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SUPERCONVERGENCE ANALYSIS OF SPECTRAL VOLUME METHODS FOR ONE-DIMENSIONAL DIFFUSION AND THIRD-ORDER WAVE EQUATIONS

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Abstract. We present a unified approach to studying the superconvergence property of the spectral volume (SV) method for high-order time-dependent partial differential equations using the local discontinuous Galerkin formulation. We choose the diffusion and third-order wave equations as our models to illustrate approach and the main idea. The SV scheme is designed with control volumes constructed using the Gauss points or Radau points in subintervals of the underlying meshes, which leads to two SV schemes referred to as GSV and RSV schemes, respectively. With a careful choice of numerical fluxes, we demonstrate that the schemes are stable and exhibit optimal error estimates. Furthermore, we establish superconvergence of the GSV and RSV for the solution itself and the auxiliary variables. To be more precise, we prove that the errors of numerical fluxes at nodes and for the cell averages are superconvergent with orders of $\mathcal{O}(h^{2k+1})$ and $\mathcal{O}(h^{2k})$ for RSV and GSV, respectively. Superconvergence for the function value and derivative value approximations is also studied and the superconvergence points are identified at Gauss points and Radau points. Numerical experiments are presented to illustrate theoretical findings.

Keywords: spectral volume method; error estimate; superconvergence; high order equation

MSC 2020: 65N30, 65N25, 65N15

1. Introduction

The spectral volume (SV) method is a class of high-order Godunov-type finite volume method [13]. It is considered the current state-of-the-art for numerically solving hyperbolic conservation laws. The SV method was first proposed by Wang, Liu et al. for hyperbolic conservation laws on unstructured grids (see, e.g., [18]). Since then it

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has attracted significant attention and has been successfully implemented for solving various partial differential equations (PDEs) such as the Euler equation [19] and the Navier-Stokes equations [17]. Similarly to discontinuous Galerkin (DG) methods [11], SV methods adopt completely discontinuous functions as solution space, offering several advantages, including the allowance of hanging nodes, easy hp adaptivity, high parallel efficiency, and more. Comparisons between the SV method and other numerical methods (e.g., DG, spectral difference) in terms of accuracy and stability have also been conducted in the literature (see, [20]).

For equations containing higher-order derivatives, SV method cannot be directly applied. This limitation arises due to the piecewise discontinuity at the element interfaces, which is not regular enough to handle higher derivatives. Inspired by works in DG methods, including the LDG method [10], [12], the interior penalty (IP) method [2], the direct discontinuous Galerkin (DDG) method [16], attempts based on the SV context have been made to solve diffusion and third-order spatial derivative equations. Sun, Wang and Liu [17] were the first to implement the SV method for the Navier-Stokes equations, using the LDG method to discretize the viscous fluxes. Later, Kannan and Wang [15] studied the SV method for a p-multigrid SV Navier-Stokes solver, employing a new penalty and BR2 viscous flux formulation.

Recently, formulations using LDG in the SV context have been separately proposed in [14] for solving equations containing third order spatial derivative. It is worth noting that relatively few studies are devoted to the mathematical analysis in terms of accuracy and error estimates for high (any) order SV schemes for high order derivative equations. Only recently, Cao et al. [1], [4] proposed arbitrary high order SV schemes for diffusion equations by using LDG and DDG formulations to discretize the viscous flux, and they establish a unified approach for the stability and error estimates of the SV schemes.

The main purpose of our paper is to study the superconvervence of the SV method for the diffusion equation and third-order wave equation. The superconvergence of numerical methods has been intensively studied during the past decades [3], [6], [7], [9]). However, little information is known about SV methods. To the best of our knowledge, no theoretic analysis on the superconvergence of the SV method has been reported yet in the literature for solving the diffusion and wave equations. To derive the superconvergence property of SV methods for high-order equations, we first present two classes of arbitrary high-order SV schemes for diffusion and third wave equations when the LDG formulation is used to discretize the high-order derivatives. This approach inherits ideas from the LDG method, where the auxiliary variable is introduced to rewrite the high-order equation into an equivalent first-order system, and then the SV method is used to solve it. By choosing suitable numerical fluxes, the two SV schemes are shown to be stable and have optimal error estimates in

the L^2 norms for both the exact solution and the auxiliary variables. By constructing some special projection of the exact solution, superconvergence properties of the SV approximation are finally established.

The remainder of the paper is organized as follows. In Section 2, we present two SV schemes based on the LDG formulation for the one-dimensional diffusion equations, and study the superconvergence property. In Section 3, we propose two SV schemes for the one-dimensional third-order equation and demonstrate their stability, optimal error estimates, and superconvergence property. Section 4 provides numerical examples to assess the accuracy of the SV method for both diffusion and third-order wave equations. Concluding remarks are presented in Section 5.

Throughout this paper, we adopt standard notations for Sobolev spaces such as $W^{m,p}(D)$ on sub-domain $D \subset \Omega$ equipped with the norm $\|\cdot\|_{m,p,D}$ and semi-norm $|\cdot|_{m,p,D}$. When $D = \Omega$, we omit the index D; and if $p = 2$, we set $H^m(D) = W^{m,p}(D)$, $\|\cdot\|_{m,D} = \|\cdot\|_{m,p,D}$, and $|\cdot|_{m,D} = |\cdot|_{m,p,D}$. Notation $A \lesssim B$ implies that A can be bounded by B multiplied by a constant independent of the mesh size h.

2. SV methods for one dimensional diffusion equations

In this section, we present and analyze the SV method for solving the one-dimensional second order equation

$$u_t = u_{xx}, \quad (x,t) \in [0, 2\pi] \times (0, T], \quad u(x,0) = u_0(x), \tag{2.1}$$

where $u_0(x)$ is smooth. For simplicity, here we only consider the periodic boundary conditions $u(0,t) = u(2\pi,t)$, $u_x(0,t) = u_x(2\pi,t)$ for all $t \geqslant 0$. The assumption is not essential, since the analysis can be applied to other boundary conditions.

To construct the SV scheme, we rewrite (2.1) into a first order system

$$u_t = q_x, \quad q = u_x. \tag{2.2}$$

2.1. SV Scheme. Let $0 = x_{1/2} < \ldots < x_{N+1/2} = 2\pi$ be $N+1$ distinct points on $\Omega = [0, 2\pi]$. For positive integer r, define $\mathbb{Z}_r = \{1, \ldots, r\}$ and $\mathbb{Z}_r^0 = \{0, \ldots, r\}$. Denote by

$$V_i = \{x\colon\ x_{i-1/2} \leqslant x \leqslant x_{i+1/2}\}, \quad x_i = (x_{i-1/2} + x_{i+1/2})/2, \quad i \in \mathbb{Z}_N,$$

the cells and cell centers, respectively. Let $h_i = x_{i+1/2} - x_{i-1/2}$ and $h = \max\limits_i h_i$. Assume that the mesh is regular, i.e., there is a constant $c > 0$ independent of h such that

$$h \leqslant c h_i, \quad i \in \mathbb{Z}_N.$$

The finite-element space of piecewise polynomials is defined as

$$U_h = \{\zeta \in L^2 \colon \zeta|_{V_i} \in \mathbb{P}_k(x),\ i \in \mathbb{Z}_N\},$$

where $\mathbb{P}_k$ denotes the space of all polynomials of degree not more than k. Denote by $v^-_{i+1/2}$ and $v^+_{i+1/2}$ the left and right limits of v at the point $x_{i+1/2}$, respectively. Define $[v] = v^+ - v^-$ and $\{v\} = (v^+ + v^-)/2$ the jump and the average of v, respectively.

Let $-1 = s_0 < s_1 < \ldots < s_k < s_{k+1} = 1$ and $-1 = \bar{s}_0 < \bar{s}_1 < \ldots < \bar{s}_k < \bar{s}_{k+1} = 1$ be $k+2$ points in the reference interval $[-1, 1]$, and

$$x_{i,j} = x_i + \frac{h_i}{2} s_j, \quad \bar{x}_{i,j} = x_i + \frac{h_i}{2} \bar{s}_j, \quad i \in \mathbb{Z}_N, \quad j \in \mathbb{Z}^0_{k+1}.$$

Obviously, $x_{i,0} = x_{i-1/2}$ and $x_{i,k+1} = x_{i+1/2}$. With these $k+2$ points, V_i is subdivided into $k+1$ sub-volumes $V_{i,j} = [x_{i,j}, x_{i,j+1}]$ and $\overline{V}_{i,j} = [\bar{x}_{i,j}, \bar{x}_{i,j+1}]$.

The SV method for (2.2) is defined as follows: Find u_h, $q_h \in U_h$ such that

$$\begin{aligned} &\int_{V_{i,j}} \partial_t u_h(x,t)\,\mathrm{d}x - \widehat{q}_h(x_{i,j+1}) + \widehat{q}_h(x_{i,j}) = 0, \\ &\int_{\overline{V}_{i,j}} q_h(x,t)\,\mathrm{d}x - \widehat{u}_h(\bar{x}_{i,j+1}) + \widehat{u}_h(\bar{x}_{i,j}) = 0, \end{aligned} \tag{2.3}$$

where $\widehat{q}_h$, $\widehat{u}_h$ denote the numerical fluxes, which are single valued functions defined on the points and should be designed based on different guiding principles for different PDEs to ensure the stability. Noticing that u_h, q_h are continuous at the interior points $x_{i,j}$, $\bar{x}_{i,j}$ with $j \in \mathbb{Z}_k$, we have

$$\widehat{q}_h(x_{ij}) = q_h(x_{ij}), \quad \widehat{u}_h(\bar{x}_{ij}) = u_h(\bar{x}_{ij}), \quad j \in \mathbb{Z}_k.$$

At the interface of each element $x_{i,j}$, $\bar{x}_{i,j}$ with $j = 0,\ k+1$, we take the following alternating fluxes:

$$(\widehat{q}_h, \widehat{u}_h) = (q_h^-, u_h^+). \tag{2.4}$$

By choosing different points s_j and $\bar{s}_j$, we obtain two different SV schemes.

GSV: $\{s_j\}_{j=1}^k$ and $\{\bar{s}_j\}_{j=1}^k$ are chosen as Gauss-Legendre points.

RSV: $\{s_j\}_{j=1}^k$ and $\{\bar{s}_j\}_{j=1}^k$ are taken as k interior Radau points according to the choice of numerical fluxes. For flux (2.4), $\{s_j\}_{j=1}^k$ are chosen as k interior right Radau points and $\{\bar{s}_j\}_{j=1}^k$ are chosen as k interior left Radau points.

Here by k interior right Radau points we mean the zeros of Radau polynomial $L_{k+1} - L_k$ except the point $s = 1$. Similarly, the k interior left Radau points denote the zeros of Radau polynomial $L_{k+1} + L_k$ except the point $s = -1$.

2.2. SV methods as a Petrov-Galerkin method. In this subsection, we reformulate the SV scheme (2.3) into its equivalent Petrov-Galerkin form, which our error estimates are based on.

With the control volumes $V_{i,j}$, $\overline{V}_{i,j}$, we define the piecewise constant function space

$$V_h = \{w^*\colon\ w^*|_{V_{i,j}} \in \mathbb{P}_0,\ i \in \mathbb{Z}_N,\ j \in \mathbb{Z}_k^0\},$$
$$\overline{V}_h = \{\overline{w}^*\colon\ \overline{w}^*|_{\overline{V}_{i,j}} \in \mathbb{P}_0,\ i \in \mathbb{Z}_N,\ j \in \mathbb{Z}_k^0\}.$$

Obviously, any functions $w^* \in V_h$ and $\overline{w}^* \in \overline{V}_h$ can be represented as

$$w^*(x,t) = \sum_{i=1}^{N}\sum_{j=0}^{k} w_{i,j}^* \chi_{V_{i,j}}(x), \quad \overline{w}^*(x,t) = \sum_{i=1}^{N}\sum_{j=0}^{k} \overline{w}_{i,j}^* \chi_{\overline{V}_{i,j}}(x),$$

where $w_{i,j}^*$, $\overline{w}_{i,j}^*$, $(i,j) \in \mathbb{Z}_N \times \mathbb{Z}_k^0$ are coefficients as functions of t. The symbol χ_A, $A \subset [a,b]$, is defined as $\chi_A = 1$ in A and $\chi_A = 0$ otherwise.

Define the so-called broken Sobolev space as follows:

$$\mathcal{H}_h = \{v\colon\ v|_{V_i} \in H^1(V_i),\ i \in \mathbb{Z}_N\}.$$

For any $(v,p,w^*,\overline{w}^*) \in \mathcal{H}_h \times \mathcal{H}_h \times V_h \times \overline{V}_h$, let

$$a_i^1(v,p;w^*) = (\partial_t v + \partial_x \widehat{p}, w^*)_i, \quad a_i^2(v,p;\overline{w}^*) = (p + \partial_x \widehat{v}, \overline{w}^*)_i. \tag{2.5}$$

Denoting

$$a(v,p;w^*,\overline{w}^*) = \sum_{i=1}^{N}(a_i^1(v,p;w^*) + a_i^2(v,p;\overline{w}^*)), \tag{2.6}$$

it is easy to check that the SV solution (u_h, q_h) of (2.3) satisfies

$$a(u_h,q_h;w^*,\overline{w}^*) = 0 \quad \forall\,(w^*,\overline{w}^*) \in V_h \times \overline{V}_h. \tag{2.7}$$

Conversely, if u_h, $q_h \in U_h$ satisfy (2.7), choosing $w^* = \chi_{V_{i,j}}$, $\overline{w}^* = \chi_{\overline{V}_{i,j}}$, then u_h, q_h satisfy (2.3). In other words, the SV method (2.3) is equivalent to method (2.7).

We end this subsection with the special transformation from the trial space to test space (see, e.g., [1], [8]). For $w \in U_h$, we define a transformation $\mathcal{F}\colon\ U_h \to V_h$ by

$$\mathcal{F}w = w^* := \sum_{i=1}^{N}\sum_{j=0}^{k} w_{i,j}^*(t)\chi_{\mathbf{V}_{i,j}}(x), \tag{2.8}$$

where the $k+1$ coefficients $w_{i,j}^*$, $j \in \mathbb{Z}_k^0$ are given as

$$w_{i,0}^* = w(x_{i-1/2}^+) + A_{i,0}w_x(x_{i-1/2}^+), \quad w_{i,j}^* - w_{i,j-1}^* = A_{i,j}w_x(x_{i,j}), \quad j \in \mathbb{Z}_k, \tag{2.9}$$

where $A_{i,j}$ denotes the numerical quadrature weights associated with the quadrature points $x_{i,j}$ to calculate the integral $\int_{V_i} f(x)\,\mathrm{d}x$. Using the inverse inequality, we get

$$\|\mathcal{F}w\|_0^2 = \|w^*\|_0^2 \lesssim \sum_{i=1}^{N}\sum_{j=0}^{k} h_i |w_{i,j}^*|^2 \lesssim \|w\|_0^2 \quad \forall\, w \in U_h. \tag{2.10}$$

Similarly, we can define a transformation $\overline{\mathcal{F}}\colon\ U_h \to \overline{V}_h$ by

$$\overline{\mathcal{F}}\overline{w} = \overline{w}^* := \sum_{i=1}^{N}\sum_{j=0}^{k} \overline{w}_{i,j}^* \chi_{V_{i,j}}(x), \quad \|\overline{\mathcal{F}}\overline{w}\|_0 \lesssim \|\overline{w}\|_0 \quad \forall\, \overline{w} \in U_h. \tag{2.11}$$

Denote

$$\partial_x^{-1} v = \int_0^x v\,\mathrm{d}x, \quad (p,v)_i = \int_{V_i} pv\,\mathrm{d}x, \quad (p,v) = \sum_{i=1}^{N} (p,v)_i.$$

Due to the special transformations $\mathcal{F}$, $\overline{\mathcal{F}}$, we have

$$(v, w^*)_i = (v,w)_i + R_i(w_x \partial_x^{-1} v), \quad (v, \overline{w}^*)_i = (v,w)_i + \overline{R}_i(w_x \partial_x^{-1} v). \tag{2.12}$$

Here $R_i(f)$, $\overline{R}_i(f)$ denote the numerical quadrature errors for any function f, i.e.,

$$R_i(f) = \int_{V_i} f\,\mathrm{d}x - \sum_{j=0}^{k+1} A_{i,j} f(x_{i,j}), \quad \overline{R}_i(f) = \int_{V_i} f\,\mathrm{d}x - \sum_{j=0}^{k+1} \bar{A}_{i,j} f(\bar{x}_{i,j}).$$

Note that $A_{i,j} = \bar{A}_{i,j}$ with $A_{i,0} = A_{i,k+1} = 0$ for GSV.

It was proved in [8] that for both GSV and RSV,

$$(v,v) \leqslant (v, \mathcal{F}v) \lesssim (v,v), \quad (v,v) \leqslant (v, \overline{\mathcal{F}}v) \lesssim (v,v), \tag{2.13}$$

which indicates $(v,\mathcal{F}v)$ and $(v,\overline{\mathcal{F}}v)$ are both equivalent to the standard L^2-norm.

It was proved in [1] that

$$(v_t, \mathcal{F}v) + (p, \overline{\mathcal{F}}p) \lesssim a(v,p;\mathcal{F}v,\overline{\mathcal{F}}p) + \|v\|_0\|p\|_0 \quad \forall\, v,p \in U_h. \tag{2.14}$$

Then the stability of the SV method follows by taking $(v,p) = (u_h, q_h)$ in the above equation and using the Gronwall inequality.

2.3. Analysis. For any function $v \in \mathcal{H}_h$, we denote by $I_h^{\pm} v \in U_h$ the standard Lagrange interpolation of v, which satisfies $k+1$ conditions in each element V_i:

$$I_h^- v(x_{i,j}) = v(x_{i,j}), \quad I_h^- v(x_{i+1/2}^-) = v(x_{i+1/2}^-), \tag{2.15}$$

$$I_h^+ v(x_{i,j}) = v(x_{i,j}), \quad I_h^+ v(x_{i-1/2}^+) = v(x_{i-1/2}^+) \quad \forall\,(i,j) \in \mathbb{Z}_N \times \mathbb{Z}_k. \tag{2.16}$$

Similarly, we can define the interpolation functions $\overline{I}_h^{\pm} v \in U_h$ of v, with the interpolation points $\bar{x}_{i,j}$ substituting $x_{i,j}$ in the above definition. Using the standard approximation theory yields that

$$\|I_h^{\pm} v - v\|_{0,m} + \|\overline{I}_h^{\pm} v - v\|_{0,m} \lesssim h^r \|v\|_{r,m}, \quad r \leqslant k+1. \tag{2.17}$$

For flux (2.4), define

$$(I_h u, I_h q) = (\overline{I}_h^{+} u, I_h^{-} q). \tag{2.18}$$

Due to the special construction of the interpolation function, we have

$$\widehat{I_h u}(\bar{x}_{i,j}) = u(\bar{x}_{i,j}), \quad \widehat{I_h q}(x_{i,j}) = q(x_{i,j}), \quad (i,j) \in \mathbb{Z}_N \times \mathbb{Z}_{k+1}^0. \tag{2.19}$$

To study the superconvergence of SV method, our analysis is along this line: we construct a special projection $(u_I, q_I) \in U_h$ of the exact solution superconvergent to the SV solution (u_h, q_h) in the L^2 (or equivalent L^2) norm. In light of (2.14), we need to design (u_I, q_I) satisfying

$$|a(u - u_I, q - q_I; v^*, \overline{v}^*)| \lesssim h^{k+l+1} \quad \forall\, v^* \in V_h, \overline{v}^* \in \overline{V}_h$$

for a positive l. Towards this end, we define

$$u_I = I_h u - w_u, \quad q_I = I_h q - w_q. \tag{2.20}$$

To derive the superconvergence result, our ultimate goal is to construct the functions w_u, $w_q \in U_h$ (referred as correction functions).

Let $w_{u,0} = u - I_h u$, $w_{q,0} = q - I_h q$, and for $1 \leqslant l \leqslant k$, we define $w_{u,l}$, $w_{q,l} \in U_h$ by

$$\begin{aligned} &(w_{q,l}, v_x)_i = -(\partial_t w_{u,l-1}, v^*)_i, \quad w_{q,l}(x_{i+1/2}^-, t) = 0, \\ &(w_{u,l}, \overline{v}_x)_i = -(w_{q,l-1}, \overline{v}^*)_i, \quad w_{u,l}(x_{i-1/2}^+, t) = 0, \end{aligned} \tag{2.21}$$

for all $v, \overline{v} \in \mathbb{P}_k \setminus \mathbb{P}_0$.

Denote by $L_{i,m}$ the standard Legendre polynomial of degree m on element V_i. Suppose $w_{u,l}$, $w_{q,l} \in U_h$, $1 \leqslant l \leqslant k$, have the following expansion in each V_i:

$$w_{u,l}|_{V_i} = \sum_{m=0}^{k} c_{i,m}^{u,l} L_{i,m}(x), \quad w_{q,l}|_{V_i} = \sum_{m=0}^{k} c_{i,m}^{q,l} L_{i,m}(x). \tag{2.22}$$

Lemma 2.1. *For all $1 \leqslant l \leqslant k$, the functions $w_{u,l}, w_{q,l}$ given in* (2.21) *are well defined. Moreover, suppose $w_{u,l}$, $w_{q,l}$ have the Legendre expansion* (2.22)*. Then*

$$|\partial_t^n c_{i,m}^{u,l}| \leqslant h^{k_{l,m}} \|\partial_t^n u\|_{k_{l,m},\infty}, \quad |c_{i,m}^{q,l}| \leqslant h^{k_{l,m}} \|q\|_{k_{l,m},\infty}, \tag{2.23}$$

where $k_{l,m} \leqslant \max(2k - m, k + l + 1)$ for GSV method, and $k_{l,m} \leqslant \max(2k + 1 - m, k + l + 1)$ for RSV method.

P r o o f. Denote $\phi_{i,m+1} = \int_{x_{i-1/2}}^{x} L_{i,m}(x)\,\mathrm{d}x$. Choosing $v = \phi_{i,m+1}$, $0 \leqslant m \leqslant k-1$ in (2.21), using (2.22) and the orthogonality of Legendre polynomials, we obtain

$$c_{i,m}^{u,l} = -\frac{2m+1}{h_i}(w_{q,l-1}, \phi_{i,m+1}^*)_i, \quad c_{i,m}^{q,l} = -\frac{2m+1}{h_i}(\partial_t w_{u,l-1}, \phi_{i,m+1}^*)_i. \tag{2.24}$$

Using the identities $w_{u,l}(x_{i-1/2}^+, t) = 0$ and $w_{q,l}(x_{i+1/2}^-, t) = 0$ yields

$$c_{i,k}^{u,l} = \sum_{m=0}^{k-1}(-1)^{m-k+1} c_{i,m}^{u,l}, \quad c_{i,k}^{q,l} = -\sum_{m=0}^{k-1} c_{i,m}^{q,l}.$$

Consequently, the coefficients $c_{i,m}^{u,l}$, $c_{i,m}^{q,l}$, $m \leqslant k$ are uniquely obtained, which indicates the uniqueness of $w_{u,l}$, $w_{q,l}$.

We prove (2.23) by mathematical induction. Note that (see Lemma 5.2 in [8])

$$|(u - I_h^{\pm} u, v^*)_i| \lesssim h^{m'} \|u\|_{m'-1,\infty,V_i} \|v\|_{0,\infty,V_i} \quad \forall\, v \in \mathbb{P}_m(V_i), \tag{2.25}$$

where $m' \leqslant \max(2k-m+1, k+2)$ for GSV and $m' \leqslant (2k-m+2, k+2)$ for RSV. By choosing $l = 1$ in (2.24) and using the above inequality, we have for all $0 \leqslant m \leqslant k-1$

$$|c_{i,m}^{q,1}| \lesssim h^{m'-1} \|\partial_t u\|_{m'-1,\infty,V_i} \|\phi_{i,m+1}\|_{0,\infty,V_i} \lesssim h^{m'} \|q\|_{m',\infty,V_i}$$

for all $m' \leqslant \max(2k-m, k+2)$ and in the last step, we have used $u_t = q_x$ and the identity $\|\phi_{i,m+1}\|_{0,\infty,V_i} \lesssim h$. As for $m = k$, we get

$$|c_{i,k}^{q,1}| \lesssim \Big| \sum_{m=0}^{k-1} c_{i,m}^{q,1} \Big| \lesssim h^{k+2} \|q\|_{k+2,\infty},$$

which indicates the second inequality of (2.23) holds true for $l = 1$. Following the same argument, we can prove the first inequality of (2.23) for $l = 1$.

Now we suppose (2.23) is valid for $l-1$ and we prove that it also holds true for l with $2 \leqslant l \leqslant k$. It was proved in [8] that

$$(\partial_t w_{u,l-1}, \phi_{i,m+1}^*)_i = (\partial_t w_{u,l-1}, \phi_{i,m+1})_i + R_i(\partial_t \partial_x^{-1} w_{u,l-1} \partial_x \phi_{i,m+1}). \tag{2.26}$$

Since the Gauss and Radau numerical quadrature is exact for polynomials of degree not more than $2k-1$ and $2k$, respectively, and $\partial_t \partial_x^{-1} w_{u,l-1} \partial_x \phi_{i,m+1} \in \mathbb{P}_{2k-1}$ for all $m \leqslant k-2$, we have for both GSV and RSV that

$$R_i(\partial_t \partial_x^{-1} w_{u,l-1} \partial_x \phi_{i,m+1}) = 0, \quad m \leqslant k-2. \tag{2.27}$$

As for $m = k-1$, we have $R_i(\partial_t \partial_x^{-1} w_{u,l-1} \partial_x \phi_{i,m+1}) = 0$ for RSV. While for GSV, we use the Gauss quadrature error formula and the inverse inequality to obtain that

$$\begin{aligned} |R_i(\partial_t \partial_x^{-1} w_{u,l-1} \partial_x \phi_{i,m+1})| &\lesssim h^{2k+1} \|\partial_t \partial_x^k w_{u,l-1}\|_{0,\infty,V_i} \|\partial_x^{k-1} L_{i,k-1}\|_{0,\infty,V_i} \\ &\lesssim h^2 |\partial_t c_{i,k}^{u,l-1}|. \end{aligned}$$

By using the inductive hypothesis $|\partial_t c_{i,k}^{u,l-1}| \lesssim h^{k+l}\|\partial_t u\|_{k+l,\infty}$, we have

$$|R_i(\partial_t \partial_x^{-1} w_{u,l-1}\partial_x \phi_{i,m+1})| \lesssim h^{k+l+2}\|\partial_t u\|_{k+l,\infty}, \quad m = k-1. \tag{2.28}$$

Combining (2.27) and (2.28), we obtain

$$h^{-1}|R_i(\partial_t \partial_x^{-1} w_{u,l-1}\partial_x \phi_{i,m+1})| \lesssim h^{k+l+1}\|q\|_{k+l+1,\infty} \quad \forall\, m \leqslant k-1. \tag{2.29}$$

In the following, we estimate $(\partial_t w_{u,l-1}, \phi_{i,m+1})_i$. For all $m \leqslant k-1$, using the orthogonality of the Lobatto and Legendre polynomials, there holds

$$\begin{aligned} |(\partial_t w_{u,l-1}, \phi_{i,m+1})_i| &= |(\partial_t (c_{i,m-1}^{u,l-1} L_{i,m-1} + c_{i,m+1}^{u,l-1} L_{i,m+1}), \phi_{i,m+1})_i| \\ &\lesssim h^2(\|\partial_t c_{i,m-1}^{u,l-1}\|_{0,\infty,V_i} + \|\partial_t c_{i,m+1}^{u,l-1}\|_{0,\infty,V_i}). \end{aligned}$$

Then we use the induction hypothesis to get for all $m \leqslant k-1$,

$$|(\partial_t w_{u,l-1}, \phi_{i,m+1})_i| \lesssim h^2 h^{k_{l-1,m+1}}\|\partial_t u\|_{k_{l-1,m+1},\infty} \lesssim h^{k_{l,m}+1}\|q\|_{k_{l,m},\infty}.$$

Substituting the above inequality and (2.29) into (2.26), we obtain that for $m \leqslant k-1$,

$$|c_{i,m}^{q,l}| \lesssim h_i^{k_{l,m}}\|q\|_{k_{l,m},\infty}.$$

As for $m = k$, we have

$$|c_{i,k}^{q,l}| \lesssim \sum_{j=1}^{k-1} |c_{i,m}^{q,l}| \lesssim h^{k+l+1}\|q\|_{k+l+1,\infty}.$$

Then the second inequality in (2.23) holds true for all positive l with $1 \leqslant l \leqslant k$. The first inequality of (2.23) follows the same argument and thus we omit it here. □

Let

$$w_u = \sum_{l=1}^{k} w_{u,l}, \quad w_q = \sum_{l=1}^{k} w_{q,l}. \tag{2.30}$$

Now we are ready to estimate $a(u-u_I, q-q_I; v^*, \overline{v}^*)$ for all v, $\overline{v} \in U_h$.

Theorem 2.1. *Let u be the solution of (2.1) satisfying $u \in W^{k+2+r,\infty}$, $\partial_t u \in W^{k+1+r,\infty}$ with $r \leqslant k$, and u_I, q_I be the interpolation functions defined in (2.20). Then for all v, $\overline{v} \in U_h$,*

$$|a(u-u_I, q-q_I; v^*, \overline{v}^*)| \lesssim h^{k+r+1}(\|u\|_{k+r+2,\infty} + \|\partial_t u\|_{k+r+1,\infty})(\|v\|_0 + \|\overline{v}\|_0), \tag{2.31}$$

where $1 \leqslant r \leqslant k-1$ for GSV and $1 \leqslant r \leqslant k$ for RSV.

P r o o f. First, we use the property of $I_h u$, $I_h q$ to derive that

$$a_i^1(u - I_h u, q - I_h q; v^*) = (\partial_t(u - I_h u), v^*)_i.$$

Second, recalling the definition of $a_i^1(\cdot,\cdot;\cdot)$ and using (2.26), we have

$$a_i^1(v, p; w^*) = (v_t, w^*)_i + (p, w_x)_i - \widehat{p} w^-|_{i+1/2} + \widehat{p} w^+|_{i-1/2} - R_i(w_x p),$$

and thus

$$a_i^1(w_u, w_q; v^*) = (\partial_t w_u, v^*)_i + (w_q, v_x)_i.$$

Consequently,

$$a_i^1(u - u_I, q - q_I; v^*) = \sum_{l=0}^{k} (\partial_t w_{u,l}, v^*)_i + \sum_{l=1}^{k} (w_{q,l}, v_x)_i.$$

For all $v \in \mathbb{P}_k \setminus \mathbb{P}_0$, we use (2.21) to derive that

$$|a_i^1(u - u_I, q - q_I; v^*)| = |(\partial_t w_{u,k}, v^*)_i| \lesssim \|\partial_t w_{u,k}\|_{0,V_i} \|v^*\|_{0,V_i}.$$

Summing up all i from 1 to N and using (2.22)–(2.23), we get for all $v \in \mathbb{P}_k \setminus \mathbb{P}_0$

(2.32)
$$|a^1(u - u_I, q - q_I; v^*)| \lesssim \|\partial_t w_{u,k}\|_0 \|v^*\|_0 \lesssim h^{k+r+1} \|\partial_t u\|_{k+r+1,\infty} \|v\|_0, \quad r \leqslant k.$$

On the other hand, for any $v \in \mathbb{P}_0$, we have from (2.23) and (2.25) that

$$\begin{aligned} |a_i^1(u - u_I, q - q_I; v^*)| &= \left| (\partial_t w_{u,0}, v^*)_i + \sum_{l=1}^{k} (\partial_t w_{u,l}, v^*)_i \right| \\ &\lesssim \sum_{l=1}^{k} |\partial_t c_{i,0}^{u,l}| (L_{i,0}, v)_i + h^{m'} \|\partial_t u\|_{m'-1,\infty} \|v\|_{0,\infty,V_i} \\ &\lesssim h^{k+3/2+r} \|\partial_t u\|_{k+1+r,\infty} \|v\|_{0,V_i} \end{aligned}$$

with $r \leqslant k-1$ for GSV and $r \leqslant k$ for RSV. Here in the last step, we have used the inverse inequality. Then

$$|a^1(u - u_I, q - q_I; v^*)| \lesssim h^{k+1+r} \|\partial_t u\|_{k+1+r,\infty} \|v\|_0, \quad v \in \mathbb{P}_0. \tag{2.33}$$

Noticing that any $v \in U_h$ can be represented as $v = v_1 + v_0$ with $v_1 \in \mathbb{P}_k \setminus \mathbb{P}_0$ and $v_0 \in \mathbb{P}_0$, we use (2.32) and (2.33) to derive that

$$|a^1(u - u_I, q - q_I; v^*)| \lesssim h^{k+1+r} \|\partial_t u\|_{k+1+r,\infty} \|v\|_0.$$

Similarly, there holds

$$|a^2(u - u_I, q - q_I; \overline{v}^*)| \lesssim h^{k+1+r} \|q\|_{k+1+r,\infty} \|\overline{v}\|_0.$$

Then the desired result (2.31) follows from the last two inequalities and $q = u_x$. □

2.4. Superconvergence results. For convenience, we use the following notations:

$$e_v = v - v_h, \quad \xi_v = v_h - v_I, \quad \eta_v = v - v_I, \quad v = (u, q).$$

We first have the following supercloseness between the numerical solutions (u_h, q_h) and the interpolation functions (u_I, q_I).

Theorem 2.2. *Let u be the solution of (2.1) satisfying $u \in W^{k+2+r,\infty}$, $\partial_t u \in W^{k+1+r,\infty}$ $(r \leqslant k)$, and (u_h, q_h) be the solution of (2.3) with $u_h(x,0) = u_I(x,0)$. Then*

$$\|u_I - u_h\|_0 + \int_0^t \|q_I - q_h\|_0 \,\mathrm{d}t \lesssim h^{k+r+1}(\|u\|_{k+r+2,\infty} + \|\partial_t u\|_{k+r+1,\infty}) \tag{2.34}$$

with $r \leqslant k-1$ for GSV and $r \leqslant k$ for RSV.

P r o o f. Choosing $(v, p) = (\xi_u, \xi_q)$ in inequality (2.14) and using the orthogonality $a(e_u, e_q; \mathcal{F}w, \overline{\mathcal{F}}w) = 0$ for all $w, \overline{w} \in U_h$, we have

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\xi_u\|_0^2 + \|\xi_q\|_0^2 &\lesssim a(\xi_u, \xi_q; \mathcal{F}\xi_u, \overline{\mathcal{F}}\xi_q) + \|\xi_u\|_0\|\xi_q\|_0 \\ &= a(\eta_u, \eta_q; \mathcal{F}\xi_u, \overline{\mathcal{F}}\xi_q) + \|\xi_u\|_0\|\xi_q\|_0.\end{aligned}$$

Then (2.34) follows from (2.31) and the Gronwall inequality. □

Denote by $e_{v,n}$, $e_{v,c}$ $(v = (u, q))$ the error of numerical fluxes at nodes and for the cell average, respectively. That is,

$$e_{v,n} = \left(\frac{1}{N}\sum_{i=1}^{N}(v - \widehat{v}_h)^2(x_{i+1/2}, t)\right)^{1/2}, \; e_{v,c} = \left(\frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i-1/2}}(v - v_h)\,\mathrm{d}x\right)^2\right)^{1/2}. \tag{2.35}$$

Theorem 2.3. *Let u be the solution of (2.1) satisfying $u \in W^{k+3,\infty}$, $\partial_t u \in W^{k+2,\infty}$, and (u_h, q_h) be the solution of (2.3) with the initial solution chosen as $u_h(x,0) = u_I(x,0)$. Then for both GSV and RSV, and $v = (u, q)$,*

$$e_{v,g} \lesssim h^{k+2}(\|u\|_{k+3,\infty} + \|\partial_t u\|_{k+2,\infty}), \quad e_{v,d} \lesssim h^{k+1}(\|u\|_{k+3,\infty} + \|\partial_t u\|_{k+2,\infty}), \tag{2.36}$$

where

$$e_{u,g} = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k} e_u^2(\bar{x}_{i,j}, t)\right)^{1/2}, \quad e_{q,g} = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k} e_q^2(x_{i,j}, t)\right)^{1/2},$$

$$e_{q,d} = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}(\partial_x e_q)^2(y_{i,j}, t)\right)^{1/2}, \quad e_{u,d} = \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}(\partial_x e_u)^2(z_{i,j}, t)\right)^{1/2}.$$

Here for GSV, $y_{i,j}$ are the roots of $\partial_x(L_{i,k}(x-x_{i+1/2}))$, and $z_{i,j}$ are the roots of $\partial_x(L_{i,k}(x+x_{i+1/2}))$, while for RSV, $z_{i,j}$ denote interior left Radau points, $y_{i,j}$ denote interior right Radau points. Furthermore, if $u \in W^{2k+2,\infty}$, $\partial_t u \in W^{2k+1,\infty}$, then

$$e_{v,n} + e_{v,c} \lesssim h^{2k+m}(\|u\|_{2k+1+m,\infty} + \|\partial_t u\|_{2k+m,\infty}) \tag{2.37}$$

with $m = 0$ for GSV and $m = 1$ for RSV.

P r o o f. Recalling the definition of w_u, w_q in (2.18) and (2.21), we have for $v = (u, q)$,

$$\widehat{v}_I(x^+_{i+1/2}, t) = \widehat{I_h v}(x_{i+1/2}, t) = v(x_{i+1/2}, t).$$

Consequently, we obtain from the inverse inequality that

$$\frac{1}{N}\sum_{i=1}^{N}(v - \widehat{v}_h)^2(x_{i+1/2}, t) = \frac{1}{N}\sum_{i=1}^{N}(v_I - \widehat{v}_h)^2(x_{i+1/2}, t) \lesssim \|v_I - v_h\|_0^2(t).$$

In light of (2.23), there holds

$$\left|\int_{x_{i-1/2}}^{x_{i+1/2}}(v - I_h v)(x)\,\mathrm{d}x\right| \lesssim h^{2k+1+m}\|u\|_{2k+1+m,\infty},$$

$$\left|\int_{x_{i-1/2}}^{x_{i+1/2}} w_v(x)\,\mathrm{d}x\right| = \left|\sum_{l=1}^{k} c_{v,0}^{i,l}(L_{i,0}, 1)_i\right| \lesssim h^{2k+1+m}\|v\|_{2k+1+m,\infty},$$

and thus

$$\left|\int_{x_{i-1/2}}^{x_{i+1/2}}(v - v_I)(x)\,\mathrm{d}x\right| \lesssim h^{2k+1+m}\|v\|_{2k+1+m,\infty}.$$

By the triangle inequality and the Cauchy-Schwarz inequality, there holds

$$|e_{v,c}|^2 = \frac{1}{N}\sum_{j=1}^{N}\left(\frac{1}{h_i}\int_{x_{i-1/2}}^{x_{i-1/2}}(\eta_v - \xi_v)\,\mathrm{d}x\right)^2 \lesssim h^{4k+2m}\|v\|^2_{2k+1+m,\infty} + \|v_I - v_h\|_0^2.$$

Taking $r = k$ (RSV) and $r = k-1$ (GSV) in (2.34), the desired result (2.37) follows.

On the other hand, we use (2.23), (2.34) and the inverse inequality to derive that

$$|e_{u,g}| \leqslant \left(\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{k}(I_h u - u_h)^2(\bar{x}_{i,j})\right)^{1/2} \lesssim \|u_I - u_h\|_0 + \|w_u\|_0 \lesssim h^{k+2}\|u\|_{k+2,\infty}.$$

Following the same argument, we have

$$|e_{q,g}| \lesssim \|q_I - q_h\|_0 + \|w_q\|_0 \lesssim h^{k+2}\|u\|_{k+2,\infty}.$$

Then the first inequality of (2.36) follows.

To show the second inequality of (2.36), we note that (see, e.g., [8])

$$|\partial_x(u - I_h u)(z_{i,j}, t)| \lesssim h^{k+1}\|u\|_{k+2,\infty}, \quad |\partial_x(q - I_h q)(y_{i,j}, t)| \lesssim h^{k+1}\|q\|_{k+2,\infty}.$$

Consequently, we use the triangle inequality to get

$$|e_{v,d}| \lesssim h^{k+1}\|v\|_{k+2,\infty} + h^{-1}\|v_I - v_h\|_0 + h^{-1}\|w_v\|_{0,\infty}, \quad v = (u, q).$$

Then the third inequality of (2.36) follows from (2.23)–(2.34). The proof is complete. □

3. SV methods for one dimensional third order wave equations

In this section, we consider the SV method for the following one dimensional third-order wave equation:

$$u_t + u_{xxx} = 0, \quad (x, t) = [0, 2\pi] \times (0, t], \quad u(x, 0) = u_0(x). \tag{3.1}$$

Without loss of generality, we still consider here the periodic boundary conditions, i.e., $u(0, t) = u(2\pi, t)$, $u_x(0, t) = u_x(2\pi, t)$, $u_{xx}(0, t) = u_{xx}(2\pi, t)$ for all $t \geqslant 0$, and the initial solution $u_0(x)$ is assumed to be smooth and 2π-periodic.

To construct the SV scheme, we rewrite (3.1) into a first order system

$$u_t = -p_x, \quad p = q_x, \quad q = u_x. \tag{3.2}$$

3.1. SV schemes and the stability. The SV method for (3.2) is defined as follows: Find u_h, p_h, $q_h \in U_h$ such that

$$\begin{aligned} &\int_{x_{i,j}}^{x_{i,j+1}} \partial_t u_h(x, t)\,\mathrm{d}x + \widehat{p}_h(x_{i,j+1}) - \widehat{p}_h(x_{i,j}) = 0, \\ &\int_{\bar{x}_{i,j}}^{\bar{x}_{i,j+1}} p_h(x, t)\,\mathrm{d}x - \widehat{q}_h(\bar{x}_{i,j+1}) + \widehat{q}_h(\bar{x}_{i,j}) = 0, \\ &\int_{\bar{x}_{i,j}}^{\bar{x}_{i,j+1}} q_h(x, t)\,\mathrm{d}x - \widehat{u}_h(\bar{x}_{i,j+1}) + \widehat{u}_h(\bar{x}_{i,j}) = 0, \end{aligned} \tag{3.3}$$

where $\widehat{p}_h$, $\widehat{q}_h$, $\widehat{u}_h$ denote the numerical fluxes chosen as

$$(\widehat{p}_h, \widehat{q}_h, \widehat{u}_h) = (p_h^-, q_h^+, u_h^+). \tag{3.4}$$

Here the points $x_{i,j}$, $\bar{x}_{i,j}$ are taken as the same with these for the diffusion equations. That is, if $x_{i,j}$, $\bar{x}_{i,j}$ are taken as Gauss points, we call the corresponding SV scheme GSV; if $x_{i,j}$, $\bar{x}_{i,j}$ are taken as interior right and left Radau points, respectively, we call the scheme RSV.

To study the stability of SV method, we first rewrite the SV scheme into its equivalent Petrov-Galerkin form. Given any $w^* \in V_h$, $\xi^* \in \overline{V}_h$, $\eta^* \in \overline{V}_h$, it is easy to check that (3.3) can be rewritten as

$$(\partial_t u_h + \partial_x \widehat{p}_h, w^*)_i = 0, \quad (p_h - \partial_x \widehat{q}_h, \xi^*)_i = 0, \quad (q_h - \partial_x \widehat{u}_h, \eta^*)_i = 0. \tag{3.5}$$

In each element V_i, $i \in \mathbb{Z}_N$, we define

$$\mathcal{D}_i(\xi, \eta; \widehat{\xi}) = (\xi, \eta_x) - \widehat{\xi}\eta^-|_{i+1/2} + \widehat{\xi}\eta^+|_{i-1/2}, \quad \mathcal{D}(\xi, \eta; \widehat{\xi}) = \sum_{i=1}^{N} \mathcal{D}_i(\xi, \eta; \widehat{\xi}). \tag{3.6}$$

Using (2.12) yields that

$$(\partial_x \widehat{v}, w^*)_i = (\partial_x \widehat{v}, w) + R_i(w_x v)_i = -\mathcal{D}_i(v, w; \widehat{v}) + R_i(w_x v) \quad \forall\, w \in U_h,\ w^* \in V_h.$$

Consequently, by defining

$$\begin{aligned} \mathcal{A}_i(\varphi, \phi, \psi; v, \xi, \eta) &= (\varphi_t, v^*)_i + (\phi, \xi^*)_i + (\psi, \eta^*)_i, \\ \mathcal{B}_i(\varphi, \phi, \psi; v, \xi, \eta) &= \mathcal{D}_i(\phi, v; \widehat{\phi}) - \mathcal{D}_i(\psi, \xi; \widehat{\psi}) - \mathcal{D}_i(\varphi, \eta; \widehat{\varphi}), \\ \mathcal{E}_i(\varphi, \phi, \psi; v, \xi, \eta) &= -R_i(\phi v_x) + R_i(\psi \xi_x) + R_i(\varphi \eta_x), \end{aligned} \tag{3.7}$$

(3.5) can be rewritten as

$$(\mathcal{A}_i - \mathcal{B}_i - \mathcal{E}_i)(u_h, p_h, q_h; v^*, \xi^*, \eta^*) = 0 \quad \forall\, v, \xi, \eta \in U_h. \tag{3.8}$$

Conversely, if u_h, q_h, $p_h \in U_h$ satisfy (3.8), then by choosing $v^* = \chi_{V_{i,j}}$, $\xi^*, \eta^* = \chi_{\overline{V}_{i,j}}$, we find that u_h, q_h, p_h also satisfy (3.3). In other words, the SV method (3.3) is equivalent to the method (3.8).

Define that

$$\begin{aligned} \mathcal{A}(\varphi, \phi, \psi; v, \xi, \eta) &= \sum_{i=1}^{N} \mathcal{A}_i(\varphi, \phi, \psi; v, \xi, \eta), \\ \mathcal{B}(\varphi, \phi, \psi; v, \xi, \eta) &= \sum_{i=1}^{N} \mathcal{B}_i(\varphi, \phi, \psi; v, \xi, \eta), \\ \mathcal{E}(\varphi, \phi, \psi; v, \xi, \eta) &= \sum_{i=1}^{N} \mathcal{E}_i(\varphi, \phi, \psi; v, \xi, \eta). \end{aligned}$$

We note that $\mathcal{E}_i(\varphi, \phi, \psi; v, \xi, \eta) = 0$ if φ, ϕ, ψ; v, ξ, η are in the finite element space, since the Gauss/Radau numerical quadrature is exact for a polynomial of

degree not more than $2k-1$. Furthermore, by a direct calculation, we have for any $v, \xi, \eta \in U_h$ that

$$
\begin{aligned}
(v_t, v^*) + \frac{1}{2}\sum_{i=1}^{N}[\eta]_{i+1/2}^2 + E_1 &= (\mathcal{A}-\mathcal{B}-\mathcal{E})(v,\xi,\eta; v,\eta,-\xi), \\
(v_{tt}, v_t^*) + \frac{1}{2}\sum_{i=1}^{N}[\eta_t]_{i+1/2}^2 + E_2 &= (\mathcal{A}-\mathcal{B}-\mathcal{E})(v_t,\xi_t,\eta_t; v_t,\eta_t,-\xi_t), \\
(\xi_t, \xi^*) + \frac{1}{2}\sum_{i=1}^{N}[v_t]_{i+1/2}^2 + E_3 &= \mathcal{A}(v,\xi_t,\eta_t; -\eta_t,\xi,v_t) \\
&\quad - (\mathcal{B}+\mathcal{E})(v_t,\xi,\eta_t; -\eta_t,\xi,v_t), \\
(\eta_t, \eta^*) + (\xi, v_t) + \sum_{i=1}^{N}[\eta][v_t]_{i+1/2} + E_4 &= \mathcal{A}(v,\xi,\eta_t; 0,v_t,\eta) \\
&\quad - (\mathcal{B}+\mathcal{E})(v_t,\xi,\eta; 0,v_t,\eta),
\end{aligned} \tag{3.9}
$$

where

$$
\begin{aligned}
E_1 &= \sum_{i=1}^{N}(R_i(\partial_x^{-1}\xi\eta_x) - R_i(\partial_x^{-1}\eta\xi_x)), \\
E_2 &= \sum_{i=1}^{N}(R_i(\partial_x^{-1}\xi_t\eta_{xt}) - R_i(\partial_x^{-1}\eta_t\xi_{xt})), \\
E_3 &= \sum_{i=1}^{N}(R_i(\partial_x^{-1}\eta_t v_{xt}) - R_i(\partial_x^{-1}v_t\eta_{xt})), \\
E_4 &= \sum_{i=1}^{N}R_i(v_{tx}\partial_x^{-1}\xi).
\end{aligned} \tag{3.10}
$$

Theorem 3.1. *Let (u_h, p_h, q_h) be the solution of* (3.3). *Then both GSV and RSV schemes are L^2 stable, i.e.,*

$$
(\|u_h\|_0^2 + \|(u_h)_t\|_0^2 + \|p_h\|_0^2 + \|q_h\|_0^2)(t) \lesssim (\|u_h\|_0^2 + \|(u_h)_t\|_0^2 + \|p_h\|_0^2 + \|q_h\|_0^2)(0). \tag{3.11}
$$

P r o o f. By choosing $(v, \xi, \eta) = (u_h, p_h, q_h)$ in (3.9) and using (3.8) and the Cauchy-Schwarz inequality, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}((u_h, u_h^*) + ((u_h)_t, (u_h^*)_t) + (p_h, p_h^*) + \frac{1}{2}(q_h, q_h^*)) \leqslant \frac{1}{4}(\|p_h\|^2 + \|(u_h)_t\|^2) + \left|\sum_{j=1}^{4} E_j\right|, \tag{3.12}
$$

where E_j, $1 \leqslant j \leqslant 4$ are given in (3.10) with $(v, \xi, \eta) = (u_h, p_h, q_h)$. The Radau numerical quadrature is exact for polynomials of degree not more than $2k$ and thus

$E_j = 0$, $1 \leqslant j \leqslant 4$ for RSV. As for GSV, we use the Gauss quadrature error to obtain

$$R_i(\partial_x^{-1}\xi\eta_x) \lesssim h^{2k}\|\partial_x^k\xi\|_{0,V_i}\|\partial_x^k\eta\|_{0,V_i} \lesssim \|\xi\|_{0,V_i}\|\eta\|_{0,V_i} \quad \forall\, \xi, \eta \in U_h.$$

Consequently,

$$|E_j| \lesssim \|u_h\|_{0,V_i}^2 + \|(u_h)_t\|_{0,V_i}^2 + \|p_h\|_{0,V_i}^2 + \|q_h\|_{0,V_i}^2, \quad 1 \leqslant j \leqslant 4.$$

Substituting the above inequality into (3.12), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}((u_h, u_h^*)+(\partial_t u_h, \partial_t u_h^*)+(p_h, p_h^*)+\frac{1}{2}(q_h, q_h^*)) \lesssim \|u_h\|^2+\frac{5}{4}\|p_h\|^2+\frac{5}{4}\|(u_h)_t\|^2+\|q_h\|^2.$$

Then (3.11) follows from the Gronwall inequality and the L^2 equivalence (2.13). □

3.2. Error estimates. Recalling that $I_h^{\pm}$ defined in (2.15), for flux (3.4), we define

$$(I_h u, I_h p, I_h q) = (I_h^+ u, I_h^- p, I_h^+ q). \tag{3.13}$$

Theorem 3.2. *Let u be the solution of* (3.1) *satisfying $\partial_t^n u \in H^{k+3}$, $0 \leqslant n \leqslant 2$, (u_h, p_h, q_h) be the solution of* (3.3) *with $u_h(x,0) = I_h u_0$. For both RSV and GSV,*

$$\|u - u_h\|_0 + \|(u-u_h)_t\|_0 + \|p - p_h\|_0 + \|q - q_h\|_0 \lesssim h^{k+1}. \tag{3.14}$$

P r o o f. Define

$$e_v = v - v_h = v - I_h v + I_h v - v_h = \eta_v + \xi_v, \quad v = (u, p, q).$$

Choose $(v, \xi, \eta) = (\xi_u, \xi_p, \xi_q)$ in (3.9), using the Cauchy-Schwarz inequality to get

$$\frac{\mathrm{d}}{\mathrm{d}t}\Big((\xi_u, \xi_u^*)+((\xi_u)_t, (\xi_u^*)_t)+(\xi_p, \xi_p^*)+\frac{1}{2}(\xi_q, \xi_q^*)\Big) \leqslant \frac{1}{4}(\|\xi_q\|_0^2+\|(\xi_u)_t\|_0^2)+\sum_{j=1}^{4}(I_j+E_j), \tag{3.15}$$

where E_j is defined in (3.10) with $(v, \xi, \eta) = (\xi_u, \xi_p, \xi_q)$, and

$$\begin{aligned}
I_1 &= (\mathcal{A}-\mathcal{B}-\mathcal{E})(\xi_u, \xi_p, \xi_q; \xi_u, -\xi_q, \xi_p),\\
I_2 &= (\mathcal{A}-\mathcal{B}-\mathcal{E})((\xi_u)_t, (\xi_p)_t, (\xi_q)_t; (\xi_u)_t, -(\xi_q)_t, (\xi_p)_t),\\
I_3 &= \mathcal{A}(\xi_u, (\xi_p)_t, (\xi_q)_t; -(\xi_q)_t, \xi_p, (\xi_u)_t) - (\mathcal{B}+\mathcal{E})((\xi_u)_t, \xi_p, (\xi_q)_t; -(\xi_q)_t, \xi_p, (\xi_u)_t),\\
I_4 &= \frac{1}{2}\mathcal{A}(\xi_u, \xi_p, (\xi_q)_t; 0, (\xi_u)_t, \xi_q) - \frac{1}{2}(\mathcal{B}+\mathcal{E})((\xi_u)_t, \xi_p, \xi_q; 0, (\xi_u)_t, \xi_q).
\end{aligned}$$

Since the exact solution u, p, q also satisfies (3.3) and (3.8), we have from the orthogonality and the property of $I_h^{\pm}$ that

$$I_1 = -(\mathcal{A} - \mathcal{B} - \mathcal{E})(\eta_u, \eta_p, \eta_q; \xi_u, -\xi_q, \xi_p) = ((\eta_u)_t, \xi_u^*) - (\eta_p, \xi_q^*) + (\eta_q, \xi_p^*).$$

Similarly, by taking time derivative for (3.3), the following orthogonality holds:

$$(\mathcal{A}_i - \mathcal{B}_i - \mathcal{E}_i)((e_u)_t, (e_p)_t, (e_q)_t; v^*, \xi^*, \eta^*) = 0 \quad \forall\, v, \xi, \eta \in U_h.$$

Then

$$\begin{aligned} I_2 &= -(\mathcal{A} - \mathcal{B} - \mathcal{E})((\eta_u)_t, (\eta_p)_t, (\eta_q)_t; (\xi_u)_t, -(\xi_q)_t, (\xi_p)_t) \\ &= ((\eta_u)_{tt}, (\xi_u^*)_t) - ((\eta_p)_t, (\xi_q^*)_t) + ((\eta_q)_t, (\xi_p^*)_t) \\ &= ((\eta_u)_{tt}, (\xi_u^*)_t) - \frac{\mathrm{d}}{\mathrm{dt}}((\eta_p)_t, \xi_q^*) + ((\eta_p)_{tt}, \xi_q^*) + \frac{\mathrm{d}}{\mathrm{dt}}((\eta_q)_t, \xi_p^*) - ((\eta_q)_{tt}, \xi_p^*). \end{aligned}$$

To estimate I_3, we take time derivative in the second and third equations of (3.3), and then substitute them into the first equation of (3.3) to obtain

$$\begin{aligned} I_3 &= \mathcal{A}(\eta_u, (\eta_p)_t, (\eta_q)_t; -(\xi_q)_t, \xi_p, (\xi_u)_t) - (\mathcal{B} + \mathcal{E})((\eta_u)_t, \eta_p, (\eta_q)_t; -(\xi_q)_t, \xi_p, (\xi_u)_t) \\ &= ((\eta_u)_t, (\xi_q^*)_t) + ((\eta_p)_t, \xi_p^*) + ((\eta_q)_t, (\xi_u^*)_t) \\ &= ((\eta_u)_{tt}, \xi_q^*) - \frac{\mathrm{d}}{\mathrm{dt}}((\eta_u)_t, \xi_q^*) + ((\eta_p)_t, \xi_p^*) + ((\eta_q)_t, (\xi_u^*)_t). \end{aligned}$$

Following the same argument, we derive

$$I_4 = \frac{1}{2}((\eta_q)_t, \xi_q^*) + \frac{1}{2}(\eta_p, (\xi_u^*)_t).$$

Use the approximation properties of $I_h^{\pm}$ and the Cauchy-Schwarz inequality to get

$$\left| \sum_{j=1}^{4} I_j \right| \lesssim h^{k+1}(\|\xi_u^*\|_0 + \|(\xi_u^*)_t\|_0 + \|\xi_q^*\|_0 + \|\xi_p^*\|_0).$$

On the other hand, follow the same argument as in Theorem 3.1 to obtain

$$\left| \sum_{j=1}^{4} E_j \right| \lesssim (\|\xi_u\|_0 + \|(\xi_u)_t\|_0 + \|\xi_q\|_0 + \|\xi_p\|_0).$$

Substituting the estimates of I_j, E_j into (3.15) and using L^2 equivalence (2.13) yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{dt}}((\xi_u, \xi_u^*) + ((\xi_u)_t, (\xi_u^*)_t) + (\xi_p, \xi_p^*) + (\xi_q, \xi_q^*)) \\ &\qquad \lesssim h^{2k+2} + \|\xi_u\|_0^2 + \|(\xi_u)_t\|_0^2 + \|\xi_p\|_0^2 + \|\xi_q\|_0^2. \end{aligned}$$

Using the L^2 equivalence (2.13) again and the Gronwall inequality, we obtain

$$\|\xi_u\|_0^2 + \|(\xi_u)_t\|_0^2 + \|\xi_p\|_0^2 + \|\xi_q\|_0^2 \lesssim h^{2k+2}.$$

Then the desired result (3.14) follows from the triangle inequality and the approximation properties of $I_h^{\pm}$. The proof is complete. □

3.3. Superconvergence results. Similarly to the diffusion case, we need to construct correction functions to obtain the superconvergence results. To this end, we first define $w_{u,0} = u - I_h u$, $w_{p,0} = p - I_h p$, $w_{q,0} = q - I_h q$. For all $1 \leqslant l \leqslant k$, let $w_{u,l}, w_{p,l}, w_{q,l} \in U_h$ satisfy

$$\begin{aligned}
&(w_{p,l}, v_x)_i = (\partial_t w_{u,l-1}, v^*), \quad w_{p,l}(x_{i+1/2}^-) = 0,\\
&(w_{q,l}, v_x)_i = -(w_{p,l-1}, v^*), \quad w_{q,l}(x_{i-1/2}^+) = 0,\\
&(w_{u,l}, v_x)_i = -(w_{q,l-1}, v^*), \quad w_{u,l}(x_{i-1/2}^+) = 0,
\end{aligned} \tag{3.16}$$

where $v \in \mathbb{P}_k(V_i) \setminus \mathbb{P}_0(V_i)$. We define the correction functions as

$$w_u = \sum_{l=1}^{k} w_{u,l}, \quad w_p = \sum_{l=1}^{k} w_{p,l}, \quad w_q = \sum_{l=1}^{k} w_{q,l}. \tag{3.17}$$

Suppose that $w_{v,l} \in U_h$, $v = (u, p, q)$ has the following expansion in each V_i:

$$w_{v,l}|_{V_i} = \sum_{m=0}^{k} c_{i,m}^{v,l} L_{i,m}(x).$$

Following the same arguments as in Lemma 2.1, we have

$$|\partial_t^n c_{i,m}^{v,l}| \leqslant h_i^{k_{l,m}} \|\partial_t^n v\|_{k_{l,m},\infty}, \quad v = (u, p, q). \tag{3.18}$$

For GSV, $k_{l,m} \leqslant \max(2k-m, k+l+1)$, and $k_{l,m} \leqslant \max(2k-m+1, k+l+1)$ for RSV.

With the correction functions, we define the special interpolation functions by

$$(u_I, p_I, q_I) := (I_h u + w_u, I_h p + w_p, I_h q + w_q). \tag{3.19}$$

Apparently, we have from (3.13)

$$(\widehat{u} - \widehat{u}_I, \widehat{p} - \widehat{p}_I, \widehat{q} - \widehat{q}_I)|_{i+1/2} = (\widehat{u} - \widehat{I_h u}, \widehat{p} - \widehat{I_h p}, \widehat{q} - \widehat{I_h q})|_{i+1/2} = 0. \tag{3.20}$$

We adopt the following notation in the rest of this paper:

$$e_v = v - v_h = v - v_I + v_I - v_h = \varepsilon_v + \bar{e}_v, \quad v = (u, p, q).$$

In light of (3.16), we have for all $v \in \mathbb{P}_k(V_i) \setminus \mathbb{P}_0(V_i)$

$$
\begin{aligned}
((\varepsilon_u)_t, v^*)_i - \mathcal{D}_i(w_p, v; w_p^-) &= -((w_{u,k})_t, v^*)_i,\\
(\varepsilon_p, v^*)_i + \mathcal{D}_i(w_q, v; w_q^+) &= -(w_{p,k}, v^*)_i,\\
(\varepsilon_q, v^*)_i + \mathcal{D}_i(w_u, v; w_u^+) &= -(w_{q,k}, v^*)_i.
\end{aligned} \tag{3.21}
$$

We have the following supercloseness result between the SV solution (u_h, p_h, q_h) and the interpolation function (u_I, p_I, q_I) defined in (3.19).

Theorem 3.3. *Let u be the solution of* (3.1) *satisfying $\partial_t^n u \in W^{2k+3,\infty}(\Omega)$ for all $n \leqslant 2$, and (u_h, p_h, q_h) be the solution of* (3.3). *Suppose $u_I, p_I, q_I \in U_h$ is defined in* (3.19). *Suppose the initial solution is chosen such that*

$$
(\|\bar{e}_u\|_0 + \|(\bar{e}_u)_t\|_0 + \|\bar{e}_p\|_0 + \|\bar{e}_q\|_0)(0) \lesssim h^{2k+1}. \tag{3.22}
$$

Then

$$
(\|\bar{e}_u\|_0 + \|(\bar{e}_u)_t\|_0 + \|\bar{e}_p\|_0 + \|\bar{e}_q\|_0)(t) \lesssim h^{2k+r} c(u), \tag{3.23}
$$

where $c(u) = \sup\limits_{\tau\in[0,t]} \sum\limits_{n=0}^{2} \|\partial_t^n u(\tau)\|_{2k+1,\infty}$, $r = 0$ *for GSV and* $r = 1$ *for RSV.*

P r o o f. Choosing $(v, \xi, \eta) = (\bar{e}_u, \bar{e}_p, \bar{e}_q)$ in (3.9) and using the orthogonality and the Cauchy-Schwarz inequality, we have

$$
\frac{\mathrm{d}}{\mathrm{d}t}\Big((\bar{e}_u, \bar{e}_u^*) + (\partial_t \bar{e}_u, \partial_t \bar{e}_u^*) + (\bar{e}_p, \bar{e}_p^*) + \frac{1}{2}(\bar{e}_q, \bar{e}_q^*)\Big) \leqslant \frac{1}{4}(\|\bar{e}_q\|_0^2 + \|(\bar{e}_u)_t\|_0^2) + \sum_{j=1}^{4}(I_j + E_j), \tag{3.24}
$$

where E_j are defined in (3.10) with $(v, \xi, \eta) = (\bar{e}_u, \bar{e}_p, \bar{e}_q)$ and

$$
\begin{aligned}
I_1 &= (\mathcal{A} - \mathcal{B} - \mathcal{E})(\varepsilon_u, \varepsilon_p, \varepsilon_q; \bar{e}_u, -\bar{e}_q, \bar{e}_p),\\
I_2 &= (\mathcal{A} - \mathcal{B} - \mathcal{E})((\varepsilon_u)_t, (\varepsilon_p)_t, (\varepsilon_q)_t; (\bar{e}_u)_t, -(\bar{e}_q)_t, (\bar{e}_p)_t),\\
I_3 &= \mathcal{A}(\varepsilon_u, (\varepsilon_p)_t, (\varepsilon_q)_t; -(\bar{e}_q)_t, \bar{e}_p, (\bar{e}_u)_t) - (\mathcal{B} + \mathcal{E})((\varepsilon_u)_t, \varepsilon_p, (\varepsilon_q)_t; -(\bar{e}_q)_t, \bar{e}_p, (\bar{e}_u)_t),\\
I_4 &= \frac{1}{2}\mathcal{A}(\varepsilon_u, \varepsilon_p, (\varepsilon_q)_t; 0, (\bar{e}_u)_t, \bar{e}_q) - \frac{1}{2}(\mathcal{B} + \mathcal{E})((\varepsilon_u)_t, \varepsilon_p, \varepsilon_q; 0, (\bar{e}_u)_t, \bar{e}_q).
\end{aligned} \tag{3.25}
$$

As for E_j, following the same argument as in Theorem 3.1, we get

$$
\Big|\sum_{j=1}^{4} E_j\Big| \lesssim (\|\bar{e}_u\|_0 + \|(\bar{e}_u)_t\|_0 + \|\bar{e}_q\|_0 + \|\bar{e}_p\|_0).
$$

We next estimate I_j, $1 \leqslant j \leqslant 4$, respectively. Notice that $\bar{e}_u$ can be decomposed into $\bar{e}_u = \bar{e}_{u0} + \bar{e}_{u1}$ with $\bar{e}_{u0}|_{V_i} \in \mathbb{P}_0$ and $\bar{e}_{u1} \in \mathbb{P}_k \setminus \mathbb{P}_0$. Then

$$\begin{aligned} I_1 := I_{10} + I_{11} = (\mathcal{A} - \mathcal{B} - \mathcal{E})(\varepsilon_u, \varepsilon_p, \varepsilon_q; \bar{e}_{u0}, -\bar{e}_{q0}, \bar{e}_{p0}) \\ + (\mathcal{A} - \mathcal{B} - \mathcal{E})(\varepsilon_u, \varepsilon_p, \varepsilon_q; \bar{e}_{u1}, -\bar{e}_{q1}, \bar{e}_{p1}). \end{aligned}$$

Similarly, we obtain $I_j = I_{j0} + I_{j1}$, $2 \leqslant j \leqslant 4$. By using the property of $I_h^{\pm}$, the definition of $w_{v,l}$ in (3.16) and (3.21), we have

$$\begin{aligned} I_{11} &= \mathcal{A}(u - I_h u, p - I_h p, q - I_h q; \bar{e}_u, -\bar{e}_q, \bar{e}_p) - (\mathcal{A} - \mathcal{B} - \mathcal{E})(w_u, w_p, w_q; \bar{e}_u, -\bar{e}_q, \bar{e}_p) \\ &= ((w_{u,k})_t, \bar{e}_u^*) - (w_{p,k}, \bar{e}_q^*) + (w_{q,k}, \bar{e}_p^*). \end{aligned}$$

Similarly, there hold

$$\begin{aligned} I_{21} &= ((w_{u,k})_{tt}, (\bar{e}_u^*)_t) - ((w_{p,k})_t, (\bar{e}_q^*)_t) + ((w_{q,k})_t, (\bar{e}_p^*)_t) \\ &= ((w_{u,k})_{tt}, (\bar{e}_u^*)_t) - \frac{\mathrm{d}}{\mathrm{d}t}((w_{p,k})_t, \bar{e}_q^*) + ((w_{p,k})_{tt}, \bar{e}_q^*) \\ &\quad + \frac{\mathrm{d}}{\mathrm{d}t}((w_{q,k})_t, \bar{e}_p^*) - ((w_{q,k})_{tt}, \bar{e}_p^*), \\ I_{31} &= ((w_{u,k})_t, (\bar{e}_q^*)_t) + ((w_{p,k})_t, \bar{e}_p^*) + ((w_{q,k})_t, (\bar{e}_u^*)_t) \\ &= ((w_{u,k})_{tt}, \bar{e}_q^*) - \frac{\mathrm{d}}{\mathrm{d}t}((w_{u,k})_t, \bar{e}_q^*) + ((w_{p,k})_t, \bar{e}_p^*) + ((w_{q,k})_t, (\bar{e}_u^*)_t), \\ I_{41} &= \frac{1}{2}((w_{q,k})_t, \bar{e}_q^*) + \frac{1}{2}(w_{p,k}, (\bar{e}_u^*)_t). \end{aligned}$$

Using (3.18) and Cauchy-Schwarz inequality, we have for all $1 \leqslant j \leqslant 4$

$$\begin{aligned} |I_{j1}| \lesssim h^{2k+1} \sum_{n=0}^{2} (\|\partial_t^n u\|_{2k+1,\infty} + \|\partial_t^n p\|_{2k+1,\infty} + \|\partial_t^n q\|_{2k+1,\infty}) \\ \times (\|\bar{e}_u^*\|_0 + \|(\bar{e}_u^*)_t\|_0 + \|\bar{e}_p^*\|_0 + \|\bar{e}_q^*\|_0). \end{aligned}$$

As for $I_{j,0}$, we again use the estimates in (3.18) and (2.25) to derive

$$\begin{aligned} |I_{j0}| \lesssim h^{2k+r} \sum_{n=0}^{2} (\|\partial_t^n u\|_{2k+1,\infty} + \|\partial_t^n p\|_{2k+1,\infty} + \|\partial_t^n q\|_{2k+1,\infty}) \\ \times (\|\bar{e}_u^*\|_0 + \|(\bar{e}_u^*)_t\|_0 + \|\bar{e}_p^*\|_0 + \|\bar{e}_q^*\|_0) \end{aligned}$$

with $r = 0$ for GSV and $r = 1$ for RSV. Consequently, we have from the identity $q = u_x$, $p = u_{xx}$,

$$|I_j| \lesssim h^{2k+r} c(u)(\|\bar{e}_u^*\|_0 + \|(\bar{e}_u^*)_t\|_0 + \|\bar{e}_p^*\|_0 + \|\bar{e}_q^*\|_0).$$

Substituting the estimates of I_j, E_j into (3.24) and using (2.13) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}((\bar{e}_u, \bar{e}_u^*) + ((\bar{e}_u)_t, (\bar{e}_u^*)_t) + (\bar{e}_p, \bar{e}_p^*) + (\bar{e}_q, \bar{e}_q^*)) \\ \lesssim h^{4k+2r} c(u)^2 + (\|\bar{e}_u^*\|_0^2 + \|(\bar{e}_u^*)_t\|_0^2 + \|\bar{e}_p^*\|_0^2 + \|\bar{e}_q^*\|_0^2).$$

By using the Gronwall inequality, (3.22) and the L^2 equivalence (2.13) again, we obtain the desired result (3.23). □

R e m a r k 3.1. We would like to point out that to ensure the initial error condition in (3.22), a nature way of initial discretization is to choose $q_h(x, 0) = q_I(x, 0)$. We refer to Lemma 4 in [5] for a detailed discussion on initial discretization.

Theorem 3.4. *Suppose all the conditions of Theorem* 3.3 *hold true. Then for both GSV and RSV and* $v = (u, p, q)$,

$$e_{v,c} + e_{v,n} \lesssim h^{2k+r} c(u), \quad e_{v,g} \lesssim h^{k+2} c(u), \quad e_{v,d} \lesssim h^{k+1} c(u), \tag{3.26}$$

where $r = 0$ *for GSV and* $r = 1$ *for RSV, and* $c(u)$ *is the same as that in Theorem* 3.3, *and* $e_{v,c}$, $e_{v,n}$, $e_{v,g}$ *and* $e_{v,d}$ *are given in Theorem* 2.3. *The superconvergenet points for the function value and derivative approximations are identified as* $\bar{x}_{i,j}$ *and* $z_{i,j}$ *for variable* u, q, *and* $x_{i,j}$ *and* $y_{i,j}$ *for variable* p.

Here we omit the proof, since it is similar to that of Theorem 2.3.

R e m a r k 3.2. Similarly to the diffusion case, if we only consider the superconvergence proerty at interior points (i.e., the errors $e_{v,g}$, $e_{v,d}$), the regularity assumption $\partial_t^n u \in W^{k+2,\infty}$, $0 \leqslant n \leqslant 2$, $u \in W^{k+4,\infty}$ is sufficient to guarantee the desired superconvergence results.

4. Numerical Experiments

In this section, we present numerical examples to validate theoretical findings. In numerical experiments, we measure the error $e_{v,c}$, $e_{v,n}$, $e_{v,d}$ and $e_{v,g}$ defined in (2.35) and Theorem 2.3. To diminish the time discretization error, we employ the third-order Runge-Kutta method with a step size $\Delta t = 0.0001h^2$. We use the $2k$-point Gauss numerical integration to calculate all integrals.

E x a m p l e 4.1. We solve the following problem:

$$u_t = u_{xx}, \quad u(x, 0) = \sin x, \quad u(0, t) = u(2\pi, t), \quad (x, t) \in [0, 2\pi] \times (0, 0.1].$$

The exact solution is $u(x, t) = e^{-t} \sin x$. In numerical experiments, we use (2.4) as the numerical flux, and create meshes by equally dividing the interval $[0, 2\pi]$ into N subintervals. We compute the numerical solution at $T = 0.1$.

k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	8	2.10E–04	–	7.08E–05	–	4.32E–04	–	6.09E–03	–
	16	1.50E–05	3.81	2.48E–06	4.83	2.94E–05	3.88	7.50E–04	3.02
2	32	9.28E–07	4.01	8.59E–08	4.85	1.82E–06	4.01	9.33E–05	3.01
	64	5.78E–08	4.00	3.58E–09	4.58	1.13E–07	4.00	1.16E–05	3.00
	128	3.61E–09	4.00	1.86E–10	4.27	7.09E–09	4.00	1.45E–06	3.00
k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	8	2.49E–06	–	1.19E–06	–	2.23E–05	–	3.00E–04	–
	16	3.74E–08	6.06	1.97E–08	5.91	7.18E–07	4.95	1.85E–05	4.02
3	32	5.83E–10	6.00	3.09E–10	6.00	2.24E–08	5.00	1.16E–06	4.00
	64	9.10E–12	6.00	4.80E–12	6.00	6.99E–10	5.00	7.21E–08	4.00
	128	1.00E–13	5.98	1.00E–13	5.96	2.18E–11	5.00	4.51E–09	4.00

Table 1. Example 4.1 errors and convergence rates of u for GSV method.

k	N	$e_{q,n}$	rate	$e_{q,c}$	rate	$e_{q,g}$	rate	$e_{q,d}$	rate
	8	4.82E–04	–	2.04E–04	–	5.36E–04	–	5.49E–03	–
	16	2.92E–05	4.04	1.49E–05	3.78	3.35E–05	4.00	7.49E–04	2.87
2	32	1.81E–06	4.01	9.26E–07	4.01	2.07E–06	4.02	9.32E–05	3.01
	64	1.13E–07	4.00	5.78E–08	4.00	1.29E–07	4.01	1.16E–05	3.00
	128	7.05E–09	4.00	3.61E–09	4.00	8.03E–09	4.00	1.45E–06	3.00
k	N	$e_{q,n}$	rate	$e_{q,c}$	rate	$e_{q,g}$	rate	$e_{q,d}$	rate
	8	1.64E–06	–	2.43E–06	–	2.25E–05	–	2.93E–04	–
	16	5.48E–08	4.90	3.71E–08	6.03	7.18E–07	4.97	1.85E–05	3.98
3	32	8.55E–10	6.00	5.82E–10	6.00	2.24E–08	5.00	1.16E–06	4.00
	64	1.34E–11	6.00	9.10E–12	6.00	6.99E–10	5.00	7.21E–08	4.00
	128	2.00E–13	5.97	1.00E–13	5.98	2.18E–11	5.00	4.51E–09	4.00

Table 2. Example 4.1 errors and convergence rates of q for GSV method.

Tables 1–4 list errors and convergence rates calculated for the GSV and RSV with $k = 2, 3$. They demonstrate superconvergence rates of $2k$th order for GSV and $(2k+1)$th order for RSV for the numerical cell averages and numerical approximation at the flux points (i.e., $e_{v,c}$ and $e_{v,n}$), $(k+2)$th order for the numerical solution at the Gauss points (Radau points) ($e_{v,g}$), and $(k+1)$th order for the partial derivatives of the approximation at the roots of $\partial_x(L_{i,k}(x - x_{i+1/2}))$ and $\partial_x(L_{i,k}(x + x_{i+1/2}))$ (interior Radau points) (i.e., $e_{v,d}$) with $v = (u, q)$. All these numerical results are consistent with the theoretical findings in Theorem 2.3.

k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	8	1.12E–05	–	1.65E–05	–	1.20E–04	–	3.79E–03	–
	16	3.42E–07	5.03	5.03E–07	5.03	7.23E–06	4.05	4.74E–04	3.00
2	32	1.04E–08	5.03	1.56E–08	5.01	4.46E–07	4.02	5.92E–05	3.00
	64	3.25E–10	5.01	4.87E–10	5.00	2.78E–08	4.00	7.40E–06	3.00
	128	1.01E–11	5.00	1.52E–11	5.00	1.74E–09	4.00	9.26E–07	3.00
k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	20	7.92E–11	–	5.94E–11	–	3.51E–08	–	4.68E–06	–
	25	1.65E–11	7.02	1.24E–11	7.02	1.15E–08	5.01	1.92E–06	4.00
3	30	4.61E–12	7.01	3.45E–12	7.01	4.61E–09	5.00	9.25E–07	4.00
	35	1.56E–12	7.01	1.17E–12	7.01	2.13E–09	5.00	5.00E–07	4.00
	40	6.14E–13	7.00	4.59E–13	7.02	1.09E–09	5.00	2.93E–07	4.00

Table 3. Example 4.1 errors and convergence rates of u for RSV method.

k	N	$e_{q,n}$	rate	$e_{q,c}$	rate	$e_{q,g}$	rate	$e_{q,d}$	rate
	8	1.07E–05	–	1.09E–05	–	1.16E–04	–	3.78E–03	–
	16	5.07E–07	4.40	3.40E–07	5.01	7.23E–06	4.01	4.74E–04	3.00
2	32	1.56E–08	5.02	1.04E–08	5.02	4.46E–07	4.02	5.92E–05	3.00
	64	4.87E–10	5.01	3.25E–10	5.01	2.78E–08	4.00	7.40E–06	3.00
	128	1.52E–11	5.00	1.01E–11	5.00	1.74E–09	4.00	9.26E–07	3.00
k	N	$e_{q,n}$	rate	$e_{q,c}$	rate	$e_{q,g}$	rate	$e_{q,d}$	rate
	20	5.97E-11	–	7.89E-11	–	3.51E–08	–	4.68E–06	–
	25	1.24E–11	7.03	1.65E–11	7.01	1.15E–08	5.01	1.92E–06	4.00
3	30	3.46E–12	7.02	4.60E–12	7.01	4.61E–09	5.00	9.25E–07	4.00
	35	1.17E–12	7.03	1.56E–12	7.01	2.13E–09	5.00	5.00E–07	4.00
	40	4.61E–13	6.98	6.13E–13	7.00	1.09E–09	5.00	2.93E–07	4.00

Table 4. Example 4.1 errors and convergence rates of q for RSV method.

E x a m p l e 4.2. Solve the third-order problem with periodic boundary condition

$$u_t + u_{xxx} = 0, \quad u(x,0) = \sin x, \quad (x,t) \in [0, 2\pi] \times (0, 0.1].$$

The exact solution is $u(x,t) = \sin(x+t)$. Tables 5–6 present errors and convergence rates calculated by GSV and RSV. We observe that $e_{u,c}$, $e_{u,n}$ converge with an order of $2k$ for GSV and $2k+1$ for RSV, a convergence rate of $k+2$ for $e_{u,g}$ and of $k+1$ for $e_{u,d}$ both RSV and GSV. These results indicate the results in Theorem 3.4. The same results can be obtained for p, q, which we omitted due to the length of the paper.

k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	8	3.21E–04	–	6.04E–05	–	5.43E–04	–	2.75E–03	–
	16	2.02E–05	3.99	4.48E–06	3.75	3.29E–05	4.05	3.31E–04	3.05
2	32	1.26E–06	4.00	3.26E–07	3.78	2.04E–06	4.01	4.08E–05	3.02
	64	7.84E–08	4.00	2.20E–08	3.89	1.27E–07	4.00	5.08E–06	3.01
	128	4.91E–09	4.00	1.47E–09	3.91	7.94E–09	4.00	6.33E–07	3.00
k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	15	6.63E–08	–	3.87E–08	–	1.09E–06	–	1.62E–05	–
	20	1.18E–08	5.99	6.91E–09	5.99	2.59E–07	5.00	5.12E–06	4.00
3	25	3.11E–09	5.99	1.81E–09	5.99	8.49E–08	5.00	2.10E–06	4.00
	30	1.04E–09	5.99	6.08E–10	5.99	3.41E–08	5.00	1.01E–06	4.00
	35	4.14E–10	5.99	2.42E–10	5.99	1.58E–08	5.00	5.45E–07	4.00

Table 5. Example 4.2 errors and convergence rates of u for GSV method.

k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	4	1.89E–03	–	1.95E–03	–	8.84E–03	–	3.31E–02	–
	8	1.16E–05	7.36	4.11E–05	5.57	3.79E–04	4.55	4.17E–03	2.99
2	16	3.71E–07	4.96	1.29E–06	5.00	2.36E–05	4.00	5.23E–04	2.99
	32	1.17E–08	4.98	4.02E–08	5.00	1.47E–06	4.00	6.54E–05	3.00
	64	3.84E-10	4.93	1.27E–09	4.98	9.21E–08	4.00	8.18E–06	3.00
k	N	$e_{u,n}$	rate	$e_{u,c}$	rate	$e_{u,g}$	rate	$e_{u,d}$	rate
	20	4.31E–10	–	5.27E–11	–	1.55E–07	–	5.17E–06	–
	25	4.51E–11	10.12	1.35E–11	6.09	5.06E–08	5.01	2.12E–06	4.00
3	30	1.26E–11	6.99	3.79E–12	6.98	2.04E–08	5.00	1.02E–06	4.00
	35	4.29E–12	7.00	1.29E–12	7.00	9.42E–09	5.00	5.52E–07	4.00
	40	1.69E–12	7.00	5.05E–13	7.01	4.83E–09	5.00	3.24E–07	4.00

Table 6. Example 4.2 errors and convergence rates of u for RSV method.

5. Concluding remarks

In this paper, we explore the superconvergence property of SV methods for partial differential equations involving higher-order derivatives. Our model problems include second- and third-order equations. We establish a superconvergence order of $(2k+1)$ and $2k$ for the error of numerical flux at nodes and cell averages, $k+2$ for the function value approximation at Gauss and also Radau points, and $k+1$ for the derivative value approximation at points $y_{i,j}$, $z_{i,j}$. These results hold true for

the exact solution u and the auxiliary variables. The key idea to derive the desired superconvergence result is to use the correction function to design special interpolation functions that are superclose to the SV solution. Our ongoing work includes the superconvergence study of the SV methods for nonlinear KdV and Cahn-Hilliard equations.

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