Sogol Niazian; Mona Aaly Kologani; Rajab Ali Borzooei
Relative co-annihilators in lattice equality algebras

Mathematica Bohemica, Vol. 149 (2024), No. 4, 585–602

Persistent URL: http://dml.cz/dmlcz/152681

149 (2024) MATHEMATICA BOHEMICA No. 4, 585–602

RELATIVE CO-ANNIHILATORS IN LATTICE EQUALITY ALGEBRAS

Sogol Niazian, Tehran, Mona Aaly Kologani, Zahedan,
Rajab Ali Borzooei, Tehran

Received August 9, 2023. Published online April 3, 2024.
Communicated by Miroslav Ploščica

Abstract. We introduce the notion of relative co-annihilator in lattice equality algebras and investigate some important properties of it. Then, we obtain some interesting relations among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Given a lattice equality algebra $\mathbb{E}$ and $\mathbb{F}$ a filter of $\mathbb{E}$, we define the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ and show that by defining some operations on it, it makes a BL-algebra.

Keywords: equality algebra; annihilator; co-annihilator; relative co-annihilator; filter

MSC 2020: 03G10, 06B99, 06B75

1. Introduction

Equality algebras were introduced by Jenei in [9] and are assumed for possible algebraic semantics of fuzzy type theory (FTT). It was proved in [4], [9] that any equality algebra has a corresponding BCK-meet-semilattice and any BCK(D)-meet-semilattice (with distributivity property) has a corresponding equality algebra. From a logical point of view, various filters have natural interpretation as various sets of provable formulas, which has a very close relationship with decision-making.

Davey studied the relationship between minimal prime ideals conditions and annihilators conditions on distributive lattices, see [5]. Turunen defined co-annihilator of a nonempty subset of a BL-algebra and proved some of its properties (see [17]). Leustean introduced the notion of co-annihilator relative to a filter F on pseudo BL-algebras (see [11]). Then Meng introduced generalized co-annihilators in BL-algebras and gave characterizations of prime and minimal prime filters (see [14]). Also, Zou et al. introduced the notion of annihilators in BL-algebras and investigated some related properties of them in [20]. Filipoiu in [6] used the notion of

DOI: 10.21136/MB.2024.0120-23

annihilator for Baer extensions of MV-algebras. In [1], [8] the notion of annihilators was studied for BCK-algebras. Leustean in [12] used the notion of co-annihilator for Baer extensions of BL-algebras. Recently, as the generalization of the co-annihilator in a BL-algebra, Saeid et al. in [13] introduced the co-annihilator of a set relative to another set in a residuated lattice, where they gave some relations between filters and co-annihilators. It is helpful for the co-annihilators to study structures and properties in algebraic systems.

In this paper, we introduce the notion of co-annihilator in equality algebras. We study basic properties of co-annihilators and investigate the relationship between them and some special types of filters. Also, we obtain some interesting relations among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Moreover, we define the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ and show that by defining some operations on it, it makes a BL-algebra.

The paper is organized as follows: In Section 2, we gather the basic notions and results on equality algebras. In Section 3, we introduce the notion of co-annihilator relative to a filter in equality algebras and get some interesting results about them. Then, we study the relation among $\vee$-irreducible filters, positive implicative filters, prime filters and relative co-annihilators. Finally, we prove that the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ can make a BL-algebra.

2. Preliminaries

In this section, we gather some basic notions and results relevant to the equality algebra, which will be needed in the next sections. For a survey of equality algebras we refer to [19].

Definition 2.1 ([9]). An algebraic structure $\mathbb{E} = (\mathbb{E}; \wedge, \sim, 1)$ of type $(2, 2, 0)$ is called an *equality algebra* if for all $\alpha, \gamma, \eta \in \mathbb{E}$ it satisfies the following conditions:
(E1) $(\mathbb{E}, \wedge, 1)$ is a commutative idempotent integral monoid,
(E2) $\alpha \sim \gamma = \gamma \sim \alpha$,
(E3) $\alpha \sim \alpha = 1$,
(E4) $\alpha \sim 1 = \alpha$,
(E5) $\alpha \leqslant \gamma \leqslant \eta$ implies $\alpha \sim \eta \leqslant \gamma \sim \eta$ and $\alpha \sim \eta \leqslant \alpha \sim \gamma$,
(E6) $\alpha \sim \gamma \leqslant (\alpha \wedge \eta) \sim (\gamma \wedge \eta)$,
(E7) $\alpha \sim \gamma \leqslant (\alpha \sim \eta) \sim (\gamma \sim \eta)$.

The operation $\wedge$ is called *meet* and $\sim$ is an *equality* operation. On an equality algebra $\mathbb{E}$ we write $\alpha \leqslant \gamma$ if and only if $\alpha \wedge \gamma = \alpha$. It is easy to see that "$\leqslant$" is a partial order relation on $\mathbb{E}$. Also, other two derived operations are defined, as the following, and we call them *implication* and *equivalence*, respectively:

$$\alpha \to \gamma = \alpha \sim (\alpha \wedge \gamma) \quad \text{and} \quad \alpha \leftrightarrow \gamma = (\alpha \to \gamma) \wedge (\gamma \to \alpha).$$

An equality algebra $\mathbb{E}$ is *bounded* if there is an element $0 \in \mathbb{E}$ such that $0 \leqslant \alpha$ for all $\alpha \in \mathbb{E}$. A *lattice equality algebra* is an equality algebra which is a lattice.

Proposition 2.2 ([9], [16], [19]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra. Then for all $\alpha, \gamma, \eta \in \mathbb{E}$, the following conditions hold:*
(i) $\alpha \to \gamma = 1$ *if and only if* $\alpha \leqslant \gamma$,
(ii) $1 \to \alpha = \alpha$, $\alpha \to 1 = 1$, *and* $\alpha \to \alpha = 1$,
(iii) $\alpha \leqslant \gamma \to \alpha$,
(iv) $\alpha \leqslant (\alpha \to \gamma) \to \gamma$,
(v) $\alpha \to (\gamma \to \eta) = \gamma \to (\alpha \to \eta)$,
(vi) $\alpha \leqslant \gamma$ *implies* $\gamma \to \eta \leqslant \alpha \to \eta$ *and* $\eta \to \alpha \leqslant \eta \to \gamma$.
If $\mathbb{E}$ is a lattice equality algebra, then
(vii) $\alpha \to \gamma = (\alpha \vee \gamma) \to \gamma$.

Definition 2.3 ([10]). Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$. Then $\mathbb{F}$ is called a *deductive system* or *filter* of $\mathbb{E}$ if for all $\alpha, \gamma \in \mathbb{E}$ we have
(i) $\alpha \in \mathbb{F}$ and $\alpha \leqslant \gamma$ imply $\gamma \in \mathbb{F}$;
(ii) $\alpha \in \mathbb{F}$ and $\alpha \sim \gamma \in \mathbb{F}$ imply $\gamma \in \mathbb{F}$.

Proposition 2.4 ([2], [4], [10]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$. Then $\mathbb{F}$ is a filter of $\mathbb{E}$ if and only if for all $\alpha, \gamma \in \mathbb{E}$*
(i) $1 \in \mathbb{F}$,
(ii) $\alpha \in \mathbb{F}$ *and* $\alpha \to \gamma \in \mathbb{F}$ *imply* $\gamma \in \mathbb{F}$.

The set of all filters of $\mathbb{E}$ is denoted by $\mathcal{F}(\mathbb{E})$. Clearly, $1 \in \mathbb{F}$ for all $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. A filter $\mathbb{F}$ of $\mathbb{E}$ is called a *proper filter* if $\mathbb{F} \neq \mathbb{E}$. Clearly, if $\mathbb{E}$ is a bounded equality algebra, then a filter is proper if and only if it does not contain 0. A proper filter $\mathbb{F}$ of $\mathbb{E}$ is called a *prime filter* if $\alpha \to \gamma \in \mathbb{F}$ or $\gamma \to \alpha \in \mathbb{F}$ for all $\alpha, \gamma \in \mathbb{E}$. A *maximal filter* (or ultra filter) is a proper filter of $\mathbb{E}$ that is not included in any other proper filter. The set of all prime (maximal) filters of $\mathbb{E}$ is denoted by $\mathrm{Prime}(\mathbb{E})$ ($\mathrm{Max}(\mathbb{E})$).

Definition 2.5 ([4]). Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\theta \subseteq \mathbb{E} \times \mathbb{E}$. Then θ is called a *congruence* relation of $\mathbb{E}$ if it is an equivalence relation on $\mathbb{E}$ and if $(\alpha_1, \gamma_1), (\alpha_2, \gamma_2) \in \theta$,

$$(\alpha_1 \wedge \alpha_2, \gamma_1 \wedge \gamma_2) \in \theta, \quad (\alpha_1 \sim \alpha_2, \gamma_1 \sim \gamma_2) \in \theta$$

for all $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in \mathbb{E}$.

The set of all congruences of $\mathbb{E}$ is denoted by $\mathrm{Con}(\mathbb{E})$. For any $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, a binary relation $\theta_{\mathbb{F}}$ associated by defining: $\alpha\, \theta_{\mathbb{F}}\, \gamma$ if and only if $\alpha \sim \gamma \in \mathbb{F}$. In [4], it is proved that there is a one-to-one correspondence between $\mathcal{F}(\mathbb{E})$ and $\mathrm{Con}(\mathbb{E})$. Denote $\mathbb{E}/\mathbb{F} = \mathbb{E}/\theta_{\mathbb{F}} := \{[\alpha]\colon\ \alpha \in \mathbb{E}\}$, where $[\alpha] := \{\gamma \in \mathbb{E}\colon\ (\alpha, \gamma) \in \theta_{\mathbb{F}}\}$.

Theorem 2.6 ([4]). *Let $(\mathbb{E}; \wedge, \sim, 1)$ be an equality algebra and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{E}/\mathbb{F}; \bar{\wedge}, \bar{\sim}, \mathbb{F})$ is an equality algebra with the following operations:*

$$[\alpha]\bar{\wedge}[\gamma] := [\alpha \wedge \gamma], \quad [\alpha] \mathrel{\bar{\sim}} [\gamma] := [\alpha \sim \gamma]$$

for all $\alpha, \gamma \in \mathbb{E}$.

Definition 2.7 ([2]). Let $\mathbb{F}$ be a nonempty subset of $\mathbb{E}$ such that $1 \in \mathbb{F}$. Then $\mathbb{F}$ is called a *positive implicative filter* if $\alpha \to (\gamma \to \eta) \in \mathbb{F}$ and $\alpha \to \gamma \in \mathbb{F}$ imply $\alpha \to \eta \in \mathbb{F}$ for all $\alpha, \gamma, \eta \in \mathbb{E}$.

Let $\mathcal{X} \subseteq \mathbb{E}$. The smallest filter of $\mathbb{E}$ containing $\mathcal{X}$ is called the *generated filter by $\mathcal{X}$ in $\mathbb{E}$* and is denoted by $\langle \mathcal{X} \rangle$. Indeed, $\langle \mathcal{X} \rangle = \bigcap\limits_{\mathcal{X} \subseteq \mathbb{F} \in \mathcal{F}(\mathbb{E})} F$.

Proposition 2.8 ([15]). *Let $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$. Then*

$$\langle \mathcal{X} \rangle = \{\alpha \in \mathbb{E}\colon\ p_1 \to (p_2 \to (\ldots \to (p_n \to \alpha)\ldots)) = 1 \\ \text{for some } n \in \mathbb{N} \text{ and } p_1, \ldots, p_n \in \mathcal{X}\}.$$

In particular, for any element $p \in \mathbb{E}$ we have

$$\langle p \rangle = \{\alpha \in \mathbb{E}\colon\ p \to^n \alpha = 1 \text{ for some } n \in \mathbb{N}\},$$

where $\alpha \to^0 \gamma = \gamma$ and $\alpha \to^n \gamma = \alpha \to (\alpha \to^{n-1} \gamma)$.

If $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $p \in \mathbb{E} \setminus \mathbb{F}$, then

$$\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E}\colon\ p \to^n \alpha \in \mathbb{F} \text{ for some } n \in \mathbb{N}\}.$$

If $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$, then

$$\langle \mathbb{F} \cup \mathbb{G} \rangle = \{\alpha \in \mathbb{E}\colon\ g \to \alpha \in \mathbb{F} \text{ for some } g \in \mathbb{G}\} \\ = \{\alpha \in \mathbb{E}\colon\ m \to \alpha \in \mathbb{G} \text{ for some } m \in \mathbb{F}\}.$$

Proposition 2.9 ([15]). *Let $\mathbb{F}$ and $\mathbb{G}$ be two proper filters of $\mathbb{E}$. Then for all $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$ and $\alpha, p, q \in \mathbb{E}$, the following statements hold:*

(i) *if $\mathcal{X} \subseteq \mathcal{Y}$, then $\langle \mathcal{X} \rangle \subseteq \langle \mathcal{Y} \rangle$;*
(ii) *if $\mathbb{F} \subseteq \mathbb{G}$, then $\langle \mathbb{F} \cup \{\alpha\} \rangle \subseteq \langle \mathbb{G} \cup \{\alpha\} \rangle$;*
(iii) *if $p \leqslant q$, then $\langle q \rangle \subseteq \langle p \rangle$;*
(iv) *if $\mathbb{F}$ is a positive implicative filter, then $\langle \mathbb{F} \cup \{p\} \rangle = \{\alpha \in \mathbb{E}\colon\ p \to \alpha \in \mathbb{F}\}$.*

Theorem 2.10 ([15]). *The algebraic structure* $(\mathcal{F}(\mathbb{E}), \subseteq, \wedge, \vee, \{1\}, \mathbb{E})$ *is a bounded distributive complete lattice, where for any* $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$,

$$\mathbb{F} \wedge \mathbb{G} := \mathbb{F} \cap \mathbb{G}, \quad \mathbb{F} \vee \mathbb{G} := \langle \mathbb{F} \cup \mathbb{G} \rangle.$$

Note. From now on, we let $(\mathbb{E}, \sim, \wedge, 0, 1)$ or $\mathbb{E}$ be a lattice equality algebra, unless otherwise stated, where for any $\alpha, \gamma \in \mathbb{E}$, the join operation $\vee$ on $\mathbb{E}$ is defined as

$$\alpha \vee \gamma := ((\alpha \to \gamma) \to \gamma) \wedge ((\gamma \to \alpha) \to \alpha).$$

Definition 2.11 ([15]). Let $\mathbb{F}$ be a proper filter of $\mathbb{E}$. Then $\mathbb{F}$ is called a $\vee$-*irreducible* filter of $\mathbb{E}$ if $\alpha \vee \gamma \in \mathbb{F}$ implies $\alpha \in \mathbb{F}$ or $\gamma \in \mathbb{F}$ for all $\alpha, \gamma \in \mathbb{E}$.

Corollary 2.12 ([15]). *Let* $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. *Then for each* $p \notin \mathbb{F}$ *there exists a* $\vee$-*irreducible filter* $\mathbb{P}$ *containing* $\mathbb{F}$ *such that* $p \notin \mathbb{P}$.

Definition 2.13 ([7]). The algebraic structure $(L, \wedge, \vee, \odot, \to, 0, 1)$ of type $(2, 2, 2, 2, 0, 0)$ is called a BL-algebra if the following conditions hold for all $x, y, z \in L$:
(BL1) $(L, \wedge, \vee, 0, 1)$ is a bounded lattice;
(BL2) $(L, \odot, 1)$ is a commutative monoid;
(BL3) $x \odot y \leqslant z$ if and only if $x \leqslant y \to z$;
(BL4) $x \wedge y = x \odot (x \to y)$;
(BL5) $(x \to y) \vee (y \to x) = 1$.

In the bounded lattice $(L, \wedge, \vee, 0, 1)$ and given a pair of elements $a, b \in L$, if $a \wedge b = 0$ and $a \vee b = 1$, then one of a and b is called a *complement* of the other. If any $a \in L$ has its complement, then L is called a *complemented* lattice. If a lattice is both complemented and distributive, then it is called a *Boolean algebra* or a *Boolean lattice* (see [3]).

3. Relative co-annihilators

In this section, we introduce the notion of relative co-annihilators on a lattice equality algebra $\mathbb{E}$ and investigate some related properties of them. Moreover, we show that for any $\mathbb{G} \in \mathcal{F}(\mathbb{E})$, its relative pseudo complement with respect to $\mathbb{F}$ is the relative co-annihilator of $\mathbb{G}$ with respect to $\mathbb{F}$.

Definition 3.1. Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathcal{X} \subseteq \mathbb{E}$. We define a *co-annihilator of* $\mathcal{X}$ *relative to* $\mathbb{F}$ as

$$\{\alpha \in \mathbb{E} \colon \alpha \vee p \in \mathbb{F} \ \forall p \in \mathcal{X}\},$$

and denote it by $(\mathbb{F} : \mathcal{X})$. When $\mathcal{X} = \{p\}$, we denote $(\mathbb{F} : \{p\})$ by $(\mathbb{F} : p)$ for short. If $\mathbb{F} = \{1\}$, then $(\{1\} : \mathcal{X}) = \mathcal{X}^{\top} = \{\alpha \in \mathbb{E} \colon \alpha \vee p = 1$ for all $p \in \mathcal{X}\}$ and $(\{1\} : p) = p^{\top}$. For more details, see [15].

E x a m p l e 3.2. Let $\mathbb{E} = \{0, p, q, r, s, 1\}$ be a set, where $0 \leqslant p \leqslant s \leqslant 1$ and $0 \leqslant q \leqslant r \leqslant s \leqslant 1$. Define the operation "$\sim$" on $\mathbb{E}$ as follows:

$\sim$	0	p	q	r	s	1
0	1	r	p	p	0	0
p	r	1	0	0	p	p
q	p	0	1	s	r	q
r	p	0	s	1	r	r
s	0	p	r	r	1	s
1	0	p	q	r	s	1

$\rightarrow$	0	p	q	r	s	1
0	1	1	1	1	1	1
p	r	1	r	r	1	1
q	p	p	1	1	1	1
r	p	p	s	1	1	1
s	0	p	r	r	1	1
1	0	p	q	r	s	1

Then $(\mathbb{E}, \sim, \wedge, 1)$ is an equality algebra. Clearly, $\mathbb{F} = \{s, 1\} \in \mathcal{F}(\mathbb{E})$. If $\mathcal{X} = \{p, s\}$ and $\mathcal{Y} = \{r\}$, then $(\mathbb{F} : \mathcal{X}) = \{q, r, s, 1\}$ and $(\mathbb{F} : r) = \{p, s, 1\}$. In addition, $\mathcal{X}^{\top} = \{1\} = r^{\top}$.

Proposition 3.3. *Let $p, q \in \mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then the following statements hold:*

(i) *If $p \leqslant q$, then $(\mathbb{F} : p) \subseteq (\mathbb{F} : q)$.*

(ii) *If $p \in \mathbb{F}$, then $(\mathbb{F} : p) = \mathbb{E}$. The converse is true when $\mathbb{E}$ is bounded.*

(iii) *$(\mathbb{F} : p) \cap (\mathbb{G} : p) = (\mathbb{F} \cap \mathbb{G} : p)$ and $(\mathbb{F} : p) \cup (\mathbb{G} : p) = (\mathbb{F} \cup \mathbb{G} : p)$.*

(iv) *$(\mathbb{F} : p \wedge q) \subseteq (\mathbb{F} : p \vee q)$.*

(v) *$(\mathbb{F} : p) \cup (\mathbb{F} : q) \subseteq (\mathbb{F} : p \vee q)$. If p, q are comparable, then the converse is true.*

(vi) *$((\mathbb{F} : p) : q) = ((\mathbb{F} : q) : p) = (\mathbb{F} : p \vee q)$.*

P r o o f. (i) Let $p \leqslant q$ and $\alpha \in (\mathbb{F} : p)$. Then $\alpha \vee p \in \mathbb{F}$ and since $\alpha \vee p \leqslant \alpha \vee q$, we get $\alpha \vee q \in \mathbb{F}$. Thus $\alpha \in (\mathbb{F} : q)$ and so, $(\mathbb{F} : p) \subseteq (\mathbb{F} : q)$.

(ii) Let $p \in \mathbb{F}$. Then for all $\alpha \in \mathbb{E}$ we have $p \leqslant p \vee \alpha$ and so, $p \vee \alpha \in \mathbb{F}$. Hence, $\alpha \in (\mathbb{F} : p)$, which means $\mathbb{E} \subseteq (\mathbb{F} : p)$. On the other hand, we always have $(\mathbb{F} : p) \subseteq \mathbb{E}$. Therefore, $(\mathbb{F} : p) = \mathbb{E}$. Now, let $\mathbb{E}$ be bounded and $\mathbb{E} = (\mathbb{F} : p)$. Then $0 \in (\mathbb{F} : p)$ and so, $p \vee 0 = p \in \mathbb{F}$.

(iii) $\alpha \in (\mathbb{F} : p) \cap (\mathbb{G} : p)$ if and only if $\alpha \vee p \in \mathbb{F} \cap \mathbb{G}$ if and only if $\alpha \in (\mathbb{F} \cap \mathbb{G} : p)$. Similarly, the next one is true.

(iv) Since $p \wedge q \leqslant p \vee q$ and (i), we have $(\mathbb{F} : p \wedge q) \subseteq (\mathbb{F} : p \vee q)$.

(v) If $\alpha \in (\mathbb{F} : p) \cup (\mathbb{F} : q)$, then $\alpha \vee p \in \mathbb{F}$ or $\alpha \vee q \in \mathbb{F}$. From $p, q \leqslant p \vee q$, we get $\alpha \vee p$, $\alpha \vee q \leqslant \alpha \vee (p \vee q)$ and by $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $\alpha \in (\mathbb{F} : p \vee q)$. Conversely, let p, q be comparable and $\alpha \in (\mathbb{F} : p \vee q)$. Since $p \leqslant q$ or $q \leqslant p$, we get $\alpha \vee q = \alpha \vee (p \vee q) \in \mathbb{F}$ or $\alpha \vee p = \alpha \vee (p \vee q) \in \mathbb{F}$. Hence, $\alpha \in (\mathbb{F} : p) \cup (\mathbb{F} : q)$.

(vi) From $\alpha \vee (p \vee q) = (\alpha \vee p) \vee q = (\alpha \vee q) \vee p$, the proof is obvious. □

The other sides of inclusions of Proposition 3.3 (iv) and (v) are not true, in general.

E x a m p l e 3.4. Let $(\mathbb{E}, \wedge, \sim, 1)$ be as in Example 3.2 and $\mathbb{F} = \{d, 1\}$. Then $(\mathbb{F} : a) = \{b, c, d, 1\}$, $(\mathbb{F} : b) = \{a, d, 1\}$. Since $a \wedge b = 0$ and $a \vee b = d$, we get

$$\mathbb{E} = (\mathbb{F} : d) = (\mathbb{F} : a \vee b) \nsubseteq (\mathbb{F} : a \wedge b) = (\mathbb{F} : 0) = \mathbb{F}.$$

Also, $\mathbb{E} = (\mathbb{F} : a \vee b) \nsubseteq (\mathbb{F} : a) \cup (\mathbb{F} : b) = \{b, c, d, 1\} \cup \{a, d, 1\} = \{a, b, c, d, 1\}$.

Proposition 3.5. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathcal{X}) \in \mathcal{F}(\mathbb{E})$.*

P r o o f. Let $p \in \mathcal{X}$. Since $1 \vee p = 1 \in \mathbb{F}$, we have $1 \in (\mathbb{F} : \mathcal{X})$. If $\alpha, \alpha \to \gamma \in (\mathbb{F} : \mathcal{X})$, then $\alpha \vee p \in \mathbb{F}$ and $(\alpha \to \gamma) \vee p \in \mathbb{F}$, for all $p \in \mathcal{X}$. Suppose $\eta := \gamma \vee p$. Since $p, \gamma \leqslant \eta$, we get $p \leqslant \eta \leqslant \alpha \to \eta$ and $\alpha \to \gamma \leqslant \alpha \to \eta$ by Proposition 2.2 (iii) and (vi), respectively. Hence, $(\alpha \to \gamma) \vee p \leqslant (\alpha \to \eta) \vee p = \alpha \to \eta$. Since $(\alpha \to \gamma) \vee p \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \to \eta \in \mathbb{F}$. Moreover, $p \leqslant \eta$, then $\alpha \vee p \leqslant \alpha \vee \eta$. From $\alpha \vee p \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \vee \eta \in \mathbb{F}$. Now, by Proposition 2.2 (vii), $\alpha \to \eta = (\alpha \vee \eta) \to \eta$. In addition, $\alpha \to \eta, \alpha \vee \eta \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we obtain $\eta \in \mathbb{F}$. Thus, $\gamma \vee p \in \mathbb{F}$, i.e., $\gamma \in (\mathbb{F} : \mathcal{X})$. Therefore $(\mathbb{F} : \mathcal{X}) \in \mathcal{F}(\mathbb{E})$. $\square$

Proposition 3.6. *Let $\mathcal{X}, \mathcal{Y} \subseteq \mathbb{E}$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then the following statements hold:*

(i) $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X})$.
(ii) $(\mathbb{F} : \mathbb{E}) = \mathbb{F}$ *and* $(\mathbb{F} : \mathbb{F}) = \mathbb{E}$.
(iii) $(\mathbb{F} : (\mathbb{F} : \mathbb{E})) = \mathbb{E}$ *and* $(\mathbb{F} : (\mathbb{F} : \mathbb{F})) = \mathbb{F}$.
(iv) *If* $\mathcal{X} \subseteq \mathcal{Y}$, *then* $(\mathbb{F} : \mathcal{Y}) \subseteq (\mathbb{F} : \mathcal{X})$.
(v) *If* $\mathbb{F} \subseteq \mathbb{G}$, *then* $(\mathbb{F} : \mathcal{X}) \subseteq (\mathbb{G} : \mathcal{X})$. *In particular,* $\mathbb{G}^{\top} \subseteq (\mathbb{F} : \mathbb{G})$.
(vi) $(\mathbb{F} : \mathcal{X}) = \mathbb{E}$ *if and only if* $\mathcal{X} \subseteq \mathbb{F}$.
(vii) $\Bigl(\mathbb{F} : \bigcup\limits_{i \in \Delta} \mathcal{X}_i\Bigr) = \bigcap\limits_{i \in \Delta} (\mathbb{F} : \mathcal{X}_i)$.
(viii) $(\mathbb{F} : \mathcal{X}) = \bigcap\limits_{p \in \mathcal{X}} (\mathbb{F} : p)$.
(ix) $\Bigl(\bigcap\limits_{i \in \Delta} \mathbb{F}_i : \mathcal{X}\Bigr) = \bigcap\limits_{i \in \Delta} (\mathbb{F}_i : \mathcal{X})$.
(x) $(\mathbb{F} : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \setminus \mathbb{F})$.
(xi) *If* $\mathbb{F} \subseteq \mathcal{X}$, *then* $\mathcal{X} \cap (\mathbb{F} : \mathcal{X}) = \mathbb{F}$.
(xii) $(\mathbb{F} : \mathcal{X}) \cap (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = \mathbb{F}$.
(xiii) $\mathcal{X} \subseteq (\mathbb{F} : (\mathbb{F} : \mathcal{X}))$.
(xiv) $(\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathcal{X}))) = (\mathbb{F} : \mathcal{X})$.
(xv) $((\mathbb{F} : \mathcal{X}) : \mathcal{Y}) = ((\mathbb{F} : \mathcal{Y}) : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$, *where* $\mathcal{X} \vee \mathcal{Y} = \{p \vee q \colon p \in \mathcal{X}, q \in \mathcal{Y}\}$.

P r o o f. (i) Let $f \in F$ and $p \in \mathcal{X}$. Then $f \leqslant p \vee f$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, so $p \vee f \in \mathbb{F}$. Hence, $f \in (\mathbb{F} : \mathcal{X})$. Therefore, $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X})$.

(ii) By (i), $\mathbb{F} \subseteq (\mathbb{F} : \mathbb{E})$. On the other hand, if $\alpha \in (\mathbb{F} : \mathbb{E})$, then for all $p \in \mathbb{E}$, $\alpha \vee p \in \mathbb{F}$. Suppose $p = \alpha$, then $\alpha = \alpha \vee \alpha \in \mathbb{F}$ and so $\alpha \in \mathbb{F}$, i.e., $(\mathbb{F} : \mathbb{E}) \subseteq \mathbb{F}$.

Therefore $(\mathbb{F}:\mathbb{E})=\mathbb{F}$. Also, for any $\alpha\in\mathbb{E}$ and $f\in\mathbb{F}$, since $f\leqslant\alpha\vee f$ and $\mathbb{F}\in\mathcal{F}(\mathbb{E})$ we have $\alpha\vee f\in\mathbb{F}$ and so $\alpha\in(\mathbb{F}:\mathbb{F})$. Hence, $\mathbb{E}\subseteq(\mathbb{F}:\mathbb{F})\subseteq\mathbb{E}$. Therefore $(\mathbb{F}:\mathbb{F})=\mathbb{E}$.

(iii) By (ii), $(\mathbb{F}:(\mathbb{F}:\mathbb{E}))=(\mathbb{F}:\mathbb{F})=\mathbb{E}$ and $(\mathbb{F}:(\mathbb{F}:\mathbb{F}))=(\mathbb{F}:\mathbb{E})=\mathbb{F}$.

(iv) Let $\mathcal{X}\subseteq\mathcal{Y}$ and $\alpha\in(\mathbb{F}:\mathcal{Y})$. Then for any $q\in\mathcal{Y}$ we have $\alpha\vee q\in\mathbb{F}$. Since $\mathcal{X}\subseteq\mathcal{Y}$, we get $\alpha\in(\mathbb{F}:\mathcal{X})$. Therefore $(\mathbb{F}:\mathcal{Y})\subseteq(\mathbb{F}:\mathcal{X})$.

(v) Let $\alpha\in(\mathbb{F}:\mathcal{X})$ and $p\in\mathcal{X}$. Then $\alpha\vee p\in\mathbb{F}\subseteq\mathbb{G}$. Hence, $(\mathbb{F}:\mathcal{X})\subseteq(\mathbb{G}:\mathcal{X})$. Specially, from $\{1\}\subseteq\mathbb{F}$ we have $\mathbb{G}^{\top}=(\{1\}:\mathbb{G})\subseteq(\mathbb{F}:\mathbb{G})$.

(vi) Let $(\mathbb{F}:\mathcal{X})=\mathbb{E}$ and $p\in\mathcal{X}$. Since $\mathcal{X}\subseteq\mathbb{E}$, clearly $p\in(\mathbb{F}:\mathcal{X})$ and so $p=p\vee p\in\mathbb{F}$. Therefore $\mathcal{X}\subseteq\mathbb{F}$. Conversely, let $\mathcal{X}\subseteq\mathbb{F}$ and $\alpha\in\mathbb{E}$. Then for all $p\in\mathcal{X}$, $p\in\mathbb{F}$ and $p\leqslant p\vee\alpha$. Since $\mathbb{F}\in\mathcal{F}(\mathbb{E})$, we get $p\vee\alpha\in\mathbb{F}$ and so $\alpha\in(\mathbb{F}:\mathcal{X})$. Therefore $\mathbb{E}=(\mathbb{F}:\mathcal{X})$.

(vii) Since $\mathcal{X}_i\subseteq\bigcup_{i\in\Delta}\mathcal{X}_i$ for all $i\in\Delta$ by (iv), $\left(\mathbb{F}:\bigcup_{i\in\Delta}\mathcal{X}_i\right)\subseteq(\mathbb{F}:\mathcal{X}_i)$ for all $i\in\Delta$. Hence, $\left(\mathbb{F}:\bigcup_{i\in\Delta}\mathcal{X}_i\right)\subseteq\bigcap_{i\in\Delta}(\mathbb{F}:\mathcal{X}_i)$. Conversely, let $\alpha\in\bigcap_{i\in\Delta}(\mathbb{F}:\mathcal{X}_i)$ and $p\in\bigcup_{i\in\Delta}\mathcal{X}_i$. Then there exists $j\in\Delta$ such that $p\in\mathcal{X}_j$. Thus, $p\vee\alpha\in\mathbb{F}$ and so $\alpha\in\left(\mathbb{F}:\bigcup_{i\in\Delta}\mathcal{X}_i\right)$. Therefore, $\left(\mathbb{F}:\bigcup_{i\in\Delta}\mathcal{X}_i\right)=\bigcap_{i\in\Delta}(\mathbb{F}:\mathcal{X}_i)$.

(viii) This is the result of (vii).

(ix) Since for all $i\in\Delta$, $\bigcap_{i\in\Delta}\mathbb{F}_i\subseteq\mathbb{F}_i$, by (v), we get $\left(\bigcap_{i\in\Delta}\mathbb{F}_i:\mathcal{X}\right)\subseteq(\mathbb{F}_i:\mathcal{X})$ and so $\left(\bigcap_{i\in\Delta}\mathbb{F}_i:\mathcal{X}\right)\subseteq\bigcap_{i\in\Delta}(\mathbb{F}_i:\mathcal{X})$. Conversely, let $\alpha\in\bigcap_{i\in\Delta}(\mathbb{F}_i:\mathcal{X})$ and $p\in\mathcal{X}$. Then for all $i\in\Delta$, $\alpha\vee p\in\mathbb{F}_i$ and so $\alpha\vee p\in\bigcap_{i\in\Delta}\mathbb{F}_i$. Hence, $\alpha\in\left(\bigcap_{i\in\Delta}\mathbb{F}_i:\mathcal{X}\right)$. Therefore $\left(\bigcap_{i\in\Delta}\mathbb{F}_i:\mathcal{X}\right)=\bigcap_{i\in\Delta}(\mathbb{F}_i:\mathcal{X})$.

(x) We know $\mathcal{X}=(\mathcal{X}\backslash\mathbb{F})\cup(\mathcal{X}\cap\mathbb{F})$. Since $\mathcal{X}\cap\mathbb{F}\subseteq\mathbb{F}$, by (vi), we get $(\mathbb{F}:\mathcal{X}\cap\mathbb{F})=\mathbb{E}$. So by (vii), we have

$$\begin{aligned}(\mathbb{F}:\mathcal{X})&=(\mathbb{F}:(\mathcal{X}\setminus\mathbb{F})\cup(\mathcal{X}\cap\mathbb{F}))=(\mathbb{F}:\mathcal{X}\setminus\mathbb{F})\cap(\mathbb{F}:\mathcal{X}\cap\mathbb{F})\\&=(\mathbb{F}:\mathcal{X}\setminus\mathbb{F})\cap\mathbb{E}=(\mathbb{F}:\mathcal{X}\setminus\mathbb{F}).\end{aligned}$$

(xi) Let $\mathbb{F}\subseteq\mathcal{X}$. By (i), $\mathbb{F}\subseteq(\mathbb{F}:\mathcal{X})$ and so $\mathbb{F}\subseteq\mathcal{X}\cap(\mathbb{F}:\mathcal{X})$. Conversely, let $\alpha\in\mathcal{X}\cap(\mathbb{F}:\mathcal{X})$. Then $\alpha\in\mathcal{X}$ and for all $p\in\mathcal{X}$, we have $\alpha\vee p\in\mathbb{F}$. Suppose $p=\alpha$, then $\alpha=\alpha\vee\alpha\in\mathbb{F}$. Hence, $\mathcal{X}\cap(\mathbb{F}:\mathcal{X})\subseteq\mathbb{F}$. Therefore $\mathcal{X}\cap(\mathbb{F}:\mathcal{X})=\mathbb{F}$.

(xii) By using (i) twice, $\mathbb{F}\subseteq(\mathbb{F}:\mathcal{X})\cap(\mathbb{F}:(\mathbb{F}:\mathcal{X}))$. For the other side of inclusion, let $\alpha\in(\mathbb{F}:\mathcal{X})\cap(\mathbb{F}:(\mathbb{F}:\mathcal{X}))$. Then $\alpha\in(\mathbb{F}:\mathcal{X})$ and from $\alpha\in(\mathbb{F}:(\mathbb{F}:\mathcal{X}))$ we get $\alpha\vee\gamma\in\mathbb{F}$ for all $\gamma\in(\mathbb{F}:\mathcal{X})$. In particular, when $\gamma:=\alpha$, we have $\alpha=\alpha\vee\alpha\in\mathbb{F}$ and so $(\mathbb{F}:\mathcal{X})\cap(\mathbb{F}:(\mathbb{F}:\mathcal{X}))\subseteq\mathbb{F}$.

(xiii) Let $p\in\mathcal{X}$ and $\alpha\in(\mathbb{F}:\mathcal{X})$. Then $\alpha\vee p\in\mathbb{F}$. Hence, $p\in(\mathbb{F}:(\mathbb{F}:\mathcal{X}))$. Therefore $\mathcal{X}\subseteq(\mathbb{F}:(\mathbb{F}:\mathcal{X}))$.

(xiv) Suppose $R = (\mathbb{F} : \mathcal{X})$. Then by (xiii), $R \subseteq (\mathbb{F} : (\mathbb{F} : R))$. Conversely, by (xiii), $\mathcal{X} \subseteq (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = (\mathbb{F} : R)$, and by (iv) we get $(\mathbb{F} : (\mathbb{F} : R)) \subseteq (\mathbb{F} : \mathcal{X}) = R$. Therefore $(\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathcal{X}))) = (\mathbb{F} : \mathcal{X})$.

(xv) Let $\alpha \in ((\mathbb{F} : \mathcal{X}) : \mathcal{Y})$. Then for all $q \in \mathcal{Y}$ and $p \in \mathcal{X}$, $(\alpha \vee b) \vee a \in \mathbb{F}$. If $\eta \in \mathcal{X} \vee \mathcal{Y}$, then there are $p \in \mathcal{X}$ and $b \in \mathcal{Y}$ such that $\eta = p \vee q$ and so $\alpha \vee \eta = \alpha \vee (p \vee q) = (\alpha \vee q) \vee p \in \mathbb{F}$. Thus $\alpha \in (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. The converse is clear. Hence, $((\mathbb{F} : \mathcal{X}) : \mathcal{Y}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. Similarly, we get $((\mathbb{F} : \mathcal{Y}) : \mathcal{X}) = (\mathbb{F} : \mathcal{X} \vee \mathcal{Y})$. Therefore the proof is complete. $\square$

The converse of Proposition 3.6 (i) and (xiii) is not true, in general.

Example 3.7. Let $\mathbb{E}$, $\mathbb{F} = \{d, 1\}$ and $\mathcal{X} = \{a, d\}$ be as in Example 3.2. Then $\mathbb{F} \subsetneqq (\mathbb{F} : \mathcal{X}) = \{b, c, d, 1\}$. Moreover, $\mathcal{X} \subsetneqq (\mathbb{F} : (\mathbb{F} : \mathcal{X})) = \{a, d, 1\}$.

Proposition 3.8. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \langle \mathcal{X} \rangle) = (\mathbb{F} : \mathcal{X})$.*

Proof. Since $\mathcal{X} \subseteq \langle \mathcal{X} \rangle$, by Proposition 3.6 (iv), $(\mathbb{F} : \langle \mathcal{X} \rangle) \subseteq (\mathbb{F} : \mathcal{X})$. For the converse, let $\alpha \in (\mathbb{F} : \mathcal{X})$ and $\alpha \notin (\mathbb{F} : \langle \mathcal{X} \rangle)$. Then there exists $p \in \langle \mathcal{X} \rangle$ such that $\alpha \vee p \notin \mathbb{F}$. By Proposition 2.8, there are $p_1, \ldots, p_n \in \mathcal{X}$ such that $p_1 \to (\ldots (p_n \to p) \ldots) = 1$ for some $n \in \mathbb{N}$. Moreover, since $\alpha \vee p \notin \mathbb{F}$, by Corollary 2.12 (i), there is a $\vee$-irreducible filter $\mathbb{P}$ of $\mathbb{E}$ containing $\mathbb{F}$ such that $\alpha \vee p \notin \mathbb{P}$. Also, since $\alpha \in (\mathbb{F} : \mathcal{X})$, we get $p_i \vee \alpha \in \mathbb{F} \subseteq \mathbb{P}$ for all $1 \leqslant i \leqslant n$. Hence, $\alpha \in \mathbb{P}$ or $p_i \in \mathbb{P}$ for all $1 \leqslant i \leqslant n$. If $\alpha \in \mathbb{P}$, then by $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we have $\alpha \vee p \in \mathbb{P}$, which is a contradiction. Thus, for any $1 \leqslant i \leqslant n$, $p_i \in \mathbb{P}$ and so by $p_1 \to (\ldots (p_n \to p) \ldots) = 1 \in \mathbb{P}$ and $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we get $p \in \mathbb{P}$. Since $p \leqslant p \vee \alpha$ and $\mathbb{P} \in \mathcal{F}(\mathbb{E})$, we get $p \vee \alpha \in \mathbb{P}$, which is a contradiction. Thus, $\alpha \in (\mathbb{F} : \langle \mathcal{X} \rangle)$. Therefore $(\mathbb{F} : \langle \mathcal{X} \rangle) = (\mathbb{F} : \mathcal{X})$. $\square$

Proposition 3.9. *Let $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$. Then*
(i) $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} \subseteq \mathbb{F}$;
(ii) $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$ *if and only if* $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$.

Proof. (i) It is clear.

(ii) Let $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$ and $\alpha \in \mathbb{H}$. Since for any $g \in \mathbb{G}$, $\alpha, g \leqslant \alpha \vee g$ and $\mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \vee g \in \mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$. Hence, $\alpha \vee g \in \mathbb{F}$ and so $\alpha \in (\mathbb{F} : \mathbb{G})$. Thus, $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Conversely, let $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Then by (i), $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{G}) \subseteq \mathbb{F}$. Therefore $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{F}$. $\square$

Proposition 3.10. *Let $\mathcal{X} \subseteq \mathbb{E}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathcal{X}) = \{\alpha \in \mathbb{E} \colon \langle \alpha \rangle \cap \langle \mathcal{X} \rangle \subseteq \mathbb{F}\}$.*

P r o o f. Suppose $B = \{\alpha \in \mathbb{E}\colon \langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}\}$. Let $\alpha \in B$. Then $\langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}$. By Proposition 3.9 (ii), we get $\langle\alpha\rangle \subseteq (\mathbb{F} : \langle\mathcal{X}\rangle)$. Since $\alpha \in \langle\alpha\rangle$ and by Proposition 3.8, we have $\alpha \in (\mathbb{F} : \mathcal{X})$. Hence, $B \subseteq (\mathbb{F} : \mathcal{X})$. Conversely, let $\alpha \in (\mathbb{F} : \mathcal{X})$. Then by Proposition 3.8, $\alpha \in (\mathbb{F} : \langle\mathcal{X}\rangle)$ and so $\langle\alpha\rangle \subseteq (\mathbb{F} : \langle\mathcal{X}\rangle)$. Now, by Proposition 3.9 (ii), we have $\langle\alpha\rangle \cap \langle\mathcal{X}\rangle \subseteq \mathbb{F}$. Hence, $(\mathbb{F} : \mathcal{X}) \subseteq B$. Therefore $(\mathbb{F} : \mathcal{X}) = B$. $\square$

Proposition 3.11. *Let $p \in \mathbb{E}$ and $\mathbb{F}$ be a positive implicative filter of $\mathbb{E}$. Then $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle = \mathbb{F}$.*

P r o o f. We know $\mathbb{F} \subseteq \langle\mathbb{F} \cup \{p\}\rangle$ and by Proposition 3.6 (i), we get $\mathbb{F} \subseteq (\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle$. Conversely, let $\alpha \in (\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle$. Then $\alpha \vee p \in \mathbb{F}$ and by Proposition 2.9 (iv), $p \to \alpha \in \mathbb{F}$. Also, by Proposition 2.2 (vii), we have $p \to \alpha = (p \vee \alpha) \to \alpha$ and since $p \to \alpha, p \vee \alpha \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we get $\alpha \in \mathbb{F}$. Hence, $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle \subseteq \mathbb{F}$. Therefore $(\mathbb{F} : p) \cap \langle\mathbb{F} \cup \{p\}\rangle = \mathbb{F}$. $\square$

Proposition 3.12. *Let $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\emptyset \neq \mathbb{G} \subseteq \mathbb{E}$. If $\mathbb{G}$ is a chain such that $\mathbb{G} \nsubseteq \mathbb{F}$, then $(\mathbb{F} : \mathbb{G})$ is a $\vee$-irreducible filter of $\mathbb{E}$.*

P r o o f. Let $\mathbb{G}$ be a chain and $\mathbb{G} \nsubseteq \mathbb{F}$. Then by Proposition 3.6 (vi), we get $(\mathbb{F} : \mathbb{G}) \neq \mathbb{E}$. If $\alpha \vee \gamma \in (\mathbb{F} : \mathbb{G})$, then $(\alpha \vee \gamma) \vee g \in \mathbb{F}$ for all $g \in \mathbb{G}$. On the contrary, let $\alpha, \gamma \notin (\mathbb{F} : \mathbb{G})$. Then there are $g_1, g_2 \in \mathbb{G}$ such that $\alpha \vee g_1 \notin \mathbb{F}$ and $\gamma \vee g_2 \notin \mathbb{F}$. Suppose $g := g_1 \wedge g_2$. Since $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and it is closed under $\wedge$-operation, we get $g \in \mathbb{G}$ and so $\alpha \vee g$, $\gamma \vee g \in \mathbb{G}$. Since $\mathbb{G}$ is a chain, without loss of generality, suppose $\alpha \vee g \leqslant \gamma \vee g$. Hence, we have

$$(\alpha \vee \gamma) \vee g = (\alpha \vee g) \vee \gamma \leqslant (\gamma \vee g) \vee \gamma = \gamma \vee g \leqslant \gamma \vee g_2.$$

Since $(\alpha \vee \gamma) \vee g \in \mathbb{F}$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $\gamma \vee g_2 \in \mathbb{F}$, which is a contradiction. Therefore $(\mathbb{F} : \mathbb{G})$ is a $\vee$-irreducible filter of $\mathbb{E}$. $\square$

Proposition 3.13. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$ such that $\mathbb{F} \subseteq \mathbb{P}$. Then $\mathcal{X} \nsubseteq \mathbb{P}$ implies $(\mathbb{F} : \mathcal{X}) \subseteq \mathbb{P}$ for any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$.*

P r o o f. Let $\mathcal{X} \nsubseteq \mathbb{P}$. Then there exists $p \in \mathcal{X}$ such that $p \notin \mathbb{P}$. Also, if $\alpha \in (\mathbb{F} : \mathcal{X})$, then $\alpha \vee p \in \mathbb{F} \subseteq \mathbb{P}$. Since $p \notin \mathbb{P}$ and $\mathbb{P}$ is a $\vee$-irreducible filter, we get $\alpha \in \mathbb{P}$. Hence, $(\mathbb{F} : \mathcal{X}) \subseteq \mathbb{P}$. $\square$

Corollary 3.14. *Let $\mathbb{F} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$. Then $\mathcal{X} \nsubseteq \mathbb{P}$ implies $(\mathbb{P} : \mathcal{X}) = \mathbb{P}$ for any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$.*

P r o o f. By Proposition 3.6 (i), we have $\mathbb{P} \subseteq (\mathbb{P} : \mathcal{X})$. Then by Proposition 3.13, the proof is complete. $\square$

Theorem 3.15. *Let $\mathbb{P} \in \mathbb{F}(\mathbb{E})$. Then $\mathbb{P}$ is a $\vee$-irreducible filter of $\mathbb{E}$ if and only if $(\mathbb{P} : \alpha) = \mathbb{P}$ for any $\alpha \notin \mathbb{P}$.*

Proof. Let $\mathbb{P}$ be a $\vee$-irreducible filter of $\mathbb{E}$ and $\alpha \notin \mathbb{P}$. By Corollary 3.14, it is enough to set $\mathcal{X} = \{\alpha\}$ and so the proof is clear. Conversely, let $\alpha \vee \gamma \in \mathbb{P}$ and $\alpha \notin \mathbb{P}$. By hypothesis, $(\mathbb{P} : \alpha) = \mathbb{P}$. Moreover, since $\alpha \vee \gamma \in \mathbb{P}$, we get $\gamma \in (\mathbb{P} : \alpha) = \mathbb{P}$. Therefore $\mathbb{P}$ is a $\vee$-irreducible filter of $\mathbb{E}$. $\square$

Definition 3.16 ([18])**.** In a lattice L with bottom element 0, an element $x \in L$ is said to have a pseudo-complement element if there exists the greatest element $x^* \in L$, disjoint from x, with the property that $x \wedge x^* = 0$. More formally, $x^* = \max\{y \in L\colon\ x \wedge y = 0\}$. The lattice L itself is called a *pseudo-complemented lattice* if every element of L has a pseudo-complement element. A *relative pseudo-complement* of a with respect to b, is a maximal element c such that $a \wedge c \leqslant b$.

Proposition 3.17. *Let $\mathbb{F}, \mathbb{G} \in \mathcal{F}(\mathbb{E})$. Then $(\mathbb{F} : \mathbb{G})$ is a relative pseudo complement of $\mathbb{G}$ with respect to $\mathbb{F}$ in the lattice $(\mathcal{F}(\mathbb{E}), \subseteq)$, where $\mathbb{F} \wedge \mathbb{G} := \mathbb{F} \cap \mathbb{G}$, $\mathbb{F} \vee \mathbb{G} := \langle \mathbb{F} \cup \mathbb{G} \rangle$.*

Proof. By Proposition 3.9 (i), $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} \subseteq \mathbb{F}$. It is enough to show that $(\mathbb{F} : \mathbb{G})$ is the greatest one. For this, suppose that there is $\mathbb{H} \in \mathcal{F}(\mathbb{E})$ such that $\mathbb{H} \cap \mathbb{G} \subseteq \mathbb{F}$ and let $\alpha \in \mathbb{H}$. Then for all $g \in \mathbb{G}$, $\alpha, g \leqslant \alpha \vee g$ and so $\alpha \vee g \in \mathbb{H} \cap \mathbb{G} \subseteq \mathbb{F}$. Thus, $\alpha \vee g \in \mathbb{F}$ for all $g \in \mathbb{G}$, i.e., $\alpha \in (\mathbb{F} : \mathbb{G})$. Hence, $\mathbb{H} \subseteq (\mathbb{F} : \mathbb{G})$. Therefore $(\mathbb{F} : \mathbb{G})$ is a relative pseudo complement of $\mathbb{G}$ with respect to $\mathbb{F}$ in the lattice $(\mathcal{F}(\mathbb{E}), \subseteq)$. $\square$

Remark 3.18. Let $\mathbb{F}$ be a proper filter of $\mathbb{E}$ and $\mathbb{H} \in \mathcal{F}(\mathbb{E}/\mathbb{F})$. If we take $\mathbb{G} := \{x \in \mathbb{E}\colon\ [x] \in \mathbb{H}\}$, then it is easy to see that $\mathbb{F} \subseteq \mathbb{G}$ and $\mathbb{H} = \mathbb{G}/\mathbb{F}$. So, any filter of quotient equality algebra $\mathbb{E}/\mathbb{F}$ has the form $\mathbb{G}/\mathbb{F}$ such that $\mathbb{G} \in \mathcal{F}(\mathbb{E})$ and $\mathbb{F} \subseteq \mathbb{G}$. That is

$$\mathcal{F}(\mathbb{E}/\mathbb{F}) = \{\mathbb{G}/\mathbb{F}\colon\ \mathbb{F} \subseteq \mathbb{G} \in \mathcal{F}(\mathbb{E})\}.$$

Proposition 3.19. *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(\mathbb{E})$ such that $\mathbb{F} \subseteq \mathbb{G}$. Then $(\mathbb{G} : \mathcal{X})/\mathbb{F} \in \mathbb{F}(\mathbb{E}/\mathbb{F})$.*

Proof. By Proposition 3.6 (i) and (v), we have $\mathbb{F} \subseteq (\mathbb{F} : \mathcal{X}) \subseteq (\mathbb{G} : \mathcal{X})$. Then by Remark 3.18, we get $(\mathbb{G} : \mathcal{X})/\mathbb{F} \in \mathbb{F}(\mathbb{E}/\mathbb{F})$. $\square$

Corollary 3.20. *Let $\mathbb{F}, \mathbb{G} \in \mathbb{F}(\mathbb{E})$ and $\mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}$ such that $\mathbb{F} \subseteq \mathbb{G}$. Then $((\mathbb{G}/\mathbb{F}) : (\mathcal{X}/\mathbb{F})) = (\mathbb{G} : \mathcal{X})/\mathbb{F}$.*

P r o o f. We have

$$\left(\frac{\mathbb{G}}{\mathbb{F}}:\frac{\mathcal{X}}{\mathbb{F}}\right)=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [p]\vee[\alpha]\in\frac{\mathbb{G}}{\mathbb{F}}\ \forall[\alpha]\in\frac{\mathcal{X}}{\mathbb{F}}\right\}=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [p\vee\alpha]\in\frac{\mathbb{G}}{\mathbb{F}}\ \forall\alpha\in\mathcal{X}\right\}$$
$$=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ p\vee\alpha\in\mathbb{G}\ \forall\alpha\in\mathcal{X}\right\}=\left\{[p]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ p\in(\mathbb{G}:\mathcal{X})\right\}=\frac{(\mathbb{G}:\mathcal{X})}{\mathbb{F}}.$$

□

Proposition 3.21. *Let* $\mathbb{F}\in\mathcal{F}(\mathbb{E})$ *and* $\emptyset\neq\mathcal{X}\subseteq\mathbb{E}$. *Then*
(i) $(\mathcal{X}/\mathbb{F})^{\top}=(\mathbb{F}:\mathcal{X})/\mathbb{F}$, *particularly,* $[\alpha]^{\top}=(\mathbb{F}:\alpha)/\mathbb{F}$ *for any* $[\alpha]\in\mathbb{E}/\mathbb{F}$,
(ii) $(\mathcal{X}/\mathbb{F})^{\top\top}=(\mathbb{F}:(\mathbb{F}:\mathcal{X}))/\mathbb{F}$.

P r o o f. (i)

$$\left(\frac{\mathcal{X}}{\mathbb{F}}\right)^{\top}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [\alpha]\vee[p]=[1]\ \forall[p]\in\frac{\mathcal{X}}{\mathbb{F}}\right\}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ [\alpha\vee p]=\mathbb{F}\ \forall p\in\mathcal{X}\right\}$$
$$=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ \alpha\vee p\in\mathbb{F}\ \forall p\in\mathcal{X}\right\}=\left\{[\alpha]\in\frac{\mathbb{E}}{\mathbb{F}}\colon\ \alpha\in(\mathbb{F}:\mathcal{X})\right\}=\frac{(\mathbb{F}:\mathcal{X})}{\mathbb{F}}.$$

Specially, suppose $\mathcal{X}=\{x\}$, then $[\alpha]^{\top}=(\mathbb{F}:\alpha)/\mathbb{F}$.

(ii) By (i), we have $(\mathcal{X}/\mathbb{F})^{\top\top}=((\mathcal{X}/\mathbb{F})^{\top})^{\top}=((\mathbb{F}:\mathcal{X})/\mathbb{F})^{\top}=(\mathbb{F}:(\mathbb{F}:\mathcal{X}))/\mathbb{F}$. □

Definition 3.22. Let $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$. Then $\mathbb{G}$ is called $\mathbb{F}$-*involutive* if $\mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. Also, if any $\mathbb{G}\in\mathcal{F}(\mathbb{E})$ is $\mathbb{F}$-involutive, then $\mathbb{E}$ is called an *involuntary* equality algebra relative to $\mathbb{F}$. The set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ is denoted by $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Indeed, $\mathcal{S}_{\mathbb{F}}(\mathbb{E})=\{\mathbb{G}\in\mathcal{F}(\mathbb{E})\colon\ \mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\}$.

E x a m p l e 3.23. Let $\mathbb{E}$ be the equality algebra as in Example 3.2, $\mathbb{F}=\{s,1\}$ and $\mathbb{G}=\{p,s,1\}$. Obviously, $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$ and $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))=\mathbb{G}$. Thus, $\mathbb{G}$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}$.

Proposition 3.24. *Let* $\mathbb{F},\mathbb{G}\in\mathcal{F}(\mathbb{E})$. *If* $\mathbb{F}\subseteq\mathbb{G}$ *and* $\mathbb{G}^{\top\top}=G$, *then* $\mathbb{G}\in\mathcal{S}_{\mathbb{F}}(\mathbb{E})$.

P r o o f. By Proposition 3.6 (xiii), we have $\mathbb{G}\subseteq(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. For the converse, let $g\notin\mathbb{G}$ so $g\notin\mathbb{G}^{\top\top}$. Thus, there exists $\alpha\in\mathbb{G}^{\top}$ such that $g\vee\alpha\neq1$. Since $\alpha\leqslant\alpha\vee g$ and $\alpha\in\mathbb{G}^{\top}\in\mathcal{F}(\mathbb{E})$, then $\alpha\vee g\in\mathbb{G}^{\top}$. By Proposition 3.6 (v), $\mathbb{G}^{\top}\subseteq(\mathbb{F}:\mathbb{G})$ and so $\alpha\vee g\in(\mathbb{F}:\mathbb{G})$. Moreover, $1\neq\alpha\vee g\in\mathbb{G}^{\top}$ and from $\mathbb{G}\cap\mathbb{G}^{\top}=\{1\}$ we have $\alpha\vee g\notin\mathbb{G}$. Since $\mathbb{F}\subseteq\mathbb{G}$, we get $\alpha\vee g\notin\mathbb{F}$. Hence, $\alpha\vee g\in(\mathbb{F}:\mathbb{G})$ and $\alpha\vee g\notin\mathbb{F}=(\mathbb{F}:\mathbb{G})\cap(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$, by Proposition 3.6 (xii). Thus, $\alpha\vee g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$ and since $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\in\mathcal{F}(\mathbb{E})$, we have $g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. Indeed, from $g\notin\mathbb{G}$ we conclude $g\notin(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$, which yields $(\mathbb{F}:(\mathbb{F}:\mathbb{G}))\subseteq\mathbb{G}$. Therefore $\mathbb{G}=(\mathbb{F}:(\mathbb{F}:\mathbb{G}))$. □

Corollary 3.25. *If* $\mathbb{F}=\{1\}$, *then* $\mathbb{G}$ *is* $\mathbb{F}$-*involutive if and only if* $\mathbb{G}=\mathbb{G}^{\top\top}$.

P r o o f. By Proposition 3.24, the proof is straightforward. □

Proposition 3.26. *If* $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, *then* $\mathbb{G}/\mathbb{F} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E}/\mathbb{F})$.

Proof. By Proposition 3.21 (ii), we get $(\mathbb{G}/\mathbb{F})^{\top\top} = (\mathbb{F} : (\mathbb{F} : \mathbb{G}))/\mathbb{F} = \mathbb{G}/\mathbb{F}$. Thus, by Proposition 3.24, $\mathbb{G}/\mathbb{F}$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}/\mathbb{F}$. $\square$

Proposition 3.27.

(i) $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = \{(\mathbb{F} : \mathbb{G})\colon \mathbb{F} \subseteq \mathbb{G} \in \mathcal{F}(\mathbb{E})\}$.

(ii) $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = \{(\mathbb{F} : \mathcal{X})\colon \mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}\}$.

(iii) *If* $\mathbb{G}, \mathbb{H} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$ *such that* $\mathbb{G} \subseteq \mathbb{H}$, *then* $\mathbb{G} \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}$.

Proof. (i) Take $B := \{(\mathbb{F} : \mathbb{G})\colon \mathbb{F} \subseteq \mathbb{G},\ \mathbb{G} \in \mathcal{F}(\mathbb{E})\}$. Then for any $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, we have $\mathbb{G} = (\mathbb{F} : (\mathbb{F} : \mathbb{G}))$. Now, suppose $\mathbb{H} := (\mathbb{F} : \mathbb{G})$. Thus, by Propositions 3.5 and 3.6 (i), we get $\mathbb{H} \in \mathcal{F}(\mathbb{E})$ such that $\mathbb{F} \subseteq \mathbb{H}$. Hence, $\mathbb{G} = (\mathbb{F} : \mathbb{H}) \in B$ and so $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) \subseteq B$. Conversely, if $(\mathbb{F} : \mathbb{G}) \in B$, then by Proposition 3.6 (xiv), we get $(\mathbb{F} : \mathbb{G}) = (\mathbb{F} : (\mathbb{F} : (\mathbb{F} : \mathbb{G})))$. Thus, $(\mathbb{F} : \mathbb{G})$ is an $\mathbb{F}$-involutive filter of $\mathbb{E}$, i.e., $(\mathbb{F} : \mathbb{G}) \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) = B$.

(ii) Suppose $C := \{(\mathbb{F} : \mathcal{X})\colon \mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}\}$. By (i), it is obvious that $\mathcal{S}_{\mathbb{F}}(\mathbb{E}) \subseteq C$. Now, let $(\mathbb{F} : \mathcal{X}) \in C$ such that $\mathbb{F} \subseteq \mathcal{X} \subseteq \mathbb{E}$. For any $\emptyset \neq \mathcal{X} \subseteq \mathbb{E}$, by Proposition 3.8, we have $(\mathbb{F} : \mathcal{X}) = (\mathbb{F} : \langle\mathcal{X}\rangle)$ such that $\mathbb{F} \subseteq \mathcal{X} \subseteq \langle\mathcal{X}\rangle \in \mathcal{F}(\mathbb{E})$ and so, $C \subseteq \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $C = \mathcal{S}_{\mathbb{F}}(\mathbb{E})$.

(iii) Since $\mathbb{G} \subseteq \mathbb{H}$, by Proposition 3.6 (iv), $(\mathbb{F} : \mathbb{H}) \subseteq (\mathbb{F} : \mathbb{G})$. By $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, Proposition 3.6 (i) and (xi), we get $\mathbb{F} \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{H}) \subseteq \mathbb{G} \cap (\mathbb{F} : \mathbb{G}) = \mathbb{F}$. Therefore $\mathbb{G} \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}$. $\square$

Proposition 3.28. *Let* $\mathbb{F}, \mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$. *Then* $(\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H})) = (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$.

Proof. Since $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{G}, \mathbb{H}$, by Proposition 3.6 (iv), $(\mathbb{F} : \mathbb{G}), (\mathbb{F} : \mathbb{H}) \subseteq (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Again by Proposition 3.6 (iv), we get $(\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H})) \subseteq (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$. Conversely, let $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H}))$ and $\gamma \in (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Then for all $g \in \mathbb{G}$ and $h \in \mathbb{H}$, we have $g, h \leqslant g \vee h$ and by $\mathbb{G}, \mathbb{H} \in \mathcal{F}(\mathbb{E})$, $g \vee h \in \mathbb{G} \cap \mathbb{H}$. Thus, $\gamma \vee (g \vee h) \in \mathbb{F}$. Since $\gamma \vee (g \vee h) \leqslant (\alpha \vee \gamma) \vee (g \vee h)$ and $\mathbb{F} \in \mathcal{F}(\mathbb{E})$, we have $(\alpha \vee \gamma \vee g) \vee h \in \mathbb{F}$ for all $h \in \mathbb{H}$ and so

$$(\alpha \vee \gamma) \vee g \in (\mathbb{F} : \mathbb{H}). \tag{3.1}$$

Also, $\alpha \leqslant (\alpha \vee \gamma) \vee g$ and $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \in \mathcal{F}(\mathbb{E})$. Thus, by Proposition 3.6 (xii),

$$(\alpha \vee \gamma) \vee g \in (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \cap (\mathbb{F} : \mathbb{H}) = \mathbb{F}. \tag{3.2}$$

Hence, for all $g \in \mathbb{G}$, $(\alpha \vee \gamma) \vee g \in \mathbb{F}$, and so $\alpha \vee \gamma \in (\mathbb{F} : \mathbb{G})$. Moreover, by $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \in \mathcal{F}(\mathbb{E})$ and $\alpha \leqslant \alpha \vee \gamma$, we have $\alpha \vee \gamma \in (\mathbb{F} : (\mathbb{F} : \mathbb{G}))$. So by

Proposition 3.6 (xii),

$$\alpha \vee \gamma \in (\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : \mathbb{G}) = \mathbb{F} \tag{3.3}$$

for any $\gamma \in (\mathbb{F} : \mathbb{G} \cap \mathbb{H})$. Thus, $\alpha \in (\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H}))$ and so $(\mathbb{F} : (\mathbb{F} : \mathbb{G})) \cap (\mathbb{F} : (\mathbb{F} : \mathbb{H})) \subseteq (\mathbb{F} : (\mathbb{F} : \mathbb{G} \cap \mathbb{H}))$. Therefore the proof is complete. □

Lemma 3.29. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ *is a complete bounded lattice, where, for any subfamily* $\{\mathbb{G}_i\}_{i\in I}$ *in* $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$*, the operations "*$\wedge$*" and "*$\vee$*" on* $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$ *are defined as follows:*

$$\bigwedge_{i\in I} \mathbb{G}_i = \bigcap_{i\in I} \mathbb{G}_i, \quad \textit{and} \quad \bigvee_{i\in I} \mathbb{G}_i = \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup i \in I \mathbb{G}_i\Big)\Big).$$

Proof. By Proposition 3.6 (iii), $\mathbb{F}$ and $\mathbb{E}$ are the least and the greatest elements of $\mathcal{S}_{\mathbb{F}}(\mathbb{E})$, respectively. Let $\{\mathbb{G}_i\}_{i\in I} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Then by Proposition 3.28, we get $\Big(\mathbb{F} : \Big(\mathbb{F} : \bigcap\limits_{i\in I} \mathbb{G}_i\Big)\Big) = \bigcap\limits_{i\in I} (\mathbb{F} : (\mathbb{F} : \mathbb{G}_i)) = \bigcap\limits_{i\in I} \mathbb{G}_i$. Thus, $\bigwedge\limits_{i\in I} \mathbb{G}_i \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Moreover, by Proposition 3.6 (xiv), we have

$$\Big(\mathbb{F} : \Big(\mathbb{F} : \bigvee_{i\in I} \mathbb{G}_i\Big)\Big) = \Big(\mathbb{F} : \underbrace{\Big(\mathbb{F} : \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup_{i\in I} \mathbb{G}_i\Big)\Big)\Big)}\Big) = \Big(\mathbb{F} : \Big(\mathbb{F} : \bigcup_{i\in I} \mathbb{G}_i\Big)\Big) = \bigvee_{i\in I} \mathbb{G}_i.$$

Hence, $\bigvee\limits_{i\in I} \mathbb{G}_i \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete bounded lattice. □

Proposition 3.30. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ *is a complemented lattice.*

Proof. Let $\mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Then $\mathbb{F} \subseteq \mathbb{G}$ and by Proposition 3.6 (xi), we get $(\mathbb{F} : \mathbb{G}) \cap \mathbb{G} = \mathbb{F}$. Also,

$$\begin{aligned}
(\mathbb{F} : \mathbb{G}) \vee \mathbb{G} &= (\mathbb{F} : \underbrace{(\mathbb{F} : [(\mathbb{F} : \mathbb{G}) \cup \mathbb{G}])}) && \text{by definition of } \vee\text{-operation} \\
&= (\mathbb{F} : (\underbrace{(\mathbb{F} : (\mathbb{F} : \mathbb{G}))} \cap (\mathbb{F} : \mathbb{G}))) && \text{by Proposition 3.6 (vii)} \\
&= (\mathbb{F} : (\mathbb{G} \cap (\mathbb{F} : \mathbb{G}))) && \text{since } \mathbb{G} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E}) \\
&= (\mathbb{F} : \mathbb{F}) && \text{by Proposition 3.6 (xi)} \\
&= \mathbb{E} && \text{Proposition 3.6 (ii).}
\end{aligned}$$

Hence, $(\mathbb{F} : \mathbb{G})$ is a complemented lattice of $\mathbb{G}$ relative to $\mathbb{F}$. Therefore

$$(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$$

is a complemented lattice. □

Theorem 3.31. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ *is a complete Boolean lattice.*

Proof. By Lemma 3.29 and Proposition 3.30, we have that $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete and complemented lattice. So, it is enough to show the distribution:

For this, let $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. Since $\mathbb{H} \cap \mathbb{K} \subseteq \mathbb{H}, \mathbb{K}$, then it is easy to see that

$$\mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}) \subseteq (\mathbb{G} \vee \mathbb{H}) \cap (\mathbb{G} \vee \mathbb{K}). \tag{3.4}$$

For the converse, we know that

$$\mathbb{H} \cap \mathbb{K} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}), \quad \mathbb{G} \cap \mathbb{K} \subseteq \mathbb{G} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.5}$$

So, by Proposition 3.27 (iii), we get

$$(\mathbb{H} \cap \mathbb{K}) \cap \underbrace{(\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))}_{B} = \mathbb{F}, \quad (\mathbb{G} \cap \mathbb{K}) \cap \underbrace{(\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))}_{B} = \mathbb{F}. \tag{3.6}$$

Hence, $\mathbb{H} \cap (\mathbb{K} \cap B) = \mathbb{F} = \mathbb{G} \cap (\mathbb{K} \cap B)$. Since by Proposition 3.17, $(\mathbb{F} : \mathbb{H})$ and $(\mathbb{F} : \mathbb{G})$ are relative pseudo complements of $\mathbb{H}$ and $\mathbb{G}$ with respect to $\mathbb{F}$, respectively, we get $(\mathbb{K} \cap B) \subseteq (\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G})$. Now, by Proposition 3.27,

$$(\mathbb{K} \cap B) \cap \underbrace{(\mathbb{F} : ((\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G})))}_{C} = \mathbb{F}. \tag{3.7}$$

Thus, $(\mathbb{K} \cap B) \cap C = \mathbb{F}$ and so $(C \cap \mathbb{K}) \cap B = \mathbb{F}$. By Proposition 3.17, $(\mathbb{F} : B)$ is a relative pseudo complement of B with respect to $\mathbb{F}$ and so

$$(C \cap \mathbb{K}) \subseteq (\mathbb{F} : B) = (\mathbb{F} : (\mathbb{F} : \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}))) = \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.8}$$

Moreover, from Propositions 3.6 (vii) and 3.8, we get

$$C = (\mathbb{F} : ((\mathbb{F} : \mathbb{H}) \cap (\mathbb{F} : \mathbb{G}))) = (\mathbb{F} : (\mathbb{F} : (\mathbb{H} \cup \mathbb{G}))) = (\mathbb{F} : (\mathbb{F} : \langle \mathbb{H} \cup \mathbb{G} \rangle)) = \mathbb{H} \vee \mathbb{G}. \tag{3.9}$$

Therefore, by (3.8) and (3.9), we get for all $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$,

$$(\mathbb{G} \vee \mathbb{H}) \cap \mathbb{K} \subseteq \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \tag{3.10}$$

Now, we have

$$\begin{aligned} (\mathbb{G} \vee \mathbb{H}) \cap \underbrace{(\mathbb{G} \vee \mathbb{K})} &\subseteq \mathbb{G} \vee (\mathbb{H} \cap (\mathbb{G} \vee \mathbb{K})) \quad \text{by (3.10)} \\ &\subseteq \mathbb{G} \vee (\mathbb{G} \vee (\mathbb{H} \cap \mathbb{K})) \quad \text{by (3.10)} \\ &= \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K}). \end{aligned}$$

Hence, by (3.4), $(\mathbb{G} \vee \mathbb{H}) \cap (\mathbb{G} \vee \mathbb{K}) = \mathbb{G} \vee (\mathbb{H} \cap \mathbb{K})$. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \vee, \wedge, \mathbb{F}, \mathbb{E})$ is a complete Boolean lattice. □

Theorem 3.32. *The algebraic structure* $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \subseteq, \rightarrow, \odot, \mathbb{F}, \mathbb{E})$ *is a BL-algebra, where operations "$\rightarrow$" and "$\odot$", for any* $\mathbb{G}, \mathbb{H} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$, *are defined as follows:*

$$\mathbb{G} \rightarrow \mathbb{H} := \mathbb{H} \vee (\mathbb{F} : \mathbb{G}), \quad \mathbb{G} \odot \mathbb{H} := \mathbb{G} \cap \mathbb{H}.$$

P r o o f. (BL1) By Lemma 3.29, $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \wedge, \vee, \mathbb{F}, \mathbb{E})$ is a bounded lattice.

(BL2) According to the definition of "$\odot$", clearly $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \odot, \mathbb{E})$ is a commutative monoid.

(BL3) Let $\mathbb{G}, \mathbb{H}, \mathbb{K} \in \mathcal{S}_{\mathbb{F}}(\mathbb{E})$. If $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$, then by definition of "$\vee$", we get $\mathbb{G} \subseteq \mathbb{K} \vee (\mathbb{F} : \mathbb{H})$. Moreover,

$$\begin{aligned}
\mathbb{G} \odot \mathbb{H} = \mathbb{G} \cap \mathbb{H} &\subseteq (\mathbb{K} \vee (\mathbb{F} : \mathbb{H})) \cap \mathbb{H} && \\
&= (\mathbb{K} \cap \mathbb{H}) \vee \underbrace{((\mathbb{F} : \mathbb{H}) \cap \mathbb{H})} && \text{by Theorem 3.31} \\
&= (\mathbb{K} \cap \mathbb{H}) \vee \mathbb{F} && \text{by Proposition 3.27 (iii)} \\
&= \mathbb{K} \cap \mathbb{H} \subseteq \mathbb{K} && \text{by Lemma 3.29.}
\end{aligned}$$

So, $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$ implies $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$. Conversely, let $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$. Then by the definition of "$\odot$", we have $\mathbb{G} \cap \mathbb{H} \subseteq \mathbb{K}$ and so

$$\begin{aligned}
\mathbb{G} = \mathbb{G} \cap \mathbb{E} = \mathbb{G} \cap [\mathbb{H} \vee (\mathbb{F} : \mathbb{H})] && \text{by Proposition 3.30} \\
= (\mathbb{G} \cap \mathbb{H}) \vee [\mathbb{G} \cap (\mathbb{F} : \mathbb{H})] && \text{by Theorem 3.31} \\
\subseteq \mathbb{K} \vee [\mathbb{G} \cap (\mathbb{F} : \mathbb{H})] \subseteq \mathbb{K} \vee (\mathbb{F} : \mathbb{H}) && \\
= \mathbb{H} \rightarrow \mathbb{K} && \text{by definition of "}\rightarrow\text{" operation.}
\end{aligned}$$

Thus, $\mathbb{G} \odot \mathbb{H} \subseteq \mathbb{K}$ implies $\mathbb{G} \subseteq \mathbb{H} \rightarrow \mathbb{K}$. Therefore (BL3) is satisfied. Moreover, we have

$$\begin{aligned}
\mathbb{G} \odot (\mathbb{G} \rightarrow \mathbb{H}) &= \mathbb{G} \cap (\mathbb{G} \rightarrow \mathbb{H}) && \\
&= \mathbb{G} \cap (\mathbb{H} \vee (\mathbb{F} : \mathbb{G})) && \\
&= (\mathbb{G} \cap \mathbb{H}) \vee \underbrace{[\mathbb{G} \cap (\mathbb{F} : \mathbb{G})]} && \text{by Theorem 3.31} \\
&= (\mathbb{G} \cap \mathbb{H}) \vee \mathbb{F} && \text{by Proposition 3.27 (iii)} \\
&= \mathbb{G} \cap \mathbb{H} && \text{by Lemma 3.29} \\
&= \mathbb{G} \odot \mathbb{H}. &&
\end{aligned}$$

Hence, (BL4) is satisfied. Also,

$$\begin{aligned}(\mathbb{G} \to \mathbb{H}) \vee (\mathbb{H} \to \mathbb{G}) &= [\mathbb{H} \vee (\mathbb{F} : \mathbb{G})] \vee [\mathbb{G} \vee (\mathbb{F} : \mathbb{H})] \\ &= [\mathbb{H} \vee (\mathbb{F} : \mathbb{H})] \vee [\mathbb{G} \vee (\mathbb{F} : \mathbb{G})] \quad \text{by associativity of “}\vee\text{”} \\ &= \mathbb{E} \vee \mathbb{E} = \mathbb{E} \quad \text{by Proposition 3.30.}\end{aligned}$$

So, (BL5) is satisfied. Therefore $(\mathcal{S}_{\mathbb{F}}(\mathbb{E}), \subseteq, \to, \odot, \mathbb{F}, \mathbb{E})$ is a BL-algebra. $\square$

4. Conclusions and future works

In this paper, the notion of relative co-annihilator in lattice equality algebras was introduced. Many properties of relative co-annihilators were investigated, the set of all $\mathbb{F}$-involutive filters of $\mathbb{E}$ was defined and showed that it can be made as a BL-algebra.

In our future work, we will continue our study of algebraic properties of this special sets and we will investigate the relation between relative co-annihilators and some special filters in equality algebras.

References

[1] *H. A. S. Abujabal, M. A. Obaid, M. Aslam, A. B. Thaheem*: On annihilators of BCK-algebras. Czech. Math. J. *45* (1995), 727–735. zbl MR doi

[2] *R. A. Borzooei, F. Zebardast, M. Aaly Kologani*: Some type filters in equality algebras. Categ. Gen. Algebr. Struct. Appl. *7* (2017), 33–55. zbl MR

[3] *S. Burris, H. P. Sankappanavar*: A Course in Universal Algebra. Graduate Texts in Mathematics 78. Springer, New York, 1981. zbl MR doi

[4] *L. C. Ciungu*: Internal states on equality algebras. Soft Comput. *19* (2015), 939–953. zbl doi

[5] *B. A. Davey*: Some annihilator conditions on distributive lattices. Algebra Univers. *4* (1974), 316–322. zbl MR doi

[6] *A. Filipoiu*: About Baer extensions of MV-algebras. Math. Jap. *40* (1994), 235–241. zbl MR

[7] *P. Hájek*: Metamathematics of Fuzzy Logic. Trends in Logic-Studia Logica Library 4. Kluwer Academic, Dordrecht, 1998. zbl MR doi

[8] *R. Halaš*: Annihilators in BCK-algebras. Czech. Math. J. *53* (2003), 1001–1007. zbl MR doi

[9] *S. Jenei*: Equality algebras. Stud. Log. *100* (2012), 1201–1209. zbl MR doi

[10] *S. Jenei, L. Kóródi*: On the variety of equality algebras. Proceedings of the 7th conference of the European Society for Fuzzy Logic and Technology. Atlantis Press, Amsterdam, 2011, pp. 153–155. zbl doi

[11] *L. Leurstean*: Some algebraic properties of non-commutative fuzzy structures. Stud. Inf. Control *9* (2000), 365–370.

[12] *L. Leurstean*: Baer extensions of BL-algebras. J. Mult.-Val. Log. Soft Comput. *12* (2006), 321–336. zbl MR

[13] *F. G. Maroof, A. Borumand Saeid, E. Eslami*: On co-annihilators in residuated lattices. J. Intell. Fuzzy Syst. *31* (2016), 1263–1270. zbl doi

[14] *B. L. Meng, X. L. Xin*: Generalized co-annihilator of BL-algebras. Open Math. *13* (2015), 639–654. zbl MR doi
[15] *S. Niazian, M. Aaly Kologani, S. Homayon Arya, R. A. Borzooei*: On co-annihilators in equality algebras. Miskolc Math. Notes *24* (2023), 933–951. zbl MR doi
[16] *A. Paad*: Ideals in bounded equality algebras. Filomat *33* (2019), 2113–2123. zbl MR doi
[17] *E. Turunen*: BL-algebras of basic fuzzy logic. Mathware Soft Comput. *6* (1999), 49–61. zbl MR
[18] *Y. Xu, D. Ruan, K. Qin, J. Liu*: Lattice-Valued Logic: An Alternative Approach to Treat Fuzziness and Incomparability. Studies in Fuzziness and Soft Computing 132. Springer, Berlin, 2003. zbl MR doi
[19] *F. Zebardast, R. A. Borzooei, M. Aaly Kologani*: Results on equality algebras. Inf. Sci. *381* (2017), 270–282. zbl MR doi
[20] *Y. X. Zou, X. L. Xin, P. F. He*: On annihilators in BL-algebras. Open Math. *14* (2016), 324–337. zbl MR doi

Authors' addresses: *Sogol Niazian* (corresponding author), Tehran Medical Sciences, Islamic Azad University, Tehran, Iran, e-mail: `s.niazian@iautmu.ac.ir`; *Mona Aali Kologani*, Hatef Higher Education Institute, Zahedan, Iran, e-mail: `mona4011@gmail.com`; *Rajab Ali Borzooei*, Shahid Behashti University, Tehran, Iran, e-mail: `borzooei@sbu.ac.ir`.