = 1$ and $\Phi(O_p(G)) = 1$.

Step 3. Every minimal normal subgroup of G has order p. Let K be a minimal normal subgroup of G. Since $O_{p'}(G) = 1$, we have $p \mid |K|$. If $K \not\leqslant N$, then KN/N is a minimal normal subgroup of G/N. Since $O_{p'}(G) = 1$ and G/N is p-supersoluble, we deduce that $|K| = p$, as desired. Hence, we may assume that $K \leqslant N$. By hypothesis, $P_i \cap N$ satisfies the Π-property in G for any $P_i \in \mathcal{M}_d(P)$. Considering the G-chief factor $K/1$, we see that $|G : N_G(P_i \cap K)|$ is a p-number, and thus $P_i \cap K \trianglelefteq G$. The minimality of K implies that either $P_i \cap K = 1$ or $P_i \cap K = K$, where $i = 1, \ldots, d$. If $P_i \cap K = K$ for any $P_i \in \mathcal{M}_d(P)$, then $K \leqslant \bigcap_{i=1}^{d} P_i = \Phi(P)$, which contradicts Step 2. So there exists a member $P_j \in \mathcal{M}_d(P)$ such that $P_j \cap K = 1$. In view of $O_{p'}(G) = 1$, we have $|P \cap K| = p$. Since $(|G|, p-1) = 1$, it follows from [6], Chapter 1, Lemma 3.39 that K is p-nilpotent. Again, $O_{p'}(G) = 1$, we conclude that $|K| = p$, as desired.

Step 4. The final contradiction. Let $T_1, T_2, \ldots, T_s$ be all minimal normal subgroups of G. By Step 3, T_i is of order p for any $i = 1, \ldots, s$. By Step 2, T_i is complemented in P, and so T_i has a complement X_i in G for any $i = 1, \ldots, s$ by

Lemma 2.3. Note that $G/C_G(T_i) \lesssim \mathrm{Aut}(T_i)$. Since $(|G|, p-1) = 1$, it follows that $G = C_G(T_i)$, i.e., $T_i \leqslant Z(G)$ for any $i = 1, \ldots, s$. Thus, $X_i \trianglelefteq G$ for any $i = 1, 2, \ldots, s$. Let Y be a subgroup of G of minimal order such that $(T_1T_2 \ldots T_s)Y = G$. Assume that $O_p(G) \cap Y > 1$. By Step 2, we see that $O_p(G) \cap Y \trianglelefteq G$. Let U be a minimal normal subgroup of G contained in $O_p(G) \cap Y$. Then $U = T_j$ for some $j \in \{1, 2, \ldots, s\}$. Hence, $G = T_jX_j = UX_j$ and $Y = U(Y \cap X_j)$. As a consequence, $G = (T_1T_2 \ldots T_s)Y = (T_1T_2 \ldots T_s)(Y \cap X_j)$. The minimal choice of Y implies that $Y \cap X_j = Y$, and thus $T_j = U \leqslant Y \leqslant X_j$, a contradiction. Therefore, $O_p(G) \cap Y = 1$. Thus, $O_p(G) = T_1T_2 \ldots T_s \leqslant Z(G)$. It follows that $1 < Y \trianglelefteq G$. By Step 3, we have $O_p(G) \cap Y > 1$, a contradiction. The proof is now complete. □

4. Final remarks and applications

In this section, we will show that the concept of the Π-property can be viewed as a generalization of many known embedding properties.

Recall that two subgroups H and K of a group G are said to be *permutable* if $HK = KH$. From [10], a subgroup H of a group G is said to be *S-permutable* (or *π-quasinormal*, *S-quasinormal*) in G if H permutes with all Sylow subgroups of G. According to [7], a subgroup H of a group G is said to be *X-permutable* with a subgroup T of G if there exists an element $x \in X$ such that $HT^x = T^xH$, where X is a nonempty subset of G. Following [1], the $\mathcal{U}$-hypercenter $Z_{\mathcal{U}}(G)$ of a group G is the product of all normal subgroups H of G such that all G-chief factors below H have prime order, where $\mathcal{U}$ denotes the class of all supersoluble groups. A subgroup H of a group G is called a *CAP-subgroup* of G if H either covers or avoids every chief factor L/K of G, that is, $HL = HK$ or $H \cap L = H \cap K$, see [4], Chapter A, Definition 10.8.

Proposition 4.1. *Let H be a subgroup of a group G. Then H satisfies* Π*-property in G if one of the following holds:*

(1) *H is normal in G;*
(2) *H is permutable in G;*
(3) *H is S-permutable in G;*
(4) *H is X-permutable with all Sylow subgroups of G, where X is a soluble normal subgroup of G;*
(5) *H is a CAP-subgroup of G;*
(6) *$H/H_G \leqslant Z_{\mathcal{U}}(G/H_G)$.*

Proof. Statements (1)–(6) were proved in [11], Propositions 2.2–2.3. □

A subgroup H of a group G is said to be *S-semipermutable* (see [3]) in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$. A subgroup H of a group G is said to be *SS-quasinormal* (see [13]) in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B.

Proposition 4.2. *Let H be a p-subgroup of a group G for a prime $p \in \pi(G)$. Then H satisfies the* Π*-property in G if one of the following holds:*
(1) *H is S-semipermutable in G;*
(2) *H is SS-quasinormal in G.*

P r o o f. (1) By [11], Proposition 2.4, the conclusion follows.

(2) Applying [13], Lemma 2.5, we see that H is S-semipermutable in G. Therefore, H satisfies the Π-property in G by statement (1). □

A subgroup H of a group G is said to be Π-*normal* (see [11]) in G if G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leqslant I \leqslant H$, where I satisfies the Π-property in G. A subgroup H of a group G is called *c-normal* (see [21]) in G if G has a normal subgroup T such that $HT = G$ and $T \cap H \leqslant H_G$, where H_G is the core of H in G. A subgroup H of a group G is called $\mathcal{U}_c$*-normal* [1] in G if there is a subnormal subgroup T of G such that $G = HT$ and $(H \cap T)H_G/H_G \leqslant Z_{\mathcal{U}}(G/H_G)$.

A subgroup H of a group G is said to be *weakly S-permutable* (see[19]) in G if there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leqslant H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are S-permutable in G. A subgroup H of a group G is said to be *weakly SS-permutable* (see [8]) in G if there exist a subnormal subgroup T of G and an SS-quasinormal subgroup H_{ss} of G contained in H such that $G = HT$ and $H \cap T \leqslant H_{ss}$. A subgroup H of a group G is called a $c^\sharp$*-normal* subgroup (see [22]) of G if G has a normal subgroup T such that $G = HT$ and $H \cap T$ is a CAP-subgroup of G. A subgroup H of a group G is *weakly S-semipermutable* [16] in G if there are a subnormal subgroup T of G and an S-semipermutable subgroup H_{ssG} of G contained in H such that $G = HT$ and $H \cap T \leqslant H_{ssG}$.

Proposition 4.3. *Let H be a p-subgroup of a group G for a prime $p \in \pi(G)$. Then $H \cap O^p(G)$ satisfies the* Π*-property in G if one of the following holds:*
(1) *H is* Π*-normal in G;*
(2) *H is c-normal in G;*
(3) *H is $\mathcal{U}_c$-normal in G;*
(4) *H is weakly S-permutable in G;*
(5) *H is a $c^\sharp$-normal subgroup of G;*
(6) *H is weakly SS-permutable in G;*
(7) *H is weakly S-semipermutable in G.*

P r o o f. (1) If H is Π-normal in G, then G has a subnormal subgroup T such that $G = HT$ and $H \cap T \leqslant I \leqslant H$, where I satisfies the Π-property in G. Since H is a p-subgroup of G, it follows from [2], Lemma 1.1.11 that $O^p(T) = O^p(G)$. Then $H \cap O^p(G) = I \cap O^p(G)$ satisfies the Π-property in G by Lemma 2.5.

Statements (2)–(7) directly follow from statement (1). By Proposition 4.1, a c-normal (or $\mathcal{U}_c$-normal, weakly S-permutable, $c^\sharp$-normal) subgroup of G is a Π-normal subgroup of G. If H is a weakly SS-permutable (or weakly S-semipermutable) p-subgroup of G, then H is Π-normal in G by Proposition 4.2. □

Applying Propositions 4.1, 4.2 and 4.3, we can obtain the following corollaries.

Corollary 4.4 ([13], Theorem 1.1)**.** *Let p be the smallest prime dividing the order of a group G and P a Sylow p-subgroup of G. If every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.5 ([13], Theorem 1.2)**.** *Let p be a prime dividing the order of a group G and P a Sylow p-subgroup of G. If $N_G(P)$ is p-nilpotent and every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G, then G is p-nilpotent.*

Corollary 4.6 ([13], Theorem 1.3)**.** *Let G be a p-solvable group for a prime p and P a Sylow p-subgroup of G. Suppose that every member of a fixed $\mathcal{M}_d(P)$ is SS-quasinormal in G. Then G is p-supersoluble.*

Corollary 4.7 ([14], Theorem 3.1)**.** *Let G be a group and P be a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If all maximal subgroups of P are S-semipermutable in G, then G is p-nilpotent.*

Corollary 4.8 ([17], Theorem 3.1)**.** *Let G be a p-soluble group and let P be a Sylow p-subgroup of G, where p is a fixed prime. Then G is p-supersoluble if and only if every member of a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.*

Corollary 4.9 ([17], Theorem 3.3)**.** *Let p be the smallest prime dividing the order of a group G and let P be a Sylow p-subgroup of G. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.*

Corollary 4.10 ([17], Theorem 3.4)**.** *Suppose that P is a Sylow p-subgroup of a group G and $N_G(P)$ is p-nilpotent for a prime $p \in \pi(G)$. Then G is p-nilpotent if and only if every member of a fixed $\mathcal{M}_d(P)$ is a CAP-subgroup of G.*

Corollary 4.11 ([18], Theorem 3.8)**.** *Let p be a prime dividing the order of a p-soluble group G and let P be a Sylow p-subgroup of G. If every member of a fixed $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-supersoluble.*

Corollary 4.12 ([18], Theorem 3.9)**.** *Let p be an odd prime dividing the order of G and let P be a Sylow p-subgroup of a group G. If $N_G(P)$ is p-nilpotent and every member of a fixed $\mathcal{M}_d(P)$ is S-semipermutable in G, then G is p-nilpotent.*

Corollary 4.13 ([23], Theorem 3.1)**.** *Let G be a group, and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Then G is p-nilpotent if and only if every member in a fixed $\mathcal{M}_d(P)$ is $c^\sharp$-normal in G.*

Corollary 4.14 ([23], Theorem 3.2)**.** *Let G be a p-soluble group, and let P be a Sylow p-subgroup of G, where p is a prime divisor of $|G|$. Then G is p-supersolvable if and only if every member in a fixed $\mathcal{M}_d(P)$ is $c^\sharp$-normal in G.*

Acknowledgments. The authors would like to thank the referee for his/her valuable suggestions and useful comments contributed to the final version of this paper.

References

[1] *A. Y. Alsheik Ahmad, J. J. Jaraden, A. N. Skiba*: On $\mathcal{U}_c$-normal subgroups of finite groups. Algebra Colloq. *14* (2007), 25–36. zbl MR doi
[2] *A. Ballester-Bolinches, R. Esteban-Romero, M. Asaad*: Products of Finite Groups. de Gruyter Expositions in Mathematics 53. Walter de Gruyter, Berlin, 2010. zbl MR doi
[3] *Z. Chen*: On a theorem of Srinivasan. J. Southwest Normal Univ., Ser. B *12* (1987), 1–4. (In Chinese.) zbl
[4] *K. Doerk, T. Hawkes*: Finite Soluble Groups. de Gruyter Expositions in Mathematics 4. Walter de Gruyter, Berlin, 1992. zbl MR doi
[5] *D. Gorenstein*: Finite Groups. Chelsea Publishing, New York, 1980. zbl MR
[6] *W. Guo*: Structure Theory for Canonical Classes of Finite Groups. Springer, Berlin, 2015. zbl MR doi
[7] *W. Guo, K.-P. Shum, A. N. Skiba*: X-quasinormal subgroups. Sib. Math. J. *48* (2007), 593–605. zbl MR doi
[8] *X. He, Y. Li, Y. Wang*: On weakly SS-permutable subgroups of a finite group. Publ. Math. Debr. *77* (2010), 65–77. zbl MR doi
[9] *B. Huppert*: Endliche Gruppen. I. Die Grundlehren der mathematischen Wissenschaften 134. Springer, Berlin, 1967. (In German.) zbl MR doi
[10] *O. H. Kegel*: Sylow-Gruppen und Subnormalteiler endlicher Gruppen. Math. Z. *78* (1962), 205–221. (In German.) zbl MR doi
[11] *B. Li*: On Π-property and Π-normality of subgroups of finite groups. J. Algebra *334* (2011), 321–337. zbl MR doi
[12] *S. Li, X. He*: On normally embedded subgroups of prime power order in finite groups. Commun. Algebra *36* (2008), 2333–2340. zbl MR doi
[13] *S. Li, Z. Shen, J. Liu, X. Liu*: The influence of SS-quasinormality of some subgroups on the structure of finite groups. J. Algebra *319* (2008), 4275–4287. zbl MR doi
[14] *Y. M. Li, X. L. He, Y. M. Wang*: On s-semipermutable subgroups of finite groups. Acta Math. Sin., Engl. Ser. *26* (2010), 2215–2222. zbl MR doi
[15] *Y. Li, L. Miao*: p-hypercyclically embedding and Π-property of subgroups of finite groups. Commun. Algebra *45* (2017), 3468–3474. zbl MR doi

[16] *Y. Li, S. Qiao, N. Su, Y. Wang*: On weakly s-semipermutable subgroups of finite groups. J. Algebra *371* (2012), 250–261. zbl MR doi
[17] *J. Liu, S. Li, Z. Shen, X. Liu*: Finite groups with some CAP-subgroups. Indian J. Pure Appl. Math. *42* (2011), 145–156. zbl MR doi
[18] *J. Lu, S. Li*: On S-semipermutable subgroups of finite groups. J. Math. Res. Expo. *29* (2009), 985–991. zbl MR doi
[19] *A. N. Skiba*: On weakly s-permutable subgroups of finite groups. J. Algebra *315* (2007), 192–209. zbl MR doi
[20] *N. Su, Y. Li, Y. Wang*: A criterion of p-hypercyclically embedded subgroups of finite groups. J. Algebra *400* (2014), 82–93. zbl MR doi
[21] *Y. Wang*: C-normality of groups and its properties. J. Algebra *180* (1996), 954–965. zbl MR doi
[22] *Y. Wang, H. Wei*: $c^\sharp$-normality of groups and its properties. Algebr. Represent. Theory *16* (2013), 193–204. zbl MR doi
[23] *G. Zhong, S.-X. Lin*: On $c^\sharp$-normal subgroups of finite groups. J. Algebra Appl. *16* (2017), Article ID 1750160, 11 pages. zbl MR doi

Authors' address: Zhengtian Qiu, Guiyun Chen, Jianjun Liu (corresponding author), School of Mathematics and Statistics, Southwest University, Beibei District, Chongqing 400715, P. R. China, e-mail: qztqzt506@163.com, gychen1963@163.com, liujj198123@163.com.