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The Clairaut's theorem on rotational surfaces in pseudo-Euclidean 4-space with index 2

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Abstract. Clairaut's theorem is expressed on the surfaces of rotation in semi Euclidean 4-space. Moreover, the general equations of time-like geodesic curves are characterized according to the results of Clairaut's theorem on the hyperbolic surfaces of rotation and the elliptic surface of rotation, respectively.

Keywords: Clairaut's theorem; surfaces of rotation; pseudo-Euclidean 4-space; geodesic curve

Classification: 53A35, 53B30, 53B50

1. Introduction

The geodesics for rotational surfaces have been studied for a long time and many examples of rotational surfaces have been discovered. A helpful tool in the understanding of the geodesics on rotational surfaces is Clairaut's theorem. It gives a well-known characterization of geodesics on surfaces of rotation.

Many studies of surfaces of rotation have received much attention from our researchers. Among them, one can cite our work [3], in which we described the rotational surfaces using curves and matrices which are the subgroups of rotating a selected axis in Galilean 4-space. We examined the tube surfaces generated by the curve in Galilean 3-space and gave certain results of describing the geodesics on the surfaces, see [2], [4]. We gave the surfaces of rotation generated by a magnetic curve. Also, we gave the conditions being geodesic on these rotational surfaces in null cone 3-space, with the help of Clairaut's theorem, see [1]. In our study [5] we expressed the hyperbolic and the elliptic rotational surfaces using a curve and matrices in 4-dimensional semi-Euclidean space. W. Goemans constructed a new type of surfaces in Euclidean and Lorentz–Minkowski 4-space and proved the classification theorems of flat double rotational surfaces, see [8]. C. M. Hoffmann and J. Zhou in [9], discussed some issues of displaying 2D surfaces in 4D space, including the behavior of surface normals under projection, the silhouette points

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due to the projection, and methods for object orientation and projection center specification.

2. Preliminaries

Let E_2^4 denote the 4-dimensional pseudo-Euclidean space with signature $(2,4)$, that is, the real vector space $\mathbb{R}^4$ endowed with the metric $\langle\,,\rangle_{E_2^4}$ which is defined by

$$\langle\,,\rangle_{E_2^4} = -\mathrm{d}x_1^2 - \mathrm{d}x_2^2 + \mathrm{d}x_3^2 + \mathrm{d}x_4^2 \tag{2.1}$$

or

$$g = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \tag{2.2}$$

where (x_1, x_2, x_3, x_4) is a standard rectangular coordinate system in E_2^4.

Recall that an arbitrary vector $v \in E_2^4 \backslash \{0\}$ can have one of three characters: it can be space-like if $g(v,v) > 0$ or $v = 0$, time-like if $g(v,v) < 0$ and null if $g(v,v) = 0$ and $v \neq 0$.

The norm of a vector v is given by $\|v\| = \sqrt{g(v,v)}$ and two vectors v and w are said to be orthogonal if $g(v,w) = 0$. A space-like or time-like curve $x(s)$ has unit speed, if $g(x', x') = \pm 1$.

Let (x_1, x_2, x_3, x_4), (y_1, y_2, y_3, y_4), (z_1, z_2, z_3, z_4) be any three vectors in E_2^4. The pseudo-Euclidean cross product is given as

$$x \wedge y \wedge z = \begin{pmatrix} -i_1 & -i_2 & i_3 & i_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{pmatrix},$$

where $i_1 = (1,0,0,0)$, $i_2 = (0,1,0,0)$, $i_3 = (0,0,1,0)$, $i_4 = (0,0,0,1)$, see [9].

The pseudo-Riemannian sphere $S_2^3(m,r)$ centered at $m \in E_2^4$ with radius $r > 0$ of E_2^4 is defined by

$$S_2^3(m,r) = \big\{x \in E_2^4 \colon \langle x - m, x - m\rangle = r^2\big\}.$$

The pseudo-hyperbolic space $H_1^3(m,r)$ centered at $m \in E_2^4$ with radius $r > 0$ of E_2^4 is defined by

$$H_1^3(m,r) = \big\{x \in E_2^4 \colon \langle x - m, x - m\rangle = -r^2\big\}.$$

The pseudo-Riemannian sphere $S^3_2(m,r)$ is diffeomorphic to $\mathbb{R}^2 \times S^1$, while $H^3_1(m,r)$ is diffeomorphic to $S^1 \times \mathbb{R}^2$. The hyperbolic space $H^3(m,r)$ is given by

$$H^3(m,r) = \{x \in E^4_2 \colon \langle x-m, x-m\rangle = -r^2,\ x_1 > 0\},$$

see [7], [12], [14].

Definition 1 ([10]). A one-parameter group of diffeomorphisms of a manifold M is a regular map $\psi\colon M \times \mathbb{R} \to M$, such that $\psi_t(x) = \psi(x,t)$, where

(1) $\psi_t\colon M \to M$ is a diffeomorphism;
(2) $\psi_0 = \mathrm{id}$;
(3) $\psi_{s+t} = \psi_s \circ \psi_t$.

This group is attached with a vector field W given by $\frac{\mathrm{d}}{\mathrm{d}t}\psi_t(x) = W(x)$, and the group of diffeomorphisms is said to be the flow of W.

Definition 2. If a one-parameter group of isometries is generated by a vector field W, then this vector field is called as a Killing vector field, see [10].

Definition 3. Let W be a vector field on a smooth manifold M and ψ_t be the local flow generated by W. For each $t \in \mathbb{R}$, the map ψ_t is a diffeomorphism of M and given a function f on M, one considers the pull-back $\psi_t f$ and one defines the Lie derivative of the function f as to W by

$$L_W f = \lim_{t\to 0}\Big(\frac{\psi_t f - f}{t}\Big) = \frac{\mathrm{d}\psi_t f}{\mathrm{d}t}_{t=0}. \tag{2.3}$$

Let $g_{\xi\varrho}$ be any pseudo-Riemannian metric, then the derivative is given as

$$L_W g_{\xi\varrho} = g_{\xi\varrho,z} W^z + g_{\xi z} W^z_\varrho + g_{z\varrho} W^z_\xi.$$

In Cartesian coordinates in Euclidean spaces where $g_{\xi\varrho,z} = 0$, the Lie derivative is given by

$$L_W g_{\xi\varrho} = g_{\xi z} W^z_\varrho + g_{z\varrho} W^z_\xi.$$

In [6], [10], [11], [15], the vector W generates a Killing field if and only if

$$L_W g = 0.$$

Theorem 1. *Let ξ be a geodesic on a surface of revolution Υ and let ϱ be the distance function of a point of Υ from the axis of rotation, and let θ be the angle between ξ and the meridians of Υ. Then, $\varrho\sin\theta$ is constant along ξ. Conversely, if $\varrho\sin\theta$ is constant along some curve ξ in the surface, and if no part of ξ is part of some parallel of Υ, then ξ is a geodesic, see* [13].

Theorem 2. *Let the pseudo-Euclidean group be a subgroup of the diffeomorphisms group in E_2^4 and let W be a vector field which generates the isometries. Then, the Killing vector field associated with the metric g is given as*

$$\begin{aligned} W(\xi,\varrho,\vartheta,\eta) &= a(\eta\partial\xi+\xi\partial\eta)+b(\vartheta\partial\varrho+\varrho\partial\vartheta)+c(\vartheta\partial\xi+\xi\partial\vartheta) \\ &\quad + d(\eta\partial\varrho+\varrho\partial\eta)+e(\vartheta\partial\eta-\eta\partial\vartheta)+f(\xi\mathrm{d}\varrho-\varrho\mathrm{d}\xi), \end{aligned}$$

where $a,b,c,d,e,f\in\mathbb{R}_0^+$, *see* [5].

Theorem 3. *Let $W(\xi,\varrho,\vartheta,\eta)$ be the Killing vector field and let γ be a curve in E_2^4, then the surfaces of rotation are given as follows:*

(1) *For the rotations $\Omega_1=\vartheta\mathrm{d}\xi+\xi\mathrm{d}\vartheta$ and $\Omega_4=\eta\mathrm{d}\varrho+\varrho\mathrm{d}\eta$, the hyperbolic surface of rotation is given as*

$$\Upsilon^1(x,\alpha,s)=\left(\begin{array}{c} f_1\cosh x+f_3\sinh x, f_2\cosh\alpha+f_4\sinh\alpha \\ f_1\sinh x+f_3\cosh x, f_2\sinh\alpha+f_4\cosh\alpha \end{array}\right).$$

(2) *For the rotations $\Omega_2=\eta\mathrm{d}\xi+\xi\mathrm{d}\eta$ and $\Omega_3=\vartheta\mathrm{d}\varrho+\varrho\mathrm{d}\vartheta$, the hyperbolic surface of rotation is given as*

$$\Upsilon^2(y,z,s)=\left(\begin{array}{c} f_1\cosh y+f_4\sinh y, f_2\cosh z+f_3\sinh z \\ f_2\sinh z+f_3\cosh z, f_1\sinh y+f_4\cosh y \end{array}\right).$$

(3) *For the rotations $\Omega_5=\xi\mathrm{d}\varrho-\varrho\mathrm{d}\xi$ and $\Omega_6=\vartheta\mathrm{d}\eta-\eta\mathrm{d}\vartheta$, the elliptic surface of rotation is given as*

$$\Upsilon^3(\beta,\theta,s)=\left(\begin{array}{c} f_1\cos\beta+f_2\sin\beta,-f_1\sin\beta+f_2\cos\beta \\ f_3\cos\theta+f_4\sin\theta,-f_3\sin\theta+f_4\cos\theta \end{array}\right),$$

where $-\infty<x,y,z,\alpha,\beta,\theta<\infty$, $s\in I$, *see* [5].

3. Clairaut's theorem on the surfaces of rotation in E_2^4

In this section, one will use three different types of surfaces of rotation given in the previous section, and will generalize Clairaut's theorem to these surfaces in E_2^4.

3.1 Clairaut's theorem on the hyperbolic surface of rotation Υ^1. In this section, one will use the hyperbolic surface of rotation parametrized as

$$\Upsilon^1(x,\alpha,t)=\left(\begin{array}{c} f_1\cosh x+f_3\sinh x, f_2\cosh\alpha+f_4\sinh\alpha \\ f_1\sinh x+f_3\cosh x, f_2\sinh\alpha+f_4\cosh\alpha \end{array}\right).$$

Also, one can take the planar curve γ for this surface of rotation to be the intersection of $\Upsilon^1(x,\alpha,t)$ with $\varrho,\vartheta = 0$. Therefore, one can write that the curve γ lies on the $\xi\eta$-plane, and the curve can be written by

$$\gamma(t) = (f_1(t), 0, 0, f_4(t)), \qquad f_1, f_4 \in C^\infty,$$

then one has the parametrization

$$\begin{aligned}
\Upsilon^1(x,\alpha,t) &= (f_1\cosh x, f_4\sinh\alpha, f_1\sinh x, f_4\cosh\alpha),\\
\Upsilon^1_x(x,\alpha,t) &= (f_1\sinh x, 0, f_1\cosh x, 0),\\
\Upsilon^1_\alpha(x,\alpha,t) &= (0, f_4\cosh\alpha, 0, f_4\sinh\alpha),\\
\Upsilon^1_t(x,\alpha,t) &= (f_1'\cosh x, f_4'\sinh\alpha, f_1'\sinh x, f_4'\cosh\alpha).
\end{aligned}$$

Hence, from the first fundamental form of the surface Υ^1, one has

$$\begin{gathered}
\langle\Upsilon^1_x,\Upsilon^1_x\rangle = f_1^2, \qquad \langle\Upsilon^1_\alpha,\Upsilon^1_\alpha\rangle = -f_4^2, \qquad \langle\Upsilon^1_t,\Upsilon^1_t\rangle = f_4'^2 - f_1'^2;\\
\langle\Upsilon^1_x,\Upsilon^1_\alpha\rangle, \langle\Upsilon^1_x,\Upsilon^1_t\rangle, \langle\Upsilon^1_\alpha,\Upsilon^1_t\rangle = 0,\\
I_{\Upsilon^1} = \begin{pmatrix} f_1^2 & 0 & 0\\ 0 & -f_4^2 & 0\\ 0 & 0 & f_4'^2 - f_1'^2\end{pmatrix}
\end{gathered} \tag{3.1}$$

and one can write Lagrangian equation

$$L = f_1^2\dot{x}^2 - f_4^2\dot{\alpha}^2 + (f_4'^2 - f_1'^2)\dot{t}^2.$$

So, the curve γ is time-like and one writes $f_4'^2 - f_1'^2 = -1$, $f_1^2 > 0$, $-f_4^2 < 0$. Then, one gets

$$L = f_1^2\dot{x}^2 - f_4^2\dot{\alpha}^2 - \dot{t}^2.$$

Hence, one can write following equations by using Clairaut's theorem,

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial x}{\partial s}}\right) &= \frac{\partial L}{\partial x} \Rightarrow \frac{\partial}{\partial s}(2f_1^2\dot{x}) = 0,\\
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \alpha}{\partial s}}\right) &= \frac{\partial L}{\partial \alpha} \Rightarrow \frac{\partial}{\partial s}(-2f_4^2\dot{\alpha}) = 0,\\
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial t}{\partial s}}\right) &= \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s}(-2\dot{t}) = 2f_1'f_1\dot{x}^2 - f_4'f_4\dot{\alpha}^2.
\end{aligned}$$

Assume that $\gamma(t)$ is a geodesic on the surface Υ^1. Hence, the curve $\gamma(t)$ can be written as

$$\dot{\gamma} = \dot{x}\,\Upsilon^1_x + \dot{\alpha}\,\Upsilon^1_\alpha + \dot{t}\,\Upsilon^1_t.$$

One can note that $\Upsilon_x^1 = f_1 N_x$ is a unit space-like vector pointing along x-axis of the meridians, and $\Upsilon_\alpha^1 = f_4 N_\alpha$ is a unit time-like vector pointing along the α-axis of the parallels. Also $\Upsilon_t^1 = N_t$ is a unit time-like vector pointing along t-axis of the parallels. It also follows that the plane spanned by N_t, N_α is space-like and $\{\Upsilon_x^1, \Upsilon_\alpha^1, \Upsilon_t^1\}$ is an orthonormal basis. Also, from (3.1), one gets

$$\dot{\gamma} = f_1 \dot{x} N_x + f_4 \dot{\alpha} N_\alpha + \dot{t} N_t. \tag{3.2}$$

Note that if γ is time-like, since $N_x^\perp \in Sp\{N_\alpha, N_t\}$ one gets

$$\begin{aligned} \dot{\gamma} &= f_1 N_x \dot{x} + (f_2 N_\alpha \dot{\alpha} + \dot{t} N_t) = N_x \cos\varphi_1 + N_x^\perp \sin\varphi_1 \\ &= \cos\varphi_1 N_x + \cosh\theta_1 \sin\varphi_1 N_\alpha + \sinh\theta_1 \sin\varphi_1 N_t. \end{aligned} \tag{3.3}$$

Also, from (3.2) and (3.3), one can write

$$f_1 \dot{x} = \cos\varphi_1, \qquad f_4 \dot{\alpha} = \cosh\theta_1 \sin\varphi_1, \qquad \dot{t} = \sinh\theta_1 \sin\varphi_1. \tag{3.4}$$

Similarly, for the curve $\gamma(t) = (0, f_2(t), f_3(t), 0)$; $f_2, f_3 \in C^\infty$. Then, the first fundamental form is written by

$$I_{S^1} = \begin{pmatrix} -f_3^2 & 0 & 0 \\ 0 & f_2^2 & 0 \\ 0 & 0 & f_3'^2 - f_2'^2 \end{pmatrix}$$

and similar calculations are obtained.

Hence, from the Lagrangian equation, one has

$$2\ddot{t} = 2f_1' f_1 \dot{x}^2 - f_4' f_4 \dot{\alpha}^2$$

and

$$\frac{\partial}{\partial s}(2f_1^2 \dot{x}) = 0 \Rightarrow x = \frac{c_1}{2f_1^2} + c_3, \qquad \frac{\partial}{\partial s}(-2f_4^2 \dot{\alpha}) = 0 \Rightarrow \alpha = \frac{c_2}{-2f_4^2} + c_4,$$

which gives that $2f_1^2 \dot{x}$ and $-2f_4^2 \dot{\alpha}$ are constant along the geodesic curve. It follows that the geodesics are given by

$$\frac{\partial}{\partial s}(2f_1^2 \dot{x}) = 0, \qquad \frac{\partial}{\partial s}(-2f_4^2 \dot{\alpha}) = 0, \qquad \frac{\partial}{\partial s}(-2\dot{t}) = 2f_1' f_1 \dot{x}^2 - f_4' f_4 \dot{\alpha}^2.$$

Now, by using the equations (3.4) and from the previous equations, one writes

$$f_1 \dot{x} = \cos\varphi_1 \Rightarrow 2f_1^2 \dot{x} = 2f_1 \cos\varphi_1 = \text{cons.} \tag{3.5}$$

$$\begin{aligned} &f_4 \dot{\alpha} = \cosh\theta_1 \sin\varphi_1 \Rightarrow -2f_4^2 \dot{\alpha} = -2f_4 \cosh\theta_1 \sin\varphi_1 = \text{cons.} \\ &-2\dot{t} = -2\sinh\theta_1 \sin\varphi_1 \neq \text{cons.} \end{aligned} \tag{3.6}$$

and for the equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial x}{\partial s}}\right) = \frac{\partial L}{\partial x}$, which means that

$$x = \int \frac{\cos\varphi_1}{f_1}\,\mathrm{d}s \tag{3.7}$$

is constant along the geodesic. Conversely, if γ is a curve with $2f_1\cos\varphi_1 =$ constant, the second equation is satisfied, differentiating L and substituting into the second Euler–Lagrangian equation yields the first Lagrangian equation. Furthermore, for the equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \alpha}{\partial s}}\right) = \frac{\partial L}{\partial \alpha}$,

$$\alpha = \int \frac{1}{f_4}\cosh\theta_1\sin\varphi_1\,\mathrm{d}s \tag{3.8}$$

is constant along the curve γ. Therefore, one gives the following theorem as a result of Clairaut's theorem on the hyperbolic surface of rotation $\Upsilon^1 \subset E^4_2$.

Theorem 4. *Let γ be a time-like geodesic curve on the hyperbolic surface of rotation Υ^1 in the E^4_2, let f_1 and f_4 be the distance functions from the axis of rotation to a point on the surface, φ_1 and θ_1 are the angles between the meridians of the surface and the time-like geodesic γ. Then $2f_1\cos\varphi_1$ and $-2f_4\cosh\theta_1\sin\varphi_1$ are constant along the curve γ. Conversely, if $2f_1\cos\varphi_1$ and $-2f_4\cosh\theta_1\sin\varphi_1$ are constant along γ, if no part of γ is a part of some parallels of the surface of rotation, then γ is time-like geodesic.*

If one wants to obtain the general equation of geodesics, one should consider the Euler–Lagrange equations

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}s} = \frac{1}{f_1}\cos\varphi_1, \qquad \dot{\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}s} = \frac{1}{f_4}\cosh\theta_1\sin\varphi_1 \tag{3.9}$$

then, adding (3.9) to Lagrangian equation L, one has

$$\begin{aligned} L &= f_1^2\Big(\frac{\mathrm{d}x}{\mathrm{d}s}\Big)^2 - f_4^2\Big(\frac{\mathrm{d}\alpha}{\mathrm{d}s}\Big)^2 - \Big(\frac{\mathrm{d}t}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}s}\Big)^2, \\ \frac{\mathrm{d}t}{\mathrm{d}x} &= f_1\sqrt{1-\cosh^2\theta_1\tan^2\varphi_1 - L\sec^2\varphi_1} \end{aligned}$$

or

$$\begin{aligned} L &= f_1^2\Big(\frac{\mathrm{d}x}{\mathrm{d}s}\Big)^2 - f_4^2\Big(\frac{\mathrm{d}\alpha}{\mathrm{d}s}\Big)^2 - \Big(\frac{\mathrm{d}t}{\mathrm{d}\alpha}\frac{\mathrm{d}\alpha}{\mathrm{d}s}\Big)^2, \\ \frac{\mathrm{d}t}{\mathrm{d}\alpha} &= f_2\sqrt{\cot^2\varphi_1\tanh^2\theta_1 - L\,\mathrm{sech}^2\varphi_1\,\mathrm{cosec}^2\varphi_1}. \end{aligned}$$

Theorem 5. *The general equation of geodesics on the hyperbolic surface of rotation $\Upsilon^1 \subset E^4_2$, and for the parameters $\dot{x} = (1/f_1)\cos\varphi_1$ and $\dot{\alpha} = (1/f_4)\cosh\theta_1 \times$*

$\sin\varphi_1$, *are given by*

$$\frac{\mathrm{d}t}{\mathrm{d}x} = f_1\sqrt{1-\cosh^2\theta_1\tan^2\varphi_1 - L\sec^2\varphi_1}$$

or

$$\frac{\mathrm{d}t}{\mathrm{d}\alpha} = f_2\sqrt{\cot^2\varphi_1\tanh^2\theta_1 - L\,\mathrm{sech}^2\varphi_1\,\mathrm{cosec}^2\varphi_1}.$$

3.2 Clairaut's theorem on the hyperbolic surface of rotation Υ^2. In this section, one will use the hyperbolic surface of rotation parametrized as

$$\Upsilon^2(y,z,t) = \begin{pmatrix} f_1\cosh y + f_4\sinh y, f_2\cosh z + f_3\sinh z \\ f_2\sinh z + f_3\cosh z, f_1\sinh y + f_4\cosh y \end{pmatrix},$$

then one can take the planar curve γ for this surface of rotation to be the intersection of $\Upsilon^2(y,z,t)$ with $\vartheta\eta = 0$. Therefore, one can write that the curve γ lies on the $\varrho\xi$-plane, and one writes

$$\gamma(t) = (f_1(t), f_2(t), 0, 0) \quad \text{or} \quad \gamma(t) = (0,0,f_3(t),f_4(t)), \qquad f_i \in C^\infty,$$

then one gets

$$\begin{aligned}\Upsilon^2(y,z,t) &= (f_1\cosh y, f_2\cosh z, f_2\sinh z, f_1\sinh y) \quad \text{or} \\ \Upsilon^2(y,z,t) &= (f_4\sinh y, f_3\sinh z, f_3\cosh z, f_4\cosh y).\end{aligned}$$

Here one will use the surface of rotation generated by the curve $\gamma(t) = (f_1, f_2, 0, 0)$. Furthermore,

$$\begin{aligned}\Upsilon^2_y &= (f_1\sinh y, 0, 0, f_1\cosh y),\\ \Upsilon^2_z &= (0, f_2\sinh z, f_2\cosh z, 0),\\ \Upsilon^2_t &= (f_1'\cosh y, f_2'\cosh z, f_2'\sinh x, f_1'\sinh y),\end{aligned}$$

by resulting in the first fundamental form:

$$\begin{gathered}\langle\Upsilon^2_y,\Upsilon^2_y\rangle = f_1^2, \qquad \langle\Upsilon^2_z,\Upsilon^2_z\rangle = f_2^2, \qquad \langle\Upsilon^2_t,\Upsilon^2_t\rangle = -f_2'^2 - f_1'^2,\\ \langle\Upsilon^2_y,\Upsilon^2_z\rangle, \langle\Upsilon^2_y,\Upsilon^2_t\rangle, \langle\Upsilon^2_z,\Upsilon^2_t\rangle = 0,\\ I_{\Upsilon^2} = \begin{pmatrix} f_1^2 & 0 & 0\\ 0 & f_2^2 & 0 \\ 0 & 0 & -f_2'^2 - f_1'^2\end{pmatrix}\end{gathered} \tag{3.10}$$

and one can write Lagrangian equation as follows

$$L = f_1^2\dot{y}^2 + f_2^2\dot{z}^2 + (-f_2'^2 - f_1'^2)\dot{t}^2.$$

Here, one is interested in the metric in E_2^4. So, one takes γ to be time-like and one writes $-f_2'^2 - f_1'^2 = -1$, $f_1^2, f_2^2 > 0$. Then,

$$L = f_1^2\dot{y}^2 + f_2^2\dot{z}^2 - \dot{t}^2.$$

Hence, one can obtain the following equations using Clairaut's theorem,

$$\begin{aligned}
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial y}{\partial s}}\right) &= \frac{\partial L}{\partial y} \Rightarrow \frac{\partial}{\partial s}(2f_1^2\dot{y}) = 0,\\
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial z}{\partial s}}\right) &= \frac{\partial L}{\partial z} \Rightarrow \frac{\partial}{\partial s}(2f_2^2\dot{z}) = 0,\\
\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial t}{\partial s}}\right) &= \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s}(2\dot{t}) = 2f_1'f_1\dot{y}^2 + 2f_2'f_2\dot{z}^2.
\end{aligned}$$

Let $\gamma(t)$ be a geodesic curve on the surface Υ^2. Hence, $\gamma(t)$ can be written as follows

$$\dot{\gamma} = \dot{y}\,\Upsilon_y^2 + \dot{z}\,\Upsilon_z^2 + \dot{t}\,\Upsilon_t^2.$$

So, one can note that $\Upsilon_t^2 = N_t$ is a unit time-like vector pointing along t-axis of the meridians, and $\Upsilon_z^2 = f_2N_z$ is a unit space-like vector pointing along the z-axis of the parallels. Also, $\Upsilon_y^2 = f_1N_y$ is a unit space-like vector pointing along y-axis of the parallels. It also follows that the plane spanned by N_y, N_z is space-like and an orthonormal basis. Also, from (3.10), one gets

$$\dot{\gamma} = \dot{t}N_t + f_1\dot{y}N_y + f_2\dot{z}N_z. \tag{3.11}$$

Note that if γ is time-like curve, since $N_t^{\perp} \in Sp\{N_y, N_z\}$, one gets

$$\begin{aligned}
\dot{\gamma} &= f_1N_y\dot{y} + f_2N_z\dot{z} + \dot{t}N_t = N_t\cosh\varphi_2 + N_t^{\perp}\sinh\varphi_2\\
&= \cosh\varphi_2 N_t + \cos\theta_2\sinh\varphi_2 N_y + \sinh\varphi_2\sin\theta_2 N_z,
\end{aligned} \tag{3.12}$$

from (3.11) and (3.12), one can write

$$f_1\dot{y} = \cos\theta_2\sinh\varphi_2, \qquad f_2\dot{z} = \sinh\varphi_2\sin\theta_2, \qquad \dot{t} = \cosh\varphi_2. \tag{3.13}$$

Hence, from the Lagrangian equation one has

$$\ddot{t} = f_1'f_1\dot{y}^2 + f_2'f_2\dot{z}^2$$

and

$$\frac{\partial}{\partial s}(2f_1^2\dot{y}) = 0 \Rightarrow y = \frac{c_1^2}{2f_1^2} + c_3^2, \qquad \frac{\partial}{\partial s}(2f_2^2\dot{z}) = 0 \Rightarrow z = \frac{c_2^2}{2f_2^2} + c_4^2,$$

which gives that $2f_1^2\dot{y}$ and $2f_2^2\dot{z}$ are constant along the geodesic. It follows that the geodesics can be written as

$$\frac{\partial}{\partial s}(2f_1^2\dot{y}) = 0, \qquad \frac{\partial}{\partial s}(2f_2^2\dot{z}) = 0, \qquad \frac{\partial}{\partial s}(-\dot{t}) = f_1' f_1 \dot{y}^2 + f_2' f_2 \dot{z}^2.$$

Now, from (3.13) one gets

$$f_1\dot{y} = \cos\theta_2 \sinh\varphi_2 \Rightarrow 2f_1^2\dot{y} = 2f_1 \cos\theta_2 \sinh\varphi_2 = \text{cons.} \tag{3.14}$$

$$f_2\dot{z} = \sinh\varphi_2 \sin\theta_2 \Rightarrow 2f_2^2\dot{z} = 2f_2 \sinh\varphi_2 \sin\theta_2 = \text{cons.} \tag{3.15}$$

$$2\dot{t} = 2\cosh\varphi_2 \neq \text{cons.}$$

for the equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial y}{\partial s}}\right) = \frac{\partial L}{\partial y}$,

$$y = \int \frac{\cos\theta_2 \sinh\varphi_2}{f_1}\,\mathrm{d}s \tag{3.16}$$

is a constant. Conversely, for the condition $2f_1 \cos\theta_2 \sinh\varphi_2 =$ constant, the second equation is satisfied, differentiating L and substituting into the second Euler–Lagrangian equation yields the first Lagrangian equation. Furthermore, for $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial z}{\partial s}}\right) = \frac{\partial L}{\partial z}$,

$$z = \int \frac{\sinh\varphi_2 \sin\theta_2}{f_2}\,\mathrm{d}s \tag{3.17}$$

is constant along the curve γ. Hence, Clairaut's theorem is expressed on the hyperbolic surface of rotation given in E_2^4.

Theorem 6. *Let γ be a time-like geodesic curve on the hyperbolic surface of rotation Υ^2 in the E_2^4, and let f_1 and f_2 be the distance functions from the axis of rotation to a point on the surface, φ_2 and θ_2 are the angles between the meridians of the surface and the time-like geodesic curve γ. Then, $2f_1 \cos\theta_2 \sinh\varphi_2$ and $2f_2 \sin\theta_2 \sinh\varphi_2$ are constant along the curve γ. Conversely, if $2f_1 \cos\theta_2 \sinh\varphi_2$ and $2f_2 \sin\theta_2 \sinh\varphi_2$ are constant along the curve γ, if no part of γ is a part of some parallels of the surface of rotation, then γ is time-like geodesic.*

In order to obtain the general equation of geodesics, one should consider the Euler–Lagrange equations

$$\dot{y} = \frac{\mathrm{d}y}{\mathrm{d}s} = \frac{\cos\theta_2 \sinh\varphi_2}{f_1}, \qquad \dot{z} = \frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\sinh\varphi_2 \sin\theta_2}{f_2}. \tag{3.18}$$

By adding the equations (3.18) at Lagrangian equation L, one has

$$L = f_1^2\Big(\frac{\mathrm{d}y}{\mathrm{d}s}\Big)^2 + f_2^2\Big(\frac{\mathrm{d}z}{\mathrm{d}s}\Big)^2 - \Big(\frac{\mathrm{d}t}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}s}\Big)^2$$

$$\Big(\frac{\mathrm{d}t}{\mathrm{d}x}\Big)^2\frac{\cos^2\theta_2\sinh^2\varphi_2}{f_1^2} = \sinh^2\varphi_2 - L \to \frac{\mathrm{d}t}{\mathrm{d}x} = \frac{f_1\sqrt{\sinh^2\varphi_2 - L}}{\cos\theta_2\sinh\varphi_2}$$

or

$$L = f_1^2\Big(\frac{\mathrm{d}y}{\mathrm{d}s}\Big)^2 + f_4^2\Big(\frac{\mathrm{d}z}{\mathrm{d}s}\Big)^2 - \Big(\frac{\mathrm{d}t}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}s}\Big)^2$$

$$\Big(\frac{\mathrm{d}t}{\mathrm{d}z}\Big)^2\Big(\frac{\sinh\varphi_2\sin\theta_2}{f_2}\Big)^2 = \sinh^2\varphi_2 - L \to \frac{\mathrm{d}t}{\mathrm{d}z} = \frac{f_2\sqrt{\sinh^2\varphi_2 - L}}{\sinh\varphi_2\sin\theta_2}.$$

Theorem 7. *The general equation of geodesics on the hyperbolic surface of rotation $\Upsilon^2 \subset E_2^4$, and for the parameters $\dot{y} = \cos\theta_2\sinh\varphi_2/f_1$ and $\dot{z} = \sinh\varphi_2\sin\theta_2/f_2$, are given by*

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{f_1}{\cos\theta_2\sinh\varphi_2}\sqrt{\sinh^2\varphi_2 - L}$$

or

$$\frac{\mathrm{d}t}{\mathrm{d}z} = \frac{f_2}{\sinh\varphi_2\sin\theta_2}\sqrt{\sinh^2\varphi_2 - L}.$$

3.3 Clairaut's theorem on the elliptic surfaces of rotation Υ^3. In this section, one will use the elliptic surface of rotation parametrized as

$$\Upsilon^3(\beta, \upsilon, s) = \begin{pmatrix} f_1\cos\beta + f_2\sin\beta, -f_1\sin\beta + f_2\cos\beta \\ f_3\cos\upsilon + f_4\sin\upsilon, -f_3\sin\upsilon + f_4\cos\upsilon \end{pmatrix}, \tag{3.19}$$

then one can take the planar curve γ for this surface of rotation to be the intersection of $\Upsilon^3(\beta, \upsilon, s)$ with $\varrho\eta = 0$. Therefore, the curve can be written by

$$\gamma(t) = (0, f_2(t), 0, f_4(t)), \qquad f_i \in C^\infty.$$

Then one has

$$\Upsilon^3(\beta, \upsilon, s) = (f_2\sin\beta, f_2\cos\beta, f_4\sin\upsilon, f_4\cos\upsilon)$$

and resulting in the first fundamental form:

$$\Upsilon^3_\beta = (f_2\cos\beta, -f_2\sin\beta, 0, 0),$$
$$\Upsilon^3_\upsilon = (0, 0, f_4\cos\upsilon, -f_4\sin\upsilon),$$
$$\Upsilon^3_t = (f_2'\sin\beta, f_2'\cos\beta, f_4'\sin\upsilon, f_4'\cos\upsilon),$$
$$\langle\Upsilon^3_\beta, \Upsilon^3_\beta\rangle = -f_2^2, \qquad \langle\Upsilon^3_\upsilon, \Upsilon^3_\upsilon\rangle = f_4^2, \qquad \langle\Upsilon^3_t, \Upsilon^3_t\rangle = -f_2'^2 + f_4'^2,$$
$$\langle\Upsilon^3_\beta, \Upsilon^3_\upsilon\rangle, \langle\Upsilon^3_\beta, \Upsilon^3_t\rangle, \langle\Upsilon^3_\upsilon, \Upsilon^3_t\rangle = 0$$

and

$$I_{\Upsilon^3} = \begin{pmatrix} -f_2^2 & 0 & 0 \\ 0 & f_4^2 & 0 \\ 0 & 0 & -f_2'^2 + f_4'^2 \end{pmatrix}.$$

Hence, one can write Lagrangian equation as follows

$$L = -f_2^2\dot{\beta}^2 + f_4^2\dot{\upsilon}^2 + (-f_2'^2 + f_4'^2)\dot{t}^2.$$

If one takes γ to be time-like, one can write $-f_2'^2 + f_4'^2 = -1$, $f_4^2 > 0$, $-f_2^2 < 0$. Then,

$$L = -f_2^2\dot{\beta}^2 + f_4^2\dot{\upsilon}^2 - \dot{t}^2.$$

Hence, one obtains the following equations using Clairaut's theorem,

$$\begin{aligned} \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \beta}{\partial s}}\right) &= \frac{\partial L}{\partial \beta} \Rightarrow \frac{\partial}{\partial s}(-2f_2^2\dot{\beta}) = 0, \\ \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial \upsilon}{\partial s}}\right) &= \frac{\partial L}{\partial \upsilon} \Rightarrow \frac{\partial}{\partial s}(2f_4^2\dot{\upsilon}) = 0, \\ \frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial t}{\partial s}}\right) &= \frac{\partial L}{\partial t} \Rightarrow \frac{\partial}{\partial s}(2\dot{t}) = -2f_2'f_2\dot{\beta}^2 + 2f_4'f_4\dot{\theta}^2. \end{aligned}$$

Let γ be a geodesic curve on Υ^3. Hence, $\gamma(t)$ can be written as follows

$$\dot{\gamma} = \dot{\beta}\,\Upsilon_\beta^2 + \dot{\upsilon}\,\Upsilon_\upsilon^2 + \dot{t}\,\Upsilon_t^2.$$

Hence, one can note that $\Upsilon_t^2 = N_t$ is a unit time-like vector pointing along t-axis of the parallels, and $\Upsilon_\beta^2 = f_2 N_\beta$ is a unit time-like vector pointing along the β-axis of the parallels. Also, $\Upsilon_\upsilon^2 = f_4 N_\upsilon$ is a unit space-like vector pointing along υ-axis of the meridians. It also follows that the plane spanned by N_β, N_β is time-like and an orthonormal basis. Also, from (4.21), one gets

$$\dot{\gamma} = \dot{t}N_t + f_2\dot{\beta}N_\beta + f_4\dot{\upsilon}N_\upsilon. \tag{3.20}$$

Note that the γ is time-like curve, since $N_\upsilon^\perp \in Sp\{N_t, N_\beta\}$, one gets

$$\begin{aligned} N_\upsilon^\perp &= \cosh\theta_3 N_\beta + \sinh\theta_3 N_t, \\ \dot{\gamma} &= \dot{t}N_t + f_2\dot{\beta}N_\beta + f_4\dot{\upsilon}N_\upsilon \\ &= \cos\varphi_3 N_t + \sin\varphi_3\cosh\theta_3 N_\beta + \sinh\theta_3\sin\varphi_3 N_\upsilon. \end{aligned} \tag{3.21}$$

Furthermore, from (3.20) and (3.21), one writes

$$f_2\dot{\beta} = \sin\varphi_3\cosh\theta_3, \qquad f_4\dot{\upsilon} = \sinh\theta_3\sin\varphi_3, \qquad \dot{t} = \cos\varphi_3. \tag{3.22}$$

Hence, from the Lagrangian equations, one has

$$\frac{\partial}{\partial s}(-\dot{t}) = f_1' f_1 \dot{y}^2 + f_2' f_2 \dot{z}^2 \ \to \ -\ddot{t} = f_2' f_2 \dot{\beta}^2 + f_4' f_4 \dot{\upsilon}^2,$$
$$\frac{\partial}{\partial s}(-2f_2^2\dot{\beta}) = 0 \ \Rightarrow \ y = \frac{-c_1^3}{2{f_2}^2} + c_3^3,$$
$$\frac{\partial}{\partial s}(2f_4^2\dot{\upsilon}) = 0 \ \Rightarrow \ z = \frac{c_2^3}{2f_4^2} + c_4^3,$$

which gives that $2f_2^2\dot{\beta}$ and $2f_4^2\dot{\upsilon}$ are constant along the geodesic, and by using (3.22) one obtains

$$f_2\dot{\beta} = \sin\varphi_3 \cosh\theta_3 \ \Rightarrow \ 2f_2^2\dot{\beta} = 2f_2 \sin\varphi_3 \cosh\theta_3 = \text{cons.} \tag{3.23}$$
$$f_4\dot{\upsilon} = \sinh\theta_3 \sin\varphi_3 \ \Rightarrow \ 2f_4^2\dot{\upsilon} = 2f_4 \sinh\theta_3 \sin\varphi_3 = \text{cons.} \tag{3.24}$$
$$2\dot{t} = 2\cos\varphi_3 \neq \text{cons.}$$

for $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial\beta}{\partial s}}\right) = \frac{\partial L}{\partial \beta}$, one has $f_2\dot{\beta} = \sin\varphi_3\cosh\theta_3$, $f_4\dot{\upsilon} = \sinh\theta_3\sin\varphi_3$, $\dot{t} = \cos\varphi_3$,

$$\beta = \int \frac{\sin\varphi_3 \cosh\theta_3}{f_2}\,\mathrm{d}s \tag{3.25}$$

is a constant. Conversely for the condition $2f_2\sin\varphi_3\cosh\theta_3 =$ constant and the equation $\frac{\partial}{\partial s}\left(\frac{\partial L}{\frac{\partial\upsilon}{\partial s}}\right) = \frac{\partial L}{\partial \upsilon}$,

$$\upsilon = \int \frac{\sinh\theta_3 \sin\varphi_3}{f_4}\,\mathrm{d}s \tag{3.26}$$

is constant along the curve γ. Hence, the following theorem can be given.

Theorem 8. *Let γ be a time-like geodesic curve on the elliptic surface of rotation $\Upsilon^3 \subset E_2^4$, and let f_2 and f_4 be the distance functions from the axis of rotation to a point on the surface, φ_3 and θ_3 are the angles between the meridians of the surface and the time-like geodesic curve γ. Then, $2f_2\sin\vartheta_3\cosh\theta_3$ and $2f_4\sinh\theta_3\sin\varphi_3$ are constant along the curve γ. Conversely, if $2f_2\sin\vartheta_3\cosh\theta_3$ and $2f_4\sinh\theta_3\sin\varphi_3$ are constant along the curve γ, if no part of γ is a part of some parallels of the surface of rotation, then γ is time-like geodesic curve.*

For the general equation of geodesics, one should consider the Euler–Lagrange equations

$$\dot{\beta} = \frac{\mathrm{d}\beta}{\mathrm{d}s} = \frac{\sin\varphi_3\cosh\theta_3}{f_2}, \qquad \dot{\upsilon} = \frac{\mathrm{d}\upsilon}{\mathrm{d}s} = \frac{\sinh\theta_3\sin\varphi_3}{f_4}. \tag{3.27}$$

By adding the equations in (3.27) at Lagrangian equation L, one has

$$L=-f_2^2\Big(\frac{\mathrm{d}\beta}{\mathrm{d}s}\Big)^2+f_4^2\Big(\frac{\mathrm{d}\upsilon}{\mathrm{d}s}\Big)^2-\Big(\frac{\mathrm{d}t}{\mathrm{d}\beta}\frac{\mathrm{d}\beta}{\mathrm{d}s}\Big)^2,$$

$$\Big(\frac{\mathrm{d}t}{\mathrm{d}\beta}\Big)^2\,\frac{\sin^2\varphi_3\cosh^2\theta_3}{f_2^2}=-\sin^2\varphi_3-L,$$

$$\frac{\mathrm{d}t}{\mathrm{d}\beta}=i\,\frac{f_2\sqrt{L+\sin^2\varphi_3}}{\sin\varphi_3\cosh\theta_3}$$

or

$$L=-f_2^2\Big(\frac{\mathrm{d}\beta}{\mathrm{d}s}\Big)^2+f_4^2\Big(\frac{\mathrm{d}\upsilon}{\mathrm{d}s}\Big)^2-\Big(\frac{\mathrm{d}t}{\mathrm{d}\upsilon}\frac{\mathrm{d}\upsilon}{\mathrm{d}s}\Big)^2,$$

$$\Big(\frac{\mathrm{d}t}{\mathrm{d}\upsilon}\Big)^2\Big(\frac{\sinh\theta_3\sin\varphi_3}{f_4}\Big)^2=-\sin^2\varphi_3-L\to\frac{\mathrm{d}t}{\mathrm{d}\upsilon}=i\,\frac{f_4}{\sinh\theta_3\sin\varphi_3}\sqrt{\sin^2\varphi_3+L}.$$

Theorem 9. *The general equation of geodesics on the elliptic surface of rotation $\Upsilon^3\subset E_2^4$, and for the parameters $\dot\beta=\sin\varphi_3\cosh\theta_3/f_2$ and $\dot\upsilon=\sinh\theta_3\sin\varphi_3/f_4$, are given by*

$$\frac{\mathrm{d}t}{\mathrm{d}\beta}=i\,\frac{f_2\sqrt{L+\sin^2\varphi_3}}{\sin\varphi_3\cosh\theta_3}\qquad \textit{or}\qquad \frac{\mathrm{d}t}{\mathrm{d}\upsilon}=i\,\frac{f_4}{\sinh\theta_3\sin\varphi_3}\sqrt{\sin^2\varphi_3+L}.$$

4. Conclusion

This study generalizes Clairaut's theorem to pseudo-Euclidean 4-space with index two, and reviews Clairaut's theorem of surfaces of rotation which define a well-known characterization of geodesics on a surface of rotation. Therefore, it is shown that the time-like geodesic curves on the hyperbolic surface of rotation Υ^1 are completely characterized by $2f_1\cos\varphi_1$ and $-2f_4\cosh\theta_1\sin\varphi_1$ being constant, the time-like geodesics on the hyperbolic surface of rotation Υ^2 are characterized by $2f_1\cos\theta_2\sinh\varphi_2$ and $2f_2\sinh\varphi_2\sin\theta_2$ being constant, and finally the time-like geodesics on the elliptic surface of rotation Υ^3 are characterized by $2f_2\sin\varphi_3\cosh\theta_3$ and $2f_4\sinh\theta_3\sin\varphi_3$ being constant, respectively.

The authors are currently working on the properties of these surfaces of rotation with a view to devising suitable metric in E_2^4 by adapting the type of conservation laws considered in the paper. In our future studies, the physical terms such as specific energy and specific angular momentum will be examined with the help of the conditions obtained by using Clairaut's theorem for geodesics on these special surfaces.

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