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About wcs-covers and wcs^*-networks on the Vietoris hyperspace $\mathcal{F}(X)$

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Abstract. We study some generalized metric properties on the hyperspace $\mathcal{F}(X)$ of finite subsets of a space X endowed with the Vietoris topology. We prove that X has a point-star network consisting of (countable) wcs-covers if and only if so does $\mathcal{F}(X)$. Moreover, X has a sequence of wcs-covers with property (P) which is a point-star network if and only if so does $\mathcal{F}(X)$, where (P) is one of the following properties: point-finite, point-countable, compact-finite, compact-countable, locally finite, locally countable. On the other hand, X has a wcs^*-network with property σ-(P) if and only if so does $\mathcal{F}(X)$. By these results, we obtain some results related to the images of metric spaces and separable metric spaces under some kinds of continuous mappings on the Vietoris hyperspace $\mathcal{F}(X)$.

Keywords: hyperspace; generalized metric property; wcs-cover; wcs^*-network

Classification: 54B20, 54C10, 54D20, 54E40

1. Introduction and preliminaries

Recently, the generalized metric properties on hyperspaces with the Vietoris topology have been studied by many authors, see [5], [9], [12], [13], [14], [15], [16], [18], [19], [20].

In [21], L. Q. Tuyen, O. V. Tuyen and L. D. R. Kočinac proved that X has a σ-(P)-strong network consisting of cs-covers (cs^*-covers) if and only if so does $\mathcal{F}(X)$, where (P) is one of the following properties: point-finite, point-countable, compact-finite, compact-countable, locally finite, locally countable. Moreover, they also proved that X is a Cauchy sn-symmetric space with a σ-(P)-property cs^*-network (cs-network, sn-network, respectively) if and only if so is $\mathcal{F}(X)$. In this paper, we study the concepts of wcs-covers and wcs^*-networks on the Vietoris hyperspace $\mathcal{F}(X)$. Throughout this paper, (P) is assumed to be one of the following properties: point-finite, point-countable, compact-finite, compact-countable, locally finite, locally countable. Moreover, all spaces are assumed to be T_1 and regular, $\mathbb{N}$ denotes the set of all positive integers. For a space X, we prove that

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(1) X has a point-star network consisting of (countable) *wcs*-covers if and only if so does $\mathcal{F}(X)$;
(2) X has a sequence of *wcs*-covers with property (P) which is a point-star network if and only if so does $\mathcal{F}(X)$;
(3) X has a *wcs**-network with property σ-(P) if and only if so does $\mathcal{F}(X)$.

By these results, we obtain that

(1) X is a pseudo-sequence-covering and π-image of a metric space if and only if so is $\mathcal{F}(X)$;
(2) X is a pseudo-sequence-covering (sequentially-quotient), s- and π-image of a metric space if and only if so is $\mathcal{F}(X)$;
(3) X is a pseudo-sequence-covering (sequentially-quotient) and π-image of a separable metric space if and only if so is $\mathcal{F}(X)$;
(4) X is a pseudo-sequence-covering and compact image of a separable metric space if and only if so is $\mathcal{F}(X)$;
(5) X is a (weak) Cauchy *sn*-symmetric space with a *wcs**-network having property σ-(P) if and only if so is $\mathcal{F}(X)$.

For a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, we say that $\{x_n\}_{n\in\mathbb{N}}$ is *eventually* in P, see [2], [17], if $\{x\}\cup\{x_n\colon n\geq m\}\subset P$ for some $m\in\mathbb{N}$, and $\{x_n\}_{n\in\mathbb{N}}$ is *frequently* in P, see [2], [17], if some subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P. Furthermore, if $\mathcal{P}$ is a family of subsets of a space X and $A\subset X$, then

$$\mathtt{St}(A,\mathcal{P})=\bigcup\{P\in\mathcal{P}\colon\ P\cap A\neq\emptyset\};$$
$$(\mathcal{P})_A=\{P\in\ P\colon P\cap A\neq\emptyset\}.$$

For $x\in X$, we use the notation $\mathtt{St}(x,\mathcal{P})$ instead of $\mathtt{St}(\{x\},\mathcal{P})$ and $(\mathcal{P})_x$ instead of $(\mathcal{P})_{\{x\}}$.

Given a space X, we define its *hyperspaces* as the following sets:

(1) $CL(X)=\{A\subset X\colon\ A \text{ is closed and nonempty}\}$;
(2) $\mathbb{K}(X)=\{A\in CL(X)\colon\ A \text{ is compact}\}$;
(3) $\mathcal{F}_n(X)=\{A\in CL(X)\colon\ |A|\leq n\}$, where $n\in\mathbb{N}$;
(4) $\mathcal{F}(X)=\{A\in CL(X)\colon\ A \text{ is finite}\}$.

The set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B}=\{\langle U_1,\dots,U_k\rangle\colon\ U_1,\dots,U_k\ \text{ are open subsets of }\ X,\ k\in\mathbb{N}\}$$

where

$$\langle U_1,\dots,U_k\rangle=\Big\{A\in CL(X)\colon\ A\subset\bigcup_{i\leq k}U_i,\ A\cap U_i\neq\emptyset\ \text{ for each }\ i\leq k\Big\}.$$

Note that, by definition, $\mathbb{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

(1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;
(2) $\mathbb{K}(X)$ is called the *hyperspace of nonempty compact subsets of* X;
(3) $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X;
(4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X) = \bigcup_{n=1}^{\infty}\mathcal{F}_n(X)$ and $\mathcal{F}_n(X) \subset \mathcal{F}_{n+1}(X)$ for each $n \in \mathbb{N}$.

Remark 1.1 ([18])**.** Let X be a space and let $n \in \mathbb{N}$.

(1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.
(2) $f_1\colon X \twoheadrightarrow \mathcal{F}_1(X)$ given by $f_1(x) = \{x\}$ is a homeomorphism.
(3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m, n \in \mathbb{N}$, $m < n$.

Notation 1.2 ([16])**.** If $U_1, \ldots, U_s$ are open subsets of a space X, then $\langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1, \ldots, U_s\rangle$ of the Vietoris topology with $\mathcal{F}(X)$.

Notation 1.3 ([20])**.** Let X be a space. If $\{x_1, \ldots, x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1, \ldots, x_r\} \in \langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$, then for each $j \leq r$, we let

$$U_{x_j} = \bigcap\{U \in \{U_1, \ldots, U_s\}\colon\ x_j \in U\}.$$

Observe that $\langle U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$.

Definition 1.4. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is called a *wcs-cover*, see [3], (or an *fcs-cover*, see [4]), if for every convergent sequence S converging to x in X, there exists a finite subfamily $\mathcal{P}'$ of $(\mathcal{P})_x$ such that S is eventually in $\bigcup\mathcal{P}'$.
(2) $\mathcal{P}$ is called a *cs^*-cover*, see [7], if for every convergent sequence S in X, there exist $P \in \mathcal{P}$ and a subsequence S' of S such that S' is eventually in P.
(3) $\mathcal{P}$ is called a *cs-cover*, see [22], if for every convergent sequence S in X, there exists $P \in \mathcal{P}$ such that S is eventually in P.
(4) $\mathcal{P}$ is called a *wcs^*-network*, see [11], for X, if for each sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to $x \in U$ with U open in X, there exists a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ of $\{x_n\}_{n\in\mathbb{N}}$ such that $\{x_{n_k}\colon\ k \in \mathbb{N}\} \subset P \subset U$ for some $P \in \mathcal{P}$.
(5) $\mathcal{P}$ is called a *cs-network*, see [6], (*cs^*-network*, respectively, see [2]) for X, if for each $x \in X$, any sequence L converging to $x \in U$ with U open in X, then L is eventually (frequently, respectively) in $P \subset U$ for some $P \in \mathcal{P}$.

Remark 1.5.

(1) Definition wcs-cover or fcs-cover is also sfp-cover in [8].
(2) cs-covers $\Rightarrow$ wcs-covers $\Rightarrow$ cs^*-covers.
(3) cs-networks $\Rightarrow$ cs^*-networks $\Rightarrow$ wcs^*-networks.

Definition 1.6. Let $\mathcal{P}$ be a family of subsets of a space X.

(1) $\mathcal{P}$ is said to be *point-finite* (or *point-countable*), if the family $(\mathcal{P})_x$ is finite (countable, respectively) for each $x \in X$.
(2) $\mathcal{P}$ is said to be *compact-finite* (*compact-countable*, respectively), if for each compact subset K of X, the family $(\mathcal{P})_K$ is finite (countable, respectively).
(3) $\mathcal{P}$ is said to be *locally finite* (*locally countable*, respectively), if for each $x \in X$, there exists an open neighborhood V of x such that the family $(\mathcal{P})_V$ is finite (countable, respectively).

Definition 1.7. For a cover $\mathcal{P}$ of a space X, we say that $\mathcal{P}$ has *property* σ-(P), if $\mathcal{P}$ can be expressed as $\bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, where each $\mathcal{P}_n$ has property (P).

Definition 1.8 ([10])**.** Let X be a space. A sequence $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ of families of subsets in X is called a *point-star network* for X, if $\{\mathtt{St}(x, \mathcal{P}_n)\}_{n\in\mathbb{N}}$ is a network at x in X for each $x \in X$.

For some undefined or related concepts, we refer the reader to [3], [10], [21].

2. Main results

Let X be a space. We say that a sequence $\{A_n\}_{n\in\mathbb{N}}$ consisting of subsets of X converges to a subset $A \subset X$, if for each open set U in X with $A \subset U$, there exists $k \in \mathbb{N}$ such that $A_n \subset U$ for each $n > k$.

Lemma 2.1 ([21, Lemma 2.1])**.** *Let X be a space and $\{F_m\}_{m\in\mathbb{N}}$ be a sequence of points of $\mathcal{F}(X)$. If $\{F_m\}_{m\in\mathbb{N}}$ converges to $F = \{x_1, \ldots, x_r\}$ in $\mathcal{F}(X)$ and $\{U_1, \ldots, U_r\}$ is a family of pairwise disjoint open subsets of X such that $x_j \in U_j$ for each $j \leq r$, then $\{F_m \cap U_j\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X for each $j \leq r$.*

Lemma 2.2 ([19, Lemma 2.1])**.** *Let $\langle U_1, \ldots, U_s\rangle, \langle V_1, \ldots, V_r\rangle \subset CL(X)$. If there exists $i_0 \leq s$ such that $U_{i_0} \cap \big(\bigcup_{j\leq r} V_j\big) = \emptyset$, then $\langle U_1, \ldots, U_s\rangle \cap \langle V_1, \ldots, V_r\rangle = \emptyset$.*

Let $\mathcal{P}$ be a family of subsets of a space X. If we put

$$\mathfrak{P} = \{\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} \colon\ P_1, \ldots P_s \in \mathcal{P},\ s \in \mathbb{N}\},$$

then observe that $\mathfrak{P}$ is a family of subsets of $\mathcal{F}(X)$.

Lemma 2.3. *Let X be a space. Then, $\mathcal{P}$ is a wcs-cover (or wcs^*-network) for X, then $\mathfrak{P}$ is a wcs-cover (wcs^*-network, respectively) for $\mathcal{F}(X)$.*

Proof: Suppose that $F = \{x_1, \ldots, x_r\} \in \mathcal{F}(X)$ and $\mathcal{U}$ is an open neighborhood of F in $\mathcal{F}(X)$. Then, there exist open subsets $U_1, \ldots, U_s$ of X such that

$$F \in \langle U_1, \ldots, U_s \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Because X is Hausdorff, by Notation 1.3, we can find pairwise disjoint open subsets $U_{x_1}, \ldots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \leq r$, and

$$F \in \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)} \subset U_1, \ldots, U_s \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Let $\{F_m\}_{m\in\mathbb{N}}$ be a sequence converging to F in $\mathcal{F}(X)$. For each $j \leq r$, it follows from Lemma 2.1 that the sequence $\{F_m \cap U_{x_j}\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X.

Case 1. Let $\mathcal{P}$ be a wcs-cover for X. Then, there exist a finite subfamily $\mathcal{P}_j$ of $(\mathcal{P})_{x_j}$ and $k_j \in \mathbb{N}$ such that

$$\{x_j\} \cup \left(\bigcup \{F_m \cap U_{x_j} \colon m \geq k_j\}\right) \subset \bigcup \mathcal{P}_j.$$

Put $k = \max\{k_j \colon j \leq r\}$ and

$$\mathfrak{P}' = \{\langle P_1, \ldots, P_r \rangle_{\mathcal{F}(X)} \colon P_j \in \mathcal{P}_j,\ j \leq r\}.$$

Then, $\mathfrak{P}'$ is a finite subfamily of $(\mathfrak{P})_F$. Furthermore, we have

$$\{F\} \cup \{F_m \colon m > k\} \subset \bigcup \mathfrak{P}'.$$

Therefore, $\mathfrak{P}$ is a wcs-cover for $\mathcal{F}(X)$.

Case 2. Let $\mathcal{P}$ be a wcs^*-network for X. Then, by induction on r, there exist $P_1, \ldots, P_r \in \mathcal{P}$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\bigcup \{F_{m_k} \cap U_{x_j} \colon k \in \mathbb{N}\} \subset P_j \subset U_{x_j}.$$

This implies that $\langle P_1, \ldots, P_r \rangle_{\mathcal{F}(X)} \in \mathfrak{P}$ and

$$\{F_{m_k} \colon k \in \mathbb{N}\} \subset \langle P_1, \ldots, P_r \rangle_{\mathcal{F}(X)} \subset \langle U_{x_1}, \ldots, U_{x_r} \rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Hence, $\mathfrak{P}$ is a wcs^*-network for $\mathcal{F}(X)$. □

Lemma 2.4 ([21, Lemma 2.3])**.** *If $\mathcal{P}$ has property (P), then so does $\mathfrak{P}$.*

Theorem 2.5. *Let X be a space.*

(1) *X has a point-star network consisting of (countable) wcs-covers if and only if so does $\mathcal{F}(X)$.*

(2) *X has a sequence of wcs-covers with property (P) which is a point-star network if and only if so does $\mathcal{F}(X)$.*

PROOF: *Necessity.* Let $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ be a sequence of *wcs*-covers which is a point-star network for X. We can assume that $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for each $n \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$, put

$$\mathfrak{P}_n = \{\langle P_1^{(n)}, \dots, P_s^{(n)}\rangle_{\mathcal{F}(X)}\colon\ P_1^{(n)}, \dots, P_s^{(n)} \in \mathcal{P}_n,\ s \in \mathbb{N}\}.$$

Take any $F = \{x_1, \dots, x_r\} \in \mathcal{F}(X)$ and an open neighborhood $\mathcal{U}$ of F in $\mathcal{F}(X)$. Similar to the proof of Lemma 2.3, we can find pairwise disjoint open subsets $U_{x_1}, \dots, U_{x_r}$ of X such that $x_j \in U_{x_j}$ for each $j \le r$ and

$$F \in \langle U_{x_1}, \dots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

For each $j \le r$, since $\{\mathcal{P}_n\}_{n\in\mathbb{N}}$ is a point-star network for X, $\{\mathtt{St}(x_j, \mathcal{P}_n)\}_{n\in\mathbb{N}}$ is a network at x_j in X. Thus, there exists $m_j \in \mathbb{N}$ such that $x_j \in \mathtt{St}(x_j, \mathcal{P}_n) \subset U_{x_j}$ whenever $n \ge m_j$. If we put $m = \max\{m_j\colon\ j \le r\}$, the

$$F \in \langle \mathtt{St}(x_1, \mathcal{P}_n), \dots, \mathtt{St}(x_r, \mathcal{P}_n)\rangle_{\mathcal{F}(X)} \subset \langle U_{x_1}, \dots, U_{x_r}\rangle_{\mathcal{F}(X)}$$

for every $n \ge m$. Furthermore, it is clear that

$$\mathtt{St}(F, \mathfrak{P}_n) \subset \langle \mathtt{St}(x_1, \mathcal{P}_n), \dots, \mathtt{St}(x_r, \mathcal{P}_n)\rangle_{\mathcal{F}(X)}.$$

Hence, $F \in \mathtt{St}(F, \mathfrak{P}_n) \subset \mathcal{U}$ for every $n \ge m$. Therefore, $\{\mathtt{St}(F, \mathfrak{P}_n)\}_{n\in\mathbb{N}}$ is a network at F in $\mathcal{F}(X)$. This shows that $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is a point-star network for $\mathcal{F}(X)$. It follows from Lemma 2.3 that $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is a sequence of *wcs*-covers for $\mathcal{F}(X)$. Moreover, if $\mathcal{P}_n$ is countable, then observe that $\mathfrak{P}_n$ is countable. On the other hand, by Lemma 2.4, if $\mathcal{P}_n$ has property (P), then $\mathfrak{P}_n$ has property (P).

Sufficiency. Assume that $\{\mathfrak{P}_n\}_{n\in\mathbb{N}}$ is a sequence of *wcs*-covers, and a point-star network for $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, we put

$$\mathfrak{Q}_n = \{\mathcal{W} \cap \mathcal{F}_1(X)\colon\ \mathcal{W} \in \mathfrak{P}_n\}.$$

Then, $\{\mathfrak{Q}_n\}_{n\in\mathbb{N}}$ is a sequence of *wcs*-covers, and a point-star network for $\mathcal{F}_1(X)$. Furthermore, if $\mathfrak{P}_n$ is countable, then $\mathfrak{Q}_n$ is countable. On the other hand, for each $n \in \mathbb{N}$, if $\mathfrak{P}_n$ has property (P), then $\mathfrak{Q}_n$ has property (P). By Remark 1.1, the proof of sufficiency is completed. □

By Theorem 2.5, [3, Theorem 2.7, Corollaries 2.9, 3.8, Proposition 3.7] and [1, Theorem 2.4], we obtain the following corollary.

Corollary 2.6. *Let X be a space.*

(1) *X is a pseudo-sequence-covering and π-image of a metric space if and only if so is $\mathcal{F}(X)$.*
(2) *X is a pseudo-sequence-covering (sequentially-quotient), s- and π-image of a metric space if and only if so is $\mathcal{F}(X)$.*
(3) *X is a pseudo-sequence-covering (sequentially-quotient) and π-image of a separable metric space if and only if so is $\mathcal{F}(X)$.*
(4) *X is a pseudo-sequence-covering and compact image of a separable metric space if and only if so is $\mathcal{F}(X)$.*

Theorem 2.7. *Let X be a space. Then, X has a wcs^*-network with property σ-(P) if and only if so does $\mathcal{F}(X)$.*

PROOF: *Necessity.* Assume that $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$ is a wcs^*-network for X, where each $\mathcal{P}_n$ has property (P) and $\mathcal{P}_n \subset \mathcal{P}_{n+1}$ for each $n \in \mathbb{N}$. It follows from Lemma 2.4 that

$$\mathfrak{P}_n = \{\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} \colon\ P_1, \ldots, P_s \in \mathcal{P}_n,\ s \in \mathbb{N}\}$$

has property (P), and $\mathfrak{P}_n \subset \mathfrak{P}_{n+1}$ for each $n \in \mathbb{N}$. If we put $\mathfrak{P} = \bigcup_{n\in\mathbb{N}} \mathfrak{P}_n$, then $\mathfrak{P}$ has property σ-(P).

Now, we will prove that

$$\mathfrak{P} = \{\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} \colon\ P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}.$$

In fact, it is easy to see that

$$\mathfrak{P} \subset \{\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} \colon\ P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}.$$

Next, let $\mathcal{W} \in \{\langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)} \colon\ P_1, \ldots, P_s \in \mathcal{P},\ s \in \mathbb{N}\}$. Then, there exist $P_1, \ldots, P_s \in \mathcal{P}$ such that $\mathcal{W} = \langle P_1, \ldots, P_s\rangle_{\mathcal{F}(X)}$. Since $\mathcal{P} = \bigcup_{n\in\mathbb{N}} \mathcal{P}_n$, there exists $n_i \in \mathbb{N}$ such that $P_i \in \mathcal{P}_{n_i}$ for each $i \le s$. If we put $m = \max\{n_i \colon\ i \le s\}$, then $P_1, \ldots, P_s \in \mathcal{P}_m$ and $m \in \mathbb{N}$. This implies that $\mathcal{W} \in \mathfrak{P}_m \subset \mathfrak{P}$.

Therefore, $\mathfrak{P}$ is a wcs^*-network for $\mathcal{F}(X)$ by Lemma 2.3.

Sufficiency. Let $\mathfrak{P} = \bigcup_{n\in\mathbb{N}} \mathfrak{P}_n$ be a wcs^*-network with property σ-(P) for $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, we put

$$\mathfrak{Q}_n = \{\mathcal{W} \cap \mathcal{F}_1(X) \colon\ \mathcal{W} \in \mathfrak{P}_n\}.$$

Then, $\mathfrak{Q} = \bigcup_{n\in\mathbb{N}} \mathfrak{Q}_n$ is a wcs^*-network with property σ-(P) for $\mathcal{F}_1(X)$. Thus, X has a wcs^*-network with property σ-(P) for X by Remark 1.1. □

By Theorem 2.7 and [21, Corollary 2.7 (2)], we obtain the following corollary.

Corollary 2.8. *Let X be a space. Then, X is a (weak) Cauchy sn-symmetric space with a wcs*-network having property σ-(P) if and only if so is $\mathcal{F}(X)$.*

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