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SUFFICIENT CONDITIONS TO DETERMINE THE LINEAR DEPENDENCY OF TWO MEROMORPHIC FUNCTIONS

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Abstract. We comprehensively explore the generalized concept of sharing sets to establish conditions for the linear dependency of two meromorphic functions. By applying this approach, we significantly extend and enhance the existing results related to URSM (unique range set of meromorphic functions). It is well known that URSMs can be represented as zeros of specific polynomials. However, our findings demonstrate that the concept of URSM can be understood from a broader perspective, where it can be characterized as a special case of the zero sets of two interconnected polynomials. Such investigations have not been conducted before, thus the text breaks the barriers of the traditional definition of URSM.

Keywords: meromorphic function; unique range set; weighted shared sets wider sense; linear dependency

MSC 2020: 30D35

1. Background and some useful definitions

In this paper, the term "meromorphic functions" refers specifically to functions in the finite complex plane. We denote the extended complex plane as $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the complex plane excluding zero as $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$, and the set of natural numbers including zero as $\overline{\mathbb{N}} = \mathbb{N} \cup \{0\}$. The set of all complex numbers is denoted as $\mathbb{C}$, the set of all integers as $\mathbb{Z}$, and throughout this paper, we adopt the standard notations and definitions of Nevanlinna theory as outlined in [16].

For any non-constant meromorphic function h, we define $S(r,h) = o(T(r,h))$ as r tends to infinity, with the condition that r does not belong to a set E consisting of positive real numbers with finite linear measure.

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Definition 1.1 ([18]). Let k be a non-negative integer or infinity. For $a \in \overline{\mathbb{C}}$ we denote by $E_k(a; f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \leqslant k$ and $k+1$ times if $m > k$. If $E_k(a; f) = E_k(a; g)$, we say that f, g share the value a with weight k and present it as (a, k). The two extreme cases $k = \infty$ and $k = 0$ are called CM and IM sharing, respectively.

Definition 1.2 ([17]). For $S \subset \overline{\mathbb{C}}$ we define $E_f(S, k) = \bigcup_{a \in S} E_k(a; f)$, where k is a non-negative integer $a \in S$ or infinity. Clearly $E_f(S) = E_f(S, \infty)$ and $\overline{E}_f(S) = E_f(S, 0)$. If $E_f(S, k) = E_g(S, k)$ holds, then we say f, g share the set S with weight k and denote it as f, g share (S, k).

Uniqueness theory, initially established by Nevanlinna, took a new trajectory after five decades when Gross (see [14]) introduced a fresh perspective by shifting the focus from value sharing to a broader concept known as set sharing.

In the work [15], a set S was identified as a unique range set of entire functions (URSE) if the condition $E_f(S) = E_g(S)$ holds for any two non-constant entire functions f and g, implying that f and g are identical. Similarly, the concept of a unique range set of meromorphic functions (URSM) can be defined in a similar manner.

If a polynomial P in $\mathbb{C}$ satisfies $P(f) \equiv cP(g)$ (or $P(f) \equiv P(g)$) for any nonzero constant c, implying that $f \equiv g$, then P is referred to as a SUPM (SUPE) (or UPM (UPE)). The zero sets of SUPM (UPM) always form unique range sets.

On the other hand, if $\overline{E}_f(S) = \overline{E}_g(S)$ implies $f \equiv g$, then S is called URSM-IM (URSE-IM). Using the definition of weighted sharing in URSM, in [7], the authors modified the same definition.

In the context of meromorphic (entire) functions, a set $S \subset \overline{\mathbb{C}}$ is considered a unique range set with weight k if for any two non-constant meromorphic (entire) functions f and g, the condition $E_f(S, k) = E_g(S, k)$ implies that $f \equiv g$. In short, such a set S is denoted as URSMk (URSEk). The study of unique range sets (URS) primarily focuses on two aspects:

a) Finding URS with the smallest possible number of elements.

b) Characterizing the properties of URS.

Inspired by the famous question of Gross (see [14]), several investigations on URSM or URSE were performed by many researchers as follows:

In 1994, Yi in [25] exhibited a URSE with 15 elements. In 1995, Li and Yang (see [20]) exhibited a URSM with 15 elements and a URSE with 7 elements. Li and Yang investigated the zero sets S of polynomials P of the form

$$P(z) = z^n + az^{n-m} + b,$$

where gcd $(n, m) = 1$, $n > m \geqslant 1$ and a and b are chosen so that P has n distinct zeros, to be a URSMs (URSEs). When $m \geqslant 2$, then the sets S generate URSMs and when $m = 1$, the sets generate URSEs. In 1996, Yi (see [26]) obtained a URSM with 13 elements. In 1998, Frank and Reinders in [12] improved the result of Yi (see [26]) and exhibited the following URSM with 11 elements:

$$P(z) = \frac{(n-1)(n-2)}{2} z^n - n(n-2) z^{n-1} + \frac{n(n-1)}{2} z^{n-2} - b,$$

where $n \geqslant 11$ and $b \neq 0, 1$. In [27], Yi established a unique range set of meromorphic functions with at least 19 elements, referred to as URSM-IM. This result was further improved by Bartels (see [9]), who obtained a URSM-IM consisting of 17 elements.

Since then, the concept of unique range sets has gained significant attention and become a prominent area of research in uniqueness theory. Numerous authors, including Banerjee [1], [2], [3], Lahiri [7], Gross and Yang [15], Fujimoto [13] have made notable contributions to this field over the years.

Subsequently, Banerjee and Mallick in [8] presented a series of results that not only improved upon previous findings but also provided generalizations of those results.

The aforementioned results primarily focused on exploring the uniqueness relationship between two meromorphic (entire) functions. However, it is worth noting that the identity relationship between two meromorphic functions is a specific instance of a linear dependency between the same functions. Therefore, the concept of uniqueness can be viewed from a broader perspective, aiming to determine the extent to which sufficient conditions on the sharing of two meromorphic (entire) functions lead to their linear dependence.

In 1998, Qiu (see [23]) conducted additional research on the scenario where the kth derivatives of two meromorphic functions share values. Since then, numerous mathematicians have obtained a multitude of elegant results concerning the kth derivatives (see [4], [5], [6], [10], [11]).

In [24] the following question was posed: Does the equality $f^{-1}(S) = g^{-1}(S)$, where $S = \{-1, 1\}$ and f, g are polynomials of same degree, imply that $f = \pm g$?

Concerning this above question Pakovitch (see [22]) gives a solution to more general question: Under what conditions of compact subsets S, T and the functions f and g, one gets

$$f^{-1}(S) = g^{-1}(T). \tag{1.1}$$

Considering the result by Pakovitch (see [22]), it becomes interesting to explore the concept of linear dependence between two meromorphic (entire) functions when their inverse images of different sets are shared. Naturally, the question arises regarding the generalization of weighted sharing of sets in light of Pakovitch's notion. Therefore, we introduce the following definition.

Definition 1.3. Let f, g be two non-constant meromorphic functions and k be a non-negative integer. If the condition $E_{f^{(k)}}(S,l) = E_{g^{(k)}}(T,l)$ implies $f^{(k)} = hg^{(k)}$, where h is a constant, then we call the pair $(S,\ T)$ linear dependent range sets of meromorphic functions in wider sense with weight l. If $h = 1$, then we call the pair $(S,\ T)$ unique range sets of meromorphic functions in wider sense with weight l and denote it by (URSMWS_m). If in (URSMWS_m), $m = 0$, then we call the pair unique range sets of meromorphic functions in the wider sense ignoring multiplicity (URSMWS-IM in short).

Next we define the following two polynomials:

$$P(z) = az^n + bz^{n-m} + d, \tag{1.2}$$

$$Q(z) = uz^n + vz^{n-m} + t, \tag{1.3}$$

where n and m are two positive integers and a, b, d, u, v, t are nonzero complex numbers such that P and Q have no multiple zero. Also let us denote two sets $S = \{z\colon\ P(z) = 0\}$ and $T = \{z\colon\ Q(z) = 0\}$.

Recently, considering the function and its derivative under the sharing of the zero set of Yi's polynomial, Li and Lin (see [19]) obtained the following results.

Theorem A ([19]). *Let f and g be two meromorphic functions and let k be a non-negative integer such that $f^{(k)}$, $g^{(k)}$ are not constant. Polynomials P and Q are defined as* (1.2), (1.3). *If $E_{f^{(k)}}(S,0) = E_{g^{(k)}}(T,0)$, then for $n > 2m+7+7/(k+1)$, where either $(n,m) = 1$, $m \geqslant 2$, or $m \geqslant 4$, then for a nonzero constant h, $f^{(k)} \equiv hg^{(k)}$, where $h^n = \mathrm{d}u/at$, $h^{n-m} = \mathrm{d}v/bt$.*

Theorem B ([19]). *Under the same conditions as in Theorem* A, *if f and g are two entire functions and $n > 2m+7$ with $(n,m) = 1$, then for a nonzero constant h, $f^{(k)} \equiv hg^{(k)}$, where $h^n = \mathrm{d}u/at$, $h^{n-m} = \mathrm{d}v/bt$.*

Theorem C ([19]). *Under the same conditions as in Theorem* A, *if $E_{f^{(k)}}(S,\infty) = E_{g^{(k)}}(T,\infty)$ and $n > 2m+4+4/(k+1)$, where either $(n,m) = 1$, $m \geqslant 2$, or $m \geqslant 4$, then for a nonzero constant h, $f^{(k)} \equiv hg^{(k)}$.*

In the same paper [19], the authors proposed the following questions:

Question 1.1. Is it possible to additionally weaken the relationship conditions n and m in Theorems A and C?

Question 1.2. What happens when Yi's polynomials P and Q are replaced by the other style of polynomials?

The primary objective of this paper is to address the aforementioned questions by offering the most exhaustive solutions based on Definition 1.3. In doing so, our results will significantly enhance the existing Theorems A–C and introduce new aspects concerning unique range sets, particularly with regards to the derivatives of the functions.

Theorem 1.1. *Let f and g be two non-constant meromorphic functions and let k be a non-negative integer and P and Q be defined as in* (1.2), (1.3). *Let $E_{f^{(k)}}(S,l) = E_{g^{(k)}}(T,l)$. If*

(i) $l = 2$, $n > 2m + 4 + 4/(k+1)$ *or*

(ii) $l = 0$, $n > 2m + 7 + 7/(k+1)$, *where either*

 (a) $k = 0$, $m \geqslant 2$ *or*

 (b) $k \geqslant 1$, $m \geqslant 1$,

then $f^{(k)} \equiv hg^{(k)}$ for a nonzero constant h such that $h^n = \mathrm{d}u/at$, $h^{n-m} = \mathrm{d}v/bt$.

Corollary 1.1. *Under the same conditions as in Theorem* 1.1, *if we consider f, g as entire functions, then for $l = 0$, we obtain Theorem* B.

The next table provides a clear comparison between Theorem A and Theorem 1.1.

Theorem A		Theorem 1.1	
shared sets with weight $E_{f^{(k)}}(S,0) = E_{g^{(k)}}(T,0)$	least cardinality	shared sets with weight $E_{f^{(k)}}(S,0) = E_{g^{(k)}}(T,0)$	least cardinality
when $k = 0$, $(n,m) = 1$	19	when $k = 0$, $(n,m) = 1$	19
when $k = 0$, $(n,m) \neq 1$	23	when $k = 0$, $(n,m) \neq 1$	19
when $k \geqslant 1$, $(n,m) = 1$	$> 11 + \dfrac{7}{k+1}$	when $k \geqslant 1$, $(n,m) = 1$	$> 9 + \dfrac{7}{k+1}$
when $k \geqslant 1$, $(n,m) \neq 1$	$> 15 + \dfrac{7}{k+1}$	when $k \geqslant 1$, $(n,m) \neq 1$	$> 9 + \dfrac{7}{k+1}$

Table 1.

R e m a r k 1.1. Form Table 1 it is clear that Theorem 1.1 improves Theorem A significantly by reducing the cardinality of shared sets under the case $k = 0$, $(n,m) \neq 1$ and the case $k \geqslant 1$ completely.

We first note that in Theorem 1.1, the sets S, T correspond to the polynomials which possess first two consecutive terms. So it will be interesting to re-investigate Theorem 1.1 for the polynomials that contain three consecutive terms as that will provide the answer to Question 1.2 as well. To this end, we introduce the following two polynomials, having no multiple zeros, of the forms

$$P_1(w) = a_1 w^n + b_1 w^{n-m} + c_1 w^{n-2m} + d_1, \tag{1.4}$$

$$Q_1(w) = a_2 w^n + b_2 w^{n-m} + c_2 w^{n-2m} + d_2, \tag{1.5}$$

where n, m are positive integers with $(n,m)=1$ and $a_i, b_i, c_i, d_i \in \mathbb{C}^*$, $i=1,2$.

We note that the case when $c_1 \cdot c_2 = 0$ is already discussed in detail in Theorem 1.1, Theorem C. In the subsequent theorems, we consider two distinct sets S_1 and T_1, which are the zero sets of the polynomials P_1 and Q_1, respectively. When $P_1 = Q_1$ is a SUPM, it is straightforward to observe that $f_1 \equiv g_1$, where $f^{(k)} = f_1$, $g^{(k)} = g_1$. However, the situation becomes more complicated when P_1 and Q_1 are different. Thus, it becomes interesting to determine the linear dependency relationship between f_1 and g_1 under such circumstances. By imposing certain restrictions on the coefficients of P_1 and Q_1, we have obtained the following result.

Theorem 1.2. *Let f and g be two meromorphic functions and let k be a non-negative integer such that $f^{(k)}$, $g^{(k)}$ are not constants. Also let us consider the sets $S_1 = \{w\colon P_1(w)=0\}$, $T_1 = \{w\colon Q_1(w)=0\}$, where P_1 is the polynomial defined as in* (1.4), *with $(n,m)=1$ and $Q_1 = KP_1 + \widehat{c}$ for some constants K $(\neq 0)$, $\widehat{c} \in \mathbb{C}$, so that Q_1 has all simple zeros. Now suppose $E_{f^{(k)}}(S_1,l) = E_{g^{(k)}}(T_1,l)$ and*

(I) $b_1^2/(4a_1c_1) \neq n(n-2m)/(n-m)^2, 1$ *with*
  (i) $l=2$ *and* $n > 4m+4+4/(k+1)$;
  (ii) $l=0$ *and* $n > 4m+7+7/(k+1)$; *or*

(II) $b_1^2/(4a_1c_1) = n(n-2m)/(n-m)^2$ *with*
  (i) $l=2$ *and* $n > 2m+4+4/(k+1)$;
  (ii) $l=0$ *and* $n > 2m+7+7/(k+1)$.

Then we have a nonzero constant h such that $f^{(k)} \equiv hg^{(k)}$, where $h^m = 1$ and $h^n = Kd_1/(Kd_1+\widehat{c})$ $(Kd_1+\widehat{c}\neq 0)$.

From the above theorem the corollary follows immediately.

Corollary 1.2. *Under the same conditions as in Theorem* 1.2 *let us assume that f and g are two entire functions, k is a non-negative integer such that $f^{(k)}$, $g^{(k)}$ are not constants. Also let $E_{f^{(k)}}(S_1,l) = E_{g^{(k)}}(T_1,l)$ and*

(I) $b_1^2/(4a_1c_1) \neq n(n-2m)/(n-m)^2, 1$ *with*
  (i) $l=2$ *and* $n > 4m+4$,
  (ii) $l=0$ *and* $n > 4m+7$; *or*

(II) $b_1^2/(4a_1c_1) = n(n-2m)/(n-m)^2$ *with*
  (i) $l=2$ *and* $n > 2m+4$,
  (ii) $l=0$ *and* $n > 2m+7$.

Then for a nonzero constant h, we have $f^{(k)} \equiv hg^{(k)}$; $h^m = 1$ and $h^n = Kd_1/(Kd_1+\widehat{c})$, where $Kd_1+\widehat{c}\neq 0$.

In the following table, we compare Theorem A with Theorem 1.2.

<table>
<tr><th colspan="2">Theorem A</th><th colspan="2">Theorem 1.2</th></tr>
<tr><td>shared sets with weight $E_{f^{(k)}}(S,0)=E_{g^{(k)}}(T,0)$</td><td>least cardinality</td><td>shared sets with weight $E_{f^{(k)}}(S_1,0)=E_{g^{(k)}}(T_1,0)$</td><td>least cardinality</td></tr>
<tr><td rowspan="2">when $k=0$, $(n,m)=1$</td><td rowspan="2">19</td><td rowspan="2">when $k=0$, $(n,m)=1$</td><td>17 when $\dfrac{b_1^2}{4a_1c_1}=\dfrac{n(n-2m)}{(n-m)^2}$</td></tr>
<tr><td>19 when $\dfrac{b_1^2}{4a_1c_1}\neq\dfrac{n(n-2m)}{(n-m)^2}, 1$</td></tr>
<tr><td rowspan="2">when $k\geqslant 1$, $(n,m)=1$</td><td rowspan="2">$>11+\dfrac{7}{k+1}$</td><td rowspan="2">when $k\geqslant 1$, $(n,m)=1$</td><td>$>9+\dfrac{7}{k+1}$ when $\dfrac{b_1^2}{4a_1c_1}=\dfrac{n(n-2m)}{(n-m)^2}$</td></tr>
<tr><td>$>11+\dfrac{7}{k+1}$ when $\dfrac{b_1^2}{4a_1c_1}\neq\dfrac{n(n-2m)}{(n-m)^2}, 1$</td></tr>
</table>

Table 2.

Remark 1.2. Table 2 reveals that Theorem 1.2 significantly reduces the cardinality of the shared set under the case $b_1^2/(4a_1c_1)=n(n-2m)/(n-m)^2$, irrespective of the choice of $k\geqslant 0$ in comparison to Theorem C.

Observation. In the case of Theorem 1.2 we have some vital observations. In Theorem 1.2, we consider the relation between the generating polynomials of two sets as $Q_1=KP_1+\widehat{c}$. Note that when $K=1$ and $\widehat{c}=0$, we have $S_1=T_1$ and the conclusion of the theorem simply reduces to $f^{(k)}=g^{(k)}$. But the case $\widehat{c}\neq 0$ deserves further attention as in this case P_1 and Q_1 are linearly independent. Here it is to be noted that for an arbitrary choice of $\widehat{c}$, one cannot simply get $E_{f^{(k)}}(S_1,l)=E_{g^{(k)}}(T_1,l)$ always, i.e., $f^{(k)}$, $g^{(k)}$ cannot share arbitrary pair S_1, T_1; rather the sharing is totally depending upon the suitable choice of $\widehat{c}$ for which h must be a common solution of $h^m=1$, $h^n=Kd_1/(Kd_1+\widehat{c})$. For example, choosing $\widehat{c}=Kd_1$, we see that $h^m=1\Rightarrow |h|=1$ and $h^n=\frac{1}{2}$, a contradiction, so in that case there are no meromorphic functions f, g such that $E_f^{(k)}(S_1,l)=E_g^{(k)}(T_1,l)$. Next let α be a root of $h^m-1=0$. So $\alpha=\mathrm{e}^{2s\pi\mathrm{i}/m}$, $s=0,1,\dots,m-1$. We choose $n=tm+1$, then $h^n=\alpha$ and so if we choose $\widehat{c}=Kd_1(1-\alpha)/\alpha$, then $h-\alpha$ will be a common factor of both h^m-1, $h^n-Kd_1/(Kd_1+\widehat{c})$. Clearly, in this case we get $f_1=\alpha g_1$.

From Theorem 1.2, we see that the case $b_1^2/(4a_1c_1) = 1$ was not explored. So the natural question arises what happens in Theorem 1.2 if only the case $b_1^2/(4a_1c_1) = 1$ is satisfied, where no such relation between the polynomials P_1 and Q_1 as mentioned in Theorem 1.2 exist. Our next theorem will elucidate in this matter. In fact, in Theorem 1.2, the coefficients associated with all positive powers of P_1 and Q_1 are proportionate and so we definitely can deduct $b_1^2/(4a_1c_1) = b_2^2/(4a_2c_2)$, which is not necessarily required in our next theorem.

Theorem 1.3. *Let f and g be two meromorphic functions and let k be a non-negative integer such that $f^{(k)}$, $g^{(k)}$ are not constants. Also let us consider the sets $S_1 = \{w\colon\ P_1(w) = 0\}$, $T_1 = \{w\colon\ Q_1(w) = 0\}$, where P_1, Q_1 are the polynomials defined as in* (1.4), (1.5), *with* $(n, m) = 1$ *and $b_1^2 = 4a_1c_1$, $b_2^2 = 4a_2c_2$; having all simple zeros. Now suppose $E_{f^{(k)}}(S_1, l) = E_{g^{(k)}}(T_1, l)$ and*

(I) *$k = 0$ and $m \geqslant 2$ with*

  (i) *$l = 2$ and $n > 4m + 4 + 4/(k + 1)$,*

  (ii) *$l = 0$ and $n > 4m + 7 + 7/(k + 1)$; or*

(II) *$km \geqslant 1$ with*

  (i) *$l = 2$ and $n \geqslant 4m + 7$,*

  (ii) *$l = 0$ and $n > 4m + 7 + 7/(k + 1)$.*

Then for a nonzero constant h, we have $f^{(k)} \equiv hg^{(k)}$.

From the proof of Theorem 1.3, the following corollary follows immediately.

Corollary 1.3. *Under the same conditions as in Theorem 1.3 let us assume that f and g are two entire functions, k is a non-negative integer such that $f^{(k)}$, $g^{(k)}$ are not constants. Also let $E_{f^{(k)}}(S_1, l) = E_{g^{(k)}}(T_1, l)$ and*

(i) *$l = 2$ and $n > 4m + 4$;*

(ii) *$l = 0$ and $n > 4m + 7$.*

Then for a nonzero constant h, we have $f^{(k)} \equiv hg^{(k)}$.

Next we provide the table to understand the overall comparison between Theorem C and Theorems 1.1–1.3 (see Table 3).

Remark 1.3. The Table 3 reveals that:

(i) In Theorem 1.1, when $k = 0$, the lower bound of n under sharing of weight 2 is same as that of Theorem C under CM sharing.

(ii) In Theorem 1.1, when $k \geqslant 1$, then as $m \geqslant 1$, the lower bound of n under sharing of weight 2 is 9 and it continuously decreases while in the case of Theorem C the minimum value of n is 12 under CM sharing.

(iii) The case $m \geqslant 4$ is never attended in Theorem 1.1 which also renders an important feature in diminishing the lower bound, but in Theorem C, $m \geqslant 4$, may occur.

(iv) In Theorem 1.2 we have considered a generalized version of the sets as in Theorem C. Hence, Theorem 1.2 is actually an extended form of Theorem C. Also for the case $b_1^2/(4a_1c_1) = 1$, Theorem 1.2 improves Theorem C by reducing the cardinality and relaxing the CM sharing of sets. Thus, Theorem 1.2 is a two step improvement of Theorem C.

(v) Additionally, the Question 1.2 posed in [19] is satisfactorily addressed in Theorems 1.2, 1.3.

	Theorem C	Theorem 1.1	Theorem 1.2	Theorem 1.3
shared sets with weight	$E_{f^{(k)}}(S,\infty) = E_{g^{(k)}}(T,\infty)$	$E_{f^{(k)}}(S,2) = E_{g^{(k)}}(T,2)$	$E_{f^{(k)}}(S_1,2) = E_{g^{(k)}}(T_1,2)$	$E_{f^{(k)}}(S_1,2) = E_{g^{(k)}}(T_1,2)$
least cardinality when $k=0$	13 when $(n,m)=1$, $m \geqslant 2$	13 when $(n,m)=1$, $m \geqslant 2$	11 when $\dfrac{b_1^2}{4a_1c_1} = \dfrac{n(n-2m)}{(n-m)^2}$	17 when $b_1^2 = 4a_1c_1$, $b_2^2 = 4a_2c_2$, $m \geqslant 2$
	17 when $(n,m) \neq 1$, $m \geqslant 4$	13 when $(n,m) \neq 1$, $m \geqslant 2$	13 when $\dfrac{b_1^2}{4a_1c_1} \neq \dfrac{n(n-2m)}{(n-m)^2}$	
least cardinality when $k \geqslant 1$	$> 8 + \dfrac{4}{k+1}$ when $(n,m)=1$, $m \geqslant 2$	$> 6 + \dfrac{4}{k+1}$ when $(n,m)=1$	$> 6 + \dfrac{4}{k+1}$ when $\dfrac{b_1^2}{4a_1c_1} = \dfrac{n(n-2m)}{(n-m)^2}$	11 when $b_1^2 = 4a_1c_1$, $b_2^2 = 4a_2c_2$
	$> 12 + \dfrac{4}{k+1}$ when $(n,m) \neq 1$, $m \geqslant 4$	$> 6 + \dfrac{4}{k+1}$ when $(n,m) \neq 1$	$> 8 + \dfrac{4}{k+1}$ when $\dfrac{b_1^2}{4a_1c_1} \neq \dfrac{n(n-2m)}{(n-m)^2}$	

Table 3.

Here we give the following definitions which will be useful for the proof of the main results of the paper.

Definition 1.4 ([7]). Let $P(z)$ be a polynomial such that $P'(z)$ has mutually k distinct zeros given by $d_1, d_2, \ldots, d_k$ with multiplicities $q_1, q_2, \ldots, q_k$, respectively. Then $P(z)$ is said to satisfy the critical injection property if $P(d_i) \neq P(d_j)$ for $i \neq j$, where $i, j \in \{1, 2, \cdots, k\}$ and the polynomial is called a critical injective polynomial. Clearly, a critical injective polynomial can have at most one multiple zero.

We have used some usual notations of counting functions like $N_E^{1)}(r,a;f)$, $\overline{N}_L(r,a;f)$, $N(r,a;f\mid=1)$, $\overline{N}(r,a;f\mid\leqslant m)$, $\overline{N}(r,a;f\mid\geqslant m)$ and $\overline{N}_*(r,a;f,g)$. For the definitions of these counting functions, we refer the reader to follow [1], [18].

2. Lemmas

Next, we present some lemmas that will be needed in the sequel. Henceforth, we denote by H the following functions:

$$H=\Big(\frac{F''}{F'}-\frac{2F'}{F}\Big)-\Big(\frac{G''}{G'}-\frac{2G'}{G}\Big).$$

Lemma 2.1. *Let F and G be non-constant meromorphic functions and let F, G share 0 IM. Then*

$$N_E^{1)}(r,0;F)\leqslant N(r,\infty;H)+S(r,F)+S(r,G).$$

Proof. We are not giving the proof as a similar proof can be found in [28]. □

Lemma 2.2. *Let F and G be non-constant meromorphic functions and let F, G share 0 IM. Then*

$$\begin{aligned}N(r,\infty;H)\leqslant{}&\overline{N}_*(r,0;F,G)+\overline{N}(r,\infty;f)+\overline{N}(r,\infty;g)\\&+\overline{N}_0(r,0;F')+\overline{N}_0(r,0;G')+S(r,F)+S(r,G),\end{aligned}$$

where $\overline{N}_0(r,0;F')$ is the reduced counting function of those zeros of F', where $F\neq 0$ and $\overline{N}_0(r,0;G')$ is similarly defined.

Proof. Here we are not giving the proof as it is similar to the proof of Lemma 2.2 in [8]. □

Lemma 2.3 ([1]). *Let F and G be non-constant meromorphic functions and let F, G share $(0,l)$. Then*

$$\begin{aligned}&\overline{N}(r,0;F)+\overline{N}(r,0;G)-N_E^{1)}(r,0;F)+\Big(l-\frac{1}{2}\Big)\overline{N}_*(r,0;F,G)\\&\quad\leqslant\frac{1}{2}(N(r,0;F)+N(r,0;G))+S(r,F)+S(r,G).\end{aligned}$$

Lemma 2.4 ([21]). *Let f be a non-constant meromorphic function and let*

$$R(f)=\frac{\sum_{k=0}^{n}a_kf^k}{\sum_{j=0}^{m}b_jf^j}$$

be an irreducible rational function in f with constant coefficients $\{a_k\}$ and $\{b_j\}$, where $a_n\neq 0$ and $b_m\neq 0$. Then

$$T(r,R(f))=dT(r,f)+S(r,f),$$

where $d=\max\{n,m\}$.

Lemma 2.5. *Let f and g be two meromorphic functions such that $f_1 = f^{(k)}$, $g_1 = g^{(k)}$ are not constants. P and Q are given by (1.2), (1.3). If $P(f_1) = Q(g_1)$ and $n \geqslant 2m+4$, where either $k=0$ and $m \geqslant 2$, or $km \geqslant 1$, then $f_1 \equiv hg_1$ for a constant h such that $h^n = u/a$, $h^{n-m} = v/b$.*

P r o o f. Doing exactly the same as in Lemma 2.8 in [19], we have

$$af_1^n + bf_1^{n-m} = ug_1^n + vg_1^{n-m}, \quad ag_1^m(h^n - \alpha) = -b(h^{n-m} - \beta), \tag{2.1}$$

where $h = f_1/g_1$ and $\alpha = u/a \neq 0$, $\beta = v/b \neq 0$. First assume that h is not a constant, then we have from above that $g_1^m = -b(h^{n-m} - \beta)/(a(h^n - \alpha))$.

Now we discuss the following two cases.

Case 1 $(n,m) = 1$.

Subcase 1.1 $k = 0$ and $m \geqslant 2$. In this case, dealing exactly in the same way as in Case 1, Lemma 2.8 in [19], we get the result.

Subcase 1.2 Here $k > 0$ and $m \geqslant 1$. Now it is given that

$$g_1^m = -\frac{b(h^{n-m} - \beta)}{a(h^n - \alpha)}, \quad \text{i.e., } (g^{(k)})^m = -\frac{b(h^{n-m} - \beta)}{a(h^n - \alpha)}.$$

First assume h is a non-constant meromorphic function. Since $(n,m) = 1$, then $h^{n-m} - \beta$ and $h^n - \alpha$ can have at most one common zero. Hence, $h^n - \alpha$ has at least $n-1$ distinct zeros, say $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. Also, let z_0 be a zero of $h - \alpha_i$ $i = 1,2,\ldots,n-1$ of order p, then it is a pole of $(g^{(k)})^m$ of order at least $m(k+1)$. Then $p \geqslant m(k+1)$.

$$\begin{aligned}(n-3)T(r,h) &\leqslant \sum_{i=1}^{n-1} \overline{N}(r,0;h-\alpha_i) + S(r,h) \leqslant \frac{n-1}{m(k+1)}T(r,h) + S(r,h) \\ &\leqslant \frac{n-1}{2}T(r,h) + S(r,h),\end{aligned} \tag{2.2}$$

a contradiction. Hence, h is constant.

Case 2 $(n,m) \neq 1$. Assume $(n,m) = d$ and $d \leqslant m$. Therefore $z^{n-m} - \beta$ and $z^n - \alpha$ can have at most m common factors. Therefore $z^n - \alpha$ can have at least $n-m$ distinct zeros, say $\beta_1, \beta_2, \ldots, \beta_{n-m}$. Next consider the following subcases.

Subcase 2.1 $k = 0$ and $m \geqslant 2$. Again, every zero of $h - \beta_i$ is a pole of g of order at least two. Now using the Second Fundamental Theorem we get

$$\begin{aligned}(n-m-2)T(r,h) &\leqslant \sum_{i=1}^{n-m} \overline{N}(r,\beta_i;h) + S(r,h) \leqslant \frac{1}{2}\sum_{i=1}^{n-m} N(r,\beta_i;h) + S(r,h) \\ &\leqslant \frac{n-m}{2}T(r,h) + S(r,h),\end{aligned}$$

a contradiction.

Subcase 2.2 $k > 0$ and $m \geqslant 1$. Now proceeding in the same way as in (2.2) we have

$$\begin{aligned}(n-m-2)T(r,h) &\leqslant \sum_{i=1}^{n-m} \overline{N}(r,\beta_i;h) + S(r,h) \leqslant \frac{1}{m(k+1)} \sum_{i=1}^{n-m} N(r,\beta_i;h) + S(r,h)\\ &\leqslant \frac{n-m}{2} T(r,h) + S(r,h),\end{aligned}$$

a contradiction. Therefore we h must be is a constant and hence, from (2.1) the result follows. □

Lemma 2.6. *Let f_1 and g_1 be defined the same as in Lemma 2.5 and k be a non-negative integer. Then for $n > 2m+7$ and*

(i) $k = 0$, $m \geqslant 2$;

(ii) $km \geqslant 1$,

$$f_1^{n-m}(af_1^m + b)g_1^{n-m}(ug_1^m + v) \not\equiv dt.$$

Proof. Let us consider two cases.

Case 1. First assume $k = 0$, $m \geqslant 2$. Also let us assume

$$f^{n-m}(af^m + b)g^{n-m}(ug^m + v) \equiv dt.$$

For the case $(n,m) = 1$ the result follows from Lemma 2.7 in [19]. Here we shall discuss the Subcase 2.2 in Lemma 2.7 of [19] when $(n,m) \neq 1$. Let z_0 be a zero of g of order p and a pole of f of order q. Then we have $(n-m)p = nq \Rightarrow p = n/(n-m)q > 1$. Therefore, here we have $p \geqslant 2$. Again, a zero of $ug^m + b$ is a pole of $f^{n-m}(af^m + b)$ of order at least n.

Now using the Second Fundamental Theorem we have

$$\begin{aligned}(m-1)T(r,g) &\leqslant \overline{N}(r,0;g) + \overline{N}(r,0;ug^m + v) + S(r,g)\\ &\leqslant \frac{1}{2}N(r,0;g) + \frac{1}{n}N(r,0;ug^m + v) + S(r,g)\\ &\leqslant \frac{1}{2}T(r,g) + \frac{m}{n}T(r,g) + S(r,g),\end{aligned}$$

a contradiction for $m \geqslant 2$.

Case 2. Consider $km \geqslant 1$. First, if possible, let us assume

$$f_1^{n-m}(af_1^m + b)g_1^{n-m}(ug_1^m + v) \equiv dt.$$

From Lemma 2.4 we get

$$T(r,f_1) = T(r,g_1) + S(r,g_1), \tag{2.3}$$

hence $S(r,f_1) = S(r,g_1)$.

Let z_1 be a zero of g_1 of order p_1 and hence a pole of f_1 of order q_1 ($\geqslant k+1$). Then we have $(n-m)p_1 = nq_1 \geqslant n(k+1)$, i.e., $p_1 \geqslant n(k+1)/(n-m) \geqslant 2n/(n-m) > 2$, therefore $p_1 \geqslant 3$. Next let z_2 be a zero of $ug_1^{n-m} + v = (g_1 - \gamma_1)\dots(g_1 - \gamma_m)$ of order p_2 and clearly z_2 will be a pole of f_1 of order q_2 ($\geqslant k+1$). Then we have $p_2 = nq_2 \geqslant n(k+1)$. Now using the Second Fundamental Theorem and (2.3) we have

$$\begin{aligned} mT(r,g_1) &\leqslant \overline{N}(r,0;g_1) + \overline{N}(r,\infty;g_1) + \overline{N}(r,0;ug_1^m + v) + S(r,g_1) \\ &\leqslant \frac{1}{3}N(r,0;g_1) + \frac{1}{n(k+1)}\sum_{i=1}^{m} N(r,0;g_1-\gamma_i) \\ &\quad + \overline{N}(r,0;f_1^{n-m}(af_1^m + b)) + S(r,g_1) \\ &\leqslant \frac{m}{2n}(T(r,f_1) + T(r,g_1)) + \frac{1}{3}(T(r,f_1) + T(r,g_1)) + S(r,g_1) \\ &\leqslant \frac{m}{n}T(r,g_1) + \frac{2}{3}T(r,g_1) + S(r,g_1), \end{aligned}$$

which gives a contradiction for $n > 2m+7$. □

Lemma 2.7. *Let f_1 and g_1 be defined the same as in Lemma 2.5 and k be a non-negative integer. If there exist two constants A ($\neq 0$) and B such that*

$$\frac{1}{P(f_1)} = \frac{A}{Q(g_1)} + B,$$

and $n > 2m+7$, where either $k=0$, $m \geqslant 2$ or $km \geqslant 1$, then $B = 0$.

Proof. Assume $B \neq 0$, proceeding exactly in the same way as in Lemma 2.7 of [19] we get

$$f_1^{n-m}(af_1^m + b)g_1^{n-m}(ug_1^m + v) = dt. \tag{2.4}$$

For the case $(n,m) = 1$ the result follows from Lemma 2.7 in [19]. When $(n,m) \neq 1$, then from Lemma 2.6 of the present paper we have a contradiction and the rest follows from Lemma 2.7 in [19]. □

Lemma 2.8. *Let $\varphi(z) = b^2(z^{n-m} - A)^2 - 4ac(z^{n-2m} - A)(z^n - A)$, where $A, a, b, c \in \mathbb{C}^*$, $(m,n) = 1$, $n > 3m$ and $b^2 \neq 4ac$. Then the following results hold.*

(i) *If e^{t_0} is any multiple zero of φ, then t_0 satisfies $\cosh(mt_0) = 1$ or $\cosh(mt_0) = b^2(n-m)^2/(2acn(n-2m)) - 1$.*

(ii) *Each multiple zero of φ is of multiplicity two whenever $b^2/(4ac) \neq n(n-2m)/(n-m)^2$.*

Proof. We omit the proof as it can be found in the proof of Lemma 2.5 in [8]. □

Lemma 2.9. *Let $\varphi(z) = b^2(z^{n-m} - A)^2 - 4ac(z^{n-2m} - A)(z^n - A)$, where $A, a, b, c \in \mathbb{C}^*$, $b^2/(4ac) = n(n-2m)/(n-m)^2$, $n > 2m$ and $(n, m) = 1$. Then it can have exactly one multiple zero of multiplicity four.*

Proof. The result can be easily obtained from Lemma 2.6 in [8]. □

Lemma 2.10. *Let $P(z) = az^n + bz^{n-m} + cz^{n-2m} + d$, where $a, b, c \in \mathbb{C}^*$. If $b^2/(4ac) = n(n-2m)/(n-m)^2$, then P is critically injective polynomial.*

Proof. Here we are not giving the proof as it can be obtained in the proof of Lemma 2.7 in [8]. □

3. Proofs of the theorems

Proof of Theorem 1.1. Let us consider $F = P(f_1)$ and $G = Q(g_1)$, where $f_1 = f^{(k)}$, $g_1 = g^{(k)}$ and consider the following cases.

Case 1. First assume $H \not\equiv 0$. We have

$$\begin{aligned}N(r,\infty;H) \leqslant{}& \overline{N}(r,0;f_1) + \overline{N}(r,0;g_1) + \overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1) + \overline{N}_*(r,0;F,G)\\ &+ \overline{N}(r,0;naf_1^m + (n-m)b) + \overline{N}(r,0;nug_1^m + (n-m)v)\\ &+ \overline{N}^1_{o_1}(r,0;f_1') + \overline{N}^1_{o_2}(r,0;g_1') + S(r,f_1) + S(r,g_1),\end{aligned}$$

where $\overline{N}^1_{o_1}(r,0;f_1')$ is the reduced counting function of those zeros of f_1' which are not zeros of $f_1P(f_1)(naf_1^m + (n-m)b)$ and $\overline{N}^1_{o_2}(r,0;g_1')$ is the reduced counting function of those zeros of g_1' which are not zeros of $g_1Q(g_1)(nug_1^m + (n-m)v)$. So

$$\begin{aligned}&(n+m)(T(r,f_1) + T(r,g_1))\\ &\quad\leqslant \overline{N}(r,0;f_1) + \overline{N}(r,0;g_1) + \overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1) + \overline{N}(r,0;F)\\ &\qquad+ \overline{N}(r,0;G) + \overline{N}(r,0;naf^m + (n-m)b) + \overline{N}(r,0;nug_1^m + (n-m)v)\\ &\qquad- N^1_{o_1}(r,0;f_1') - N^1_{o_2}(r,0;g_1') + S(r,f_1) + S(r,g_1).\end{aligned}$$

Using Lemmas 2.3, 2.1 from above, we have

$$\begin{aligned}(n-1)T(r) \leqslant{}& \frac{1}{2}(N(r,0;F) + N(r,0;G)) + N^{1)}_E(r,0;F) + \Big(\frac{1}{2} - l\Big)\overline{N}_*(r,0;F,G)\\ &+ \overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1) - N^1_{o_1}(r,0;f_1') - N^1_{o_2}(r,0;g_1') + S(r),\end{aligned}$$

where $T(r) = T(r,f_1) + T(r,g_1)$ and $S(r) = S(r,f_1) + S(r,g_1) = o(T(r))$.

$$\begin{aligned}\Big(\frac{n}{2} - 1\Big)T(r) \leqslant{}& \overline{N}(r,0;f_1) + \overline{N}(r,0;g_1) + 2(\overline{N}(r,\infty;f_1) + \overline{N}(r,\infty;g_1))\\ &+ \Big(\frac{3}{2} - l\Big)\overline{N}_*(r,0;F,G) + \overline{N}(r,0;naf_1^m + (n-m)b)\\ &+ \overline{N}(r,0;nug_1^m + (n-m)v) + S(r).\end{aligned}\tag{3.1}$$

From (3.1) we have

$$
\begin{aligned}
(3.2)\quad \Big(\frac{n}{2}-2\Big)T(r) &\leqslant \Big(\frac{2}{k+1}+m\Big)T(r)\\
&\quad+\Big(\frac{3}{2}-l\Big)(\overline{N}_L(r,0;F)+\overline{N}_L(r,0;G))+S(r)\\
&\leqslant \Big(\frac{2}{k+1}+m\Big)T(r)\\
&\quad+\Big(\frac{3}{2}-l\Big)(\overline{N}(r,0;F\,|\geqslant l+2)+\overline{N}(r,0;G\,|\geqslant l+2))+S(r)\\
&\leqslant \Big(\frac{2}{k+1}+m\Big)T(r)+\Big(\frac{3}{2}-l\Big)\frac{1}{l+1}(\overline{N}(r,0;f_1)+\overline{N}(r,\infty;f_1)\\
&\quad+\overline{N}(r,0;g_1)+\overline{N}(r,\infty;g_1))+S(r)\\
&\leqslant \Big(\frac{2}{k+1}+m\Big)T(r)+\Big(\frac{3}{2}-l\Big)\frac{1}{l+1}\Big(1+\frac{1}{k+1}\Big)T(r)+S(r).
\end{aligned}
$$

Now from (3.2) for

(i) $l=2$ and $n>2m+4+4/(k+1)$,

(ii) $l=0$ and $n>2m+7+7/(k+1)$,

we arrive at a contradiction.

Case 2. Let $H\equiv 0$ and then integrating we have

$$\frac{1}{F}=\frac{C}{G}+D\Rightarrow\frac{1}{P(f_1)}=\frac{C}{Q(g_1)}+D,\quad\text{i.e.,}\ \frac{Q(g_1)}{P(f_1)}=C+DQ(g_1),$$

where $C\ (\neq 0)$, D are finite constants. Clearly, if the zeros of $P(f_1)$ and $Q(g_1)$ have different multiplicities, then either $C=0$ or ∞, a contradiction. Hence, the zeros of $P(f_1)$ and $Q(g_1)$ are of the same multiplicity, i.e., $P_1(f_1)$ and $Q_1(g_1)$ share 0 CM. Now, from the proof of Theorem 1.7 in [19] and with the help of Lemmas 2.5, 2.6, and 2.7, the result follows immediately. □

Proof of Theorem 1.2. Let us consider Q_1 defined as in (1.5). Then from the assumption $Q_1=KP_1+\widehat{c}$, comparing the coefficient we have $a_2=Ka_1, b_2=Kb_1$ and $c_2=Kc_1, d_2=Kd_1+\widehat{c}$.

Also let us assume $F=P_1(f_1)$ and $G=Q_1(g_1)$. Clearly, here F and G share $(0,l)$.

Case 1. Let $b_1^2/(4a_1c_1)\neq n(n-2m)/(n-m)^2,1$. Now, $F'=(P_1(f_1))'=P_1'(f_1)f_1'=f_1^{n-2m-1}(na_1f_1^{2m}+b_1(n-m)f_1^m+c_1(n-2m))f_1'$, $G'=g_1^{n-2m-1}(na_2g_1^{2m}+b_2(n-m)g_1^m+c_2(n-2m))g_1'$. We know $T(r,F)=nT(r,f_1)+S(r,f_1)$ and $T(r,G)=nT(r,g_1)+S(r,g_1)$.

Next let α_i, $i=1,2,\dots,k_1$ be the distinct zeros of $na_1z^{2m}+b_1(n-m)z^m+c_1(n-2m)$.

Subcase 1.1 First let us consider $H \not\equiv 0$. Now from Lemma 2.2 we have

$$(3.3)\qquad \begin{aligned} N(r,\infty;H) \leqslant\ & \overline{N}(r,0;f_1)+\overline{N}(r,0;g_1)+\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1) \\ & +\overline{N}_*(r,0;F,G)+\sum_{i=1}^{k_1}\overline{N}(r,\alpha_i;f_1)+\sum_{i=1}^{k_1}\overline{N}(r,\alpha_i;g_1) \\ & +\overline{N}^{2}_{o_1}(r,0;f_1')+\overline{N}^{2}_{o_2}(r,0;g_1')+S(r,f_1)+S(r,g_1), \end{aligned}$$

where $\overline{N}^{2}_{o_1}(r,0;f_1')$ is the reduced counting function of those zeros of f_1' which are not zeros of $f_1P_1(f_1)\prod\limits_{i=1}^{k_1}(f_1-\alpha_i)$; similarly $\overline{N}^{2}_{o_2}(r,0;g_1')$ can be defined. Clearly $P_1(\alpha_i), Q_1(\alpha_i)\neq 0$ and $P_1(0)\cdot Q_1(0)\neq 0$. Applying the Second Fundamental Theorem to f_1 and g_1 we have

$$\begin{aligned} &(n+k_1)(T(r,f_1)+T(r,g_1)) \\ &\quad\leqslant \overline{N}(r,0;f_1)+\overline{N}(r,0;g_1)+\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1) \\ &\qquad +\overline{N}(r,0;F)+\overline{N}(r,0;G)+\sum_{i=1}^{k_1}(\overline{N}(r,\alpha_i;f_1)+\overline{N}(r,\alpha_i;g_1)) \\ &\qquad - N^{2}_{o_1}(r,0;f_1')-N^{2}_{o_2}(r,0;g_1')+S(r,f_1)+S(r,g_1), \end{aligned}$$

i.e.,

$$(3.4)\qquad \begin{aligned} (n-1)T(r) \leqslant\ & \overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+\overline{N}(r,0;G)+\overline{N}(r,0;F) \\ & - N^{2}_{o_1}(r,0;f_1')-N^{2}_{o_2}(r,0;g_1')+S(r). \end{aligned}$$

Applying Lemma 2.3 from (3.4) we have

$$\begin{aligned} (n-1)T(r) &\leqslant \overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+N^{1)}_{E}(r,0;F)+\Big(\frac{1}{2}-l\Big)\overline{N}_*(r,0;F,G) \\ &\quad +\frac{1}{2}(N(r,0;F)+N(r,0;G))-N^{2}_{o_1}(r,0;f_1')-N^{2}_{o_2}(r,0;g_1')+S(r) \\ &\leqslant \overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+N^{1)}_{E}(r,0;F)+\Big(\frac{1}{2}-l\Big)\overline{N}_*(r,0;F,G) \\ &\quad +\frac{1}{2}(T(r,F)+T(r,G))-N^{2}_{o_1}(r,0;f_1')-N^{2}_{o_2}(r,0;g_1')+S(r). \end{aligned}$$

Using Lemma 2.1 from above we get

$$\begin{aligned} (n-1)T(r) \leqslant\ & \frac{n}{2}T(r)+\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+N(r,\infty;H) \\ & +\Big(\frac{1}{2}-l\Big)\overline{N}_*(r,0;F,G)-N^{2}_{o_1}(r,0;f_1')-N^{2}_{o_2}(r,0;g_1')+S(r). \end{aligned}$$

Next using (3.3) from above we get

(3.5)

$$\begin{aligned}
\Big(\frac{n}{2}-1\Big)T(r) &\leqslant \overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1)+N(r,\infty;H)\\
&\quad+\Big(\frac{1}{2}-l\Big)\overline{N}_*(r,0;F,G)-N^2_{o_1}(r,0;f_1')-N^2_{o_2}(r,0;g_1')+S(r)\\
&\leqslant \overline{N}(r,0;f_1)+\overline{N}(r,0;g_1)+2(\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1))\\
&\quad+\Big(\frac{3}{2}-l\Big)\overline{N}_*(r,0;F,G)+\sum_{i=1}^{k_1}\overline{N}(r,\alpha_i;f_1)+\sum_{i=1}^{k_1}\overline{N}(r,\alpha_i;g_1)+S(r)\\
&\leqslant 2(\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1))+\Big(\frac{3}{2}-l\Big)(\overline{N}_L(r,0;F)+\overline{N}_L(r,0;G))\\
&\quad+(k_1+1)T(r)+S(r)\\
&\leqslant 2(\overline{N}(r,\infty;f_1)+\overline{N}(r,\infty;g_1))\\
&\quad+\Big(\frac{3}{2}-l\Big)(\overline{N}(r,0;F\,|\geqslant l+2)+\overline{N}(r,0;G\,|\geqslant l+2))\\
&\quad+(k_1+1)T(r)+S(r)\\
&\leqslant (k_1+1)T(r)+\frac{2}{k+1}T(r)+\Big(\frac{3}{2}-l\Big)\frac{1}{l+1}\\
&\quad\times(\overline{N}(r,0;f_1)+\overline{N}(r,\infty;f_1)+\overline{N}(r,0;g_1)+\overline{N}(r,\infty;g_1))+S(r)\\
&\leqslant (k_1+1)T(r)+\frac{2}{k+1}T(r)+\Big(\frac{3}{2}-l\Big)\frac{1}{l+1}\Big(1+\frac{1}{k+1}\Big)T(r)+S(r).
\end{aligned}$$

Now $k_1\leqslant 2m$, then from (3.5) for

(i) $l=2$ and $n>4m+4+4/(k+1)$,

(ii) $l=0$ and $n>4m+7+7/(k+1)$,

we have a contradiction.

Subcase 1.2. $H\equiv 0$. Then integrating we have

$$\frac{1}{P_1(f_1)}=\frac{D_1}{Q_1(g_1)}+D_2,$$

where D_1 $(\neq 0)$, D_2 are finite constants. Using Lemma 2.4 we have

$$T(r,f_1)=T(r,g_1)+O(1). \tag{3.6}$$

Now from above we have

$$\frac{1}{P_1(f_1)}=\frac{D_1+D_2Q_1(g_1)}{Q_1(g_1)},\qquad \frac{Q_1(g_1)}{P_1(f_1)}=D_1+D_2Q_1(g_1).$$

Clearly, if the zeros of $P_1(f_1)$ and $Q_1(g_1)$ have different multiplicity, then either $D_1=0$ or ∞, a contradiction. Hence, the zeros of $P_1(f_1)$ and $Q_1(g_1)$ are of the same multiplicity, i.e., $P_1(f_1)$ and $Q_1(g_1)$ share 0 CM.

Using the Second Fundamental Theorem in view of (3.6) we have

$$\begin{aligned}(3.7)\qquad nT(r,g_1)+O(1)&=T(r,Q_1(g_1))\\&\leqslant \overline{N}(r,\infty;Q_1(g_1))+\overline{N}(r,0;Q_1(g_1)-d_2)\\&\quad+\overline{N}(r,0;Q_1(g_1)+D_1/D_2)+S(r,g_1)\\&\leqslant \overline{N}(r,\infty;g_1)+\overline{N}(r,0;g_1^{n-2m}(a_2g_1^{2m}+b_2g_1^m+c_2))\\&\quad+\overline{N}(r,\infty;f_1)+S(r,g_1)\\&\leqslant (2m+3)T(r,g_1)+S(r,g_1),\end{aligned}$$

a contradiction. Therefore either $d_2=-D_1/D_2$ or $D_2=0$.

Subcase 1.2.1. First assume $d_2=-D_1/D_2$, then

$$\frac{1}{P(f_1)}=\frac{D_2g_1^{n-2m}(a_2g_1^{2m}+b_2g_1^m+c_2)}{Q_1(g_1)}.$$

Since $b_2^2/(4a_2c_2)\neq n(n-2m)/(n-m)^2, 1$ therefore $a_2z^{2m}+b_2z^m+c_2$ has all simple zeros which are poles of f_1 and hence of multiplicity at least of $\geqslant n(k+1)$. Now let z_0 be a zero of g_1 of order p and a pole of f_1 of order q. Then we have $(n-2m)p=nq\Rightarrow p=n/(n-2m)q>1$, i.e., $p\geqslant 2$.

So using the Second Fundamental Theorem we have

$$\begin{aligned}(3.8)\quad 2mT(r,g_1)&\leqslant \overline{N}(r,0;g_1)+\overline{N}(r,0;a_2g_1^{2m}+b_2g_1^m+c_2)+\overline{N}(r,\infty;g_1)+S(r,g_1)\\&\leqslant\Big(\frac{1}{2}+\frac{2m}{n}+\frac{1}{k+1}\Big)T(r,g_1)+S(r,g_1),\end{aligned}$$

a contradiction for $n\geqslant 4m+4$.

Subcase 1.2.2. Let $D_2=0$. Hence, finally we have $P_1(f_1)=DQ(g_1)$, where $D=1/D_1$. So

$$\begin{aligned}(3.9)\qquad f_1^{n-2m}(a_1f_1^{2m}+b_1f_1^m+c_1)&=D(a_2g_1^n+b_2g_1^{n-m}+c_2g_1^{n-2m}+d_2-d_1/D)\\&=D(G-d_1/D).\end{aligned}$$

Subcase 1.2.2.1. Let $d_2\neq d_1/D$. Applying the Second Fundamental Theorem to G and using (3.6), from (3.9) we have

(3.10)

$$\begin{aligned}nT(r,g_1)+O(1)&=T(r,G)\\&\leqslant \overline{N}(r,0;G-d_2)+\overline{N}(r,\infty;G)+\overline{N}(r,0;G-d_1/D)+S(r,g_1)\\&\leqslant \overline{N}(r,0;g_1)+\overline{N}(r,0;a_2g_1^{2m}+b_2g_1^m+c_2)+\overline{N}(r,\infty;g_1)\\&\quad+\overline{N}(r,0;f_1)+\overline{N}(r,0;a_1f_1^{2m}+b_1f_1^m+c_1)+S(r,g_1)\\&\leqslant\Big(2+4m+\frac{1}{k+1}\Big)T(r,g_1)+S(r,g_1),\end{aligned}$$

a contradiction as $n\geqslant 4m+4$.

Subcase 1.2.2.2. If $d_2 - d_1/D = 0$, i.e., $D = d_1/d_2$, then

$$f_1^{n-2m}(a_1 f_1^{2m} + b_1 f_1^m + c_1) = D g_1^{n-2m}(a_2 g_1^{2m} + b_2 g_1^m + c_2).$$

Suppose $h = f_1/g_1$. Then we have

$$a_1 g_1^{2m}(h^n - KD) + b_1 g_1^m (h^{n-m} - KD) + c_1(h^{n-2m} - KD) = 0. \tag{3.11}$$

Subcase 1.2.2.2.1. If h is a non-constant meromorphic function, then from (3.11) we have

$$g_1^{2m} + \frac{b_1(h^{n-m} - KD)}{a_1(h^n - KD)} g_1^m + \frac{c_1(h^{n-2m} - KD)}{a_1(h^n - KD)} = 0,$$

i.e.,

$$\begin{aligned} \Big(g_1^m + \frac{b_1(h^{n-m} - K')}{2a_1(h^n - K')}\Big)^2 &= \frac{b_1^2(h^{n-m} - K')^2 - 4a_1c_1(h^n - K')(h^{n-2m} - K')}{4a_1^2(h^n - K')^2} \\ &= \frac{\varphi_o(h)}{4a_1^2(h^n - K')^2}, \end{aligned} \tag{3.12}$$

where $K' = KD$.

As by the statement of the theorem $n > 3m$, from Lemma 2.8, if φ_o has multiple zero e^{t_0}, then $\cosh(mt_0) = 1$ or $\cosh(mt_0) = b_1^2(n-m)^2/(2a_1c_1n(n-2m)) - 1$, and each multiple zero of φ_o has multiplicity two.

Let e^{t_0} be a multiple zero of φ_o. Then either $\frac{1}{2}(\mathrm{e}^{mt_0} + \mathrm{e}^{-mt_0}) = 1 \Rightarrow (\mathrm{e}^{t_0})^m = 1$ or $\frac{1}{2}(\mathrm{e}^{mt_0} + \mathrm{e}^{-mt_0}) = b_1^2(n-m)^2/(2a_1c_1n(n-2m)) - 1 = p$ (say) $\Rightarrow (\mathrm{e}^{t_0})^m = p \pm \sqrt{p^2 - 1}$, i.e., there exist at most $m + 2m = 3m$ multiple zeros of order two. Hence, φ_o has at least $2n - 2m - 2 \cdot 3m = 2n - 8m$ distinct simple zeros, say (ν_i, $i = 1, 2, \ldots, 2n - 8m$).

Applying the Second Fundamental Theorem to h, we have

$$(2n-8m-2)T(r,h) \leqslant \sum_{i=1}^{2n-8m} \overline{N}(r,\nu_i;h) + S(r,h) \leqslant (n-4m)T(r,h) + S(r,h), \tag{3.13}$$

a contradiction for $n \geqslant 4m + 3$.

Subcase 1.2.2.2.2. Consider h is a constant, then from (3.11) we get $h^n = KD = h^{n-m} = h^{n-2m} \Rightarrow h^m = 1$ and hence $|f_1| = |g_1|$.

Case 2. Consider $b_1^2/(4a_1c_1) = n(n-2m)/(n-m)^2$. If $b_1^2/(4a_1c_1) = n(n-2m)/(n-m)^2$, then we have

$$na_1 f_1^{2m} + b_1(n-m) f_1^m + c_1(n-2m) = na_1\Big(f_1^m + \frac{b_1(n-m)}{2na_1}\Big)^2 = na_1 \prod_{i=1}^m (f_1 - \alpha_i)^2.$$

Therefore, $F' = \prod_{i=1}^m (f_1 - \alpha_i)^2 f_1'$ and $G' = \prod_{i=1}^m (g_1 - \alpha_i)^2 g_1'$.

Subcase 2.1. Let $H \not\equiv 0$, we have

$$\begin{aligned} N(r,\infty;H) \leqslant{} & \overline{N}(r,0;f_1) + \overline{N}(r,0;g_1) + \overline{N}_*(r,0;F,G) \\ & + \sum_{i=1}^{m}(\overline{N}(r,\alpha_i;f_1) + \overline{N}(r,\alpha_i;g_1)) + \overline{N}^2_{o_1}(r,0;f_1') \\ & + \overline{N}^2_{o_2}(r,0;g_1') + S(r,f_1) + S(r,g_1), \end{aligned}$$

where $\overline{N}^2_{o_1}(r,0;f_1')$ is the reduced counting function of those zeros of f_1' which are not zeros of $f_1P_1(f_1)\prod_{i=1}^{k}(f_1-\alpha_i)$; similarly, $\overline{N}^2_{o_2}(r,0;g_1')$ can be defined.

Now proceeding in the same way as in (3.4), (3.5) for $k_1 = m$ and

(i) $l = 2$ and $n > 2m + 4 + 4/(k+1)$,

(ii) $l = 0$ and $n > 2m + 7 + 7/(k+1)$,

we have a contradiction.

Subcase 2.2. Assume $H \equiv 0$. Then integrating we have

$$\frac{1}{(P_1(f_1))} = \frac{A_1}{(Q_1(g_1))} + A_2,$$

where A_1 $(\neq 0)$, A_2 are constants. Also proceeding similarly as in (3.7) and (3.8), we have $A_2 = 0$ and then $P_1(f_1) = AQ_1(g_1)$, where $A = 1/A_1$.

Now,

$$f_1^{n-2m}(a_1f_1^{2m} + b_1f_1^m + c_1) = A(a_2g_1^n + b_2g_2^{n-m} + c_2g_1^{n-2m} + d_2 - d_1/A). \tag{3.14}$$

Subcase 2.2.1. Let $d_2 \neq d_1/A$. From Lemma 2.10 the polynomial $a_2z^n + b_2z^{n-m} + c_2z^{n-2m} + d_2 - d_1/A$ is critically injective. Hence, it can have at most one multiple zero of multiplicity three and at least $n-2$ distinct zeros (δ_i, $i = 1,2,\ldots,n-2$).

Applying the Second Fundamental Theorem to g_1 and using (3.6), we have from (3.14) that

$$\begin{aligned} (n-3)T(r,g_1) &\leqslant \sum_{i=1}^{n-2}\overline{N}(r,\delta_i;g_1) + \overline{N}(r,\infty;g_1) + S(r,g_1) \\ &\leqslant \overline{N}(r,0;f_1) + \overline{N}(r,0;a_1f_1^{2m} + b_1f_1^m + c_1) \\ &\quad + \frac{1}{k+1}N(r,\infty;g_1) + S(r,g_1) \\ &\leqslant (2m+1)T(r,f_1) + \frac{1}{k+1}T(r,g_1) + S(r,g_1) \\ &\leqslant \Big(2m + 1 + \frac{1}{k+1}\Big)T(r,g_1) + S(r,g_1), \end{aligned} \tag{3.15}$$

a contradiction for $n > 2m + 4 + 1/(k+1)$.

Subcase 2.2.2. $A = d_1/d_2$. Finally, from (3.14) we get

$$f_1^{n-2m}(a_1 f^{2m} + b_1 f_1^m + c_1) = A g_1^{n-2m}(a_2 g_1^{2m} + b_2 g_1^m + c_2),$$

i.e.,

$$a_1 g_1^{2m}(h^n - KA) + b_1 g_1^m (h^{n-m} - KA) + c_1(h^{n-2m} - KA) = 0. \tag{3.16}$$

Subcase 2.2.2.1. If h is a non-constant meromorphic function, then we get

$$\begin{aligned}\left(g_1^m + \frac{b_1(h^{n-m} - KA)}{2a_1(h^n - KA)}\right)^2 &= \frac{b_1^2(h^{n-m} - KA)^2 - 4a_1c_1(h^n - KA)(h^{n-2m} - KA)}{4a_1^2(h^n - KA)^2} \\ &= \frac{\varphi_1(h)}{4a_1^2(h^n - KA)^2}.\end{aligned}$$

Next from Lemma 2.9, φ_1 can have at most one multiple zero of multiplicity four and at least $2n - 2m - 4$ distinct simple zeros, say y_i, $(i = 1, 2, \ldots, 2n - 2m - 4)$. Then applying the Second Fundamental Theorem to 'h' we have

(3.17)

$$(2n - 2m - 6)T(r, h) \leqslant \sum_{i=1}^{2n-2m-4} \overline{N}(r, y_i; h) + S(r, h) \leqslant (n - m - 2)T(r, h) + S(r, h),$$

a contradiction for $n > m + 4$.

Subcase 2.2.1.2. If h is a constant, then from (3.16), $h^n = KA = h^{n-m} = h^{n-2m}$ and so $h^n = KA = Kd_1/d_2 = Kd_1/(Kd_1 + \widehat{c})$, $h^m = 1$. Hence, we get the result. In particular, if $\widehat{c} = 0$, then we get $f_1 \equiv g_1$. $\square$

P r o o f of Theorem 1.3. Assume $F = P_1(f_1)$ and $G = Q_1(g_1)$.

Case 1. Assume $H \not\equiv 0$, then proceeding in the same way as in (3.4), (3.5) we have a contradiction.

Case 2. Let $H \equiv 0$. Then integrating we have

$$\frac{1}{P_1(f_1)} = \frac{E_1}{Q_1(g_1)} + E_2, \tag{3.18}$$

where E_1 $(\neq 0)$, E_2 are two constants.

Next we show that $E_2 = 0$. Now proceeding in the same manner as in (3.7) we have $E_2 = 0$ or $E_1/E_2 = -d_2$.

Subcase 2.1. Assume $E_1/E_2 = -d_2$, then

$$\frac{1}{P_1(f_1)} = \frac{E_2(g_1^{n-2m}(a_2 g_1^{2m} + b_2 g_1^m + c_2))}{Q_1(g_1)} = \frac{E_2(a_2 g_1^{n-2m}(g_1^m + b_2/2a_2)^2)}{Q_1(g_1)}.$$

Now assume z_0 be a zero $z^m + b_2/2a_2$ of order p, then it is also a pole of $P_1(f_1)$. Then we have $2p \geqslant n(k+1) \Rightarrow p \geqslant \frac{1}{2}n(k+1)$. Also assume z_1 be a zero of order q and hence a pole of f_1, therefore $(n-2m)q \geqslant n(k+1) \Rightarrow q \geqslant n(k+1)/(n-2m) > (k+1)$, i.e., $q \geqslant k+2$. Using the Second Fundamental Theorem we have

$$\begin{aligned}(3.19)\quad mT(r,g_1) &\leqslant \overline{N}(r,0;g_1) + \overline{N}(r,\infty;g_1) + \overline{N}(r,0;g_1^m + b_2/2a_2) + S(r,g_1)\\ &\leqslant \Big(\frac{1}{k+2} + \frac{1}{k+1} + \frac{2m}{n(k+1)}\Big)T(r,g_1) + S(r,g_1)\end{aligned}$$

for $k=0$, $m \geqslant 2$, $n \geqslant 4m+5$ and for $k, m \geqslant 1$ and $n \geqslant 4m+5$ we arrive at a contradiction.

Subcase 2.2. $E_2 = 0$. From (3.18) we have $P_1(f_1) = EQ_1(g_1)$ $(E = 1/E_1)$. Again,

$$(3.20)\quad f_1^{n-2m}(a_1 f_1^{2m} + b_1 f_1^m + c_1) = E\Big(g_1^{n-2m}(a_2 g_1^{2m} + b_2 g_1^m + c_2) + d_2 - \frac{d_1}{E}\Big).$$

Subcase 2.2.1. Let us consider $d_2 - d_1/E \neq 0$. Applying the Second Fundamental Theorem to G and using (3.20), (3.6) we have

$$\begin{aligned}(3.21)\quad & nT(r,g_1) + O(1)\\ &= T(r,G) \leqslant \overline{N}(r,0;G-d_2) + \overline{N}(r,\infty;G) + \overline{N}(r,0;G-d_1/E) + S(r,g_1)\\ &\leqslant \overline{N}(r,0;g_1) + \overline{N}(r,0;a_2 g_1^{2m} + b_2 g_1^m + c_2) + \overline{N}(r,\infty;g_1)\\ &\quad + \overline{N}(r,0;f_1) + \overline{N}(r,0;a_1 f_1^{2m} + b_1 f_1^m + c_1) + S(r,g_1)\\ &\leqslant \Big(2m + 2 + \frac{1}{k+1}\Big)T(r,g_1) + S(r,g_1),\end{aligned}$$

a contradiction as $n > 2m+3$.

Subcase 2.2.2. Let $d_2 - d_1/E = 0$. Now

$$(3.22)\quad f_1^{n-2m}(a_1 f_1^{2m} + b_1 f_1^m + c_1) = g_1^{n-2m}(a_3 g_1^{2m} + b_3 g_1^m + c_3),$$

where $a_3 = Ea_2$, $b_3 = Eb_2$ and $c_3 = Ec_2$ and $b_2^2 = 4a_2c_2$ implies $b_3^2 = 4a_3c_3$.

Now from above we have

$$\begin{aligned}(3.23)\quad & (a_1 f_1^n - a_3 g_1^n) + (b_1 f_1^{n-m} - b_3 g_1^{n-m}) + (c_1 f^{n-2m} - c_3 g_1^{n-2m}) = 0\\ &\Rightarrow g_1^{2m}(a_1 h^n - a_3) + g_1^m(b_1 h^{n-m} - b_3) + (c_1 h^{n-2m} - c_3) = 0,\end{aligned}$$

where $h = f_1/g_1$.

Subcase 2.2.2.1. Let h be a non-constant meromorphic function and finally from (3.23) we get

$$g_1^{2m} + \frac{b_1 h^{n-m} - b_3}{a_1 h^n - a_3} g_1^m + \frac{c_1 h^{n-2m} - c_3}{a_1 h^n - a_3} = 0.$$

Using the facts $b_1^2 = 4a_1c_1$ and $b_2^2 = 4a_2c_2$, from above we get

(3.24)
$$\begin{aligned}\Big(g_1^m + \frac{b_1h^{n-m} - b_3}{2(a_1h^n - a_3)}\Big)^2 &= \frac{(b_1h^{n-m} - b_3)^2 - 4(a_1h^n - a_3)(c_1h^{n-2m} - c_3)}{4(a_1h^n - a_3)^2}\\ &= \frac{h^{n-2m}(4a_1c_3h^{2m} - 2b_1b_3h^m + 4a_3c_1)}{4(a_1h^n - a_3)^2}\\ &= \frac{h^{n-2m}b_1^2(4a_1(c_3/b_1^2)h^{2m} - 2b_1(b_3/b_1^2)h^m + 4a_3c/b_1^2)}{4(a_1h^n - a_3)^2}\\ &= \frac{h^{n-2m}b_1^2((c_3/c_1)h^{2m} - 2(b_3/b_1)h^m + a_3/a_1)}{4a_1^2(h^n - (a_3/a_1))^2}\\ &= \frac{h^{n-2m}b_1^2c_*(h^m - b_*/c_*)^2}{4a_1^2(h^n - a_*)^2} = \frac{h^{n-2m}c_3(h^m - \beta)^2}{a_1(h^n - a_*)^2},\end{aligned}$$

where $a_* = a_3/a_1$, $b_* = b_3/b_1$, $c_* = c_3/c_1$ and $b_*/c_* = \beta$.

Subcase 2.2.2.1.1. Suppose n is even. Assume $n = 2r$. From (3.24) we have

$$\begin{aligned}\Big(g_1^m + \frac{b_1h^{n-m} - b_3}{2a_1(h^n - a_*)}\Big)^2 &= \frac{h^{n-2m}c_3(h^m - \beta)^2}{a_1(h^n - a_*)^2},\\ g_1^m + \frac{b_1h^{2r-m} - b_3}{2a_1(h^{2r} - a_*)} &= \pm\Big(\frac{c_3}{a_1}\Big)^{1/2}\frac{h^{r-m}(h^m - \beta)}{h^{2r} - a_*},\end{aligned}$$

i.e.,

$$\begin{aligned}g_1^m &= -\frac{b_1h^{2r-m} - b_3}{2a_1(h^{2r} - a_*)} \pm \frac{(c_3/a_1)^{1/2}h^{r-m}(h^m - \beta)}{h^{2r} - a_*}\\ &= \frac{-(b_1h^{2r-m} - b_3) \pm 2a_1(c_3/a_1)^{1/2}h^{r-m}(h^m - \beta)}{2a_1(h^{2r} - a_*)}.\end{aligned} \tag{3.25}$$

Now it can be seen that the numerator and denominator can have at most $r + m$ common factors. Therefore the denominator has at least $r - m$ distinct zeros, say, $\gamma_1, \gamma_2, \dots, \gamma_{r-m}$. Now using the Second Fundamental Theorem we have

$$\begin{aligned}(r - m - 2)T(r, h) &\leqslant \sum_{i=1}^{r-m} \overline{N}(r, 0; h - \gamma_i) + S(r, h)\\ &\leqslant \frac{1}{m(k+1)}\sum_{i=1}^{r-m} \overline{N}(r, 0; h - \gamma_i) + S(r, h)\\ &\leqslant \frac{r - m}{m(k+1)}T(r, h) + S(r, h) \leqslant \frac{r - m}{2}T(r, h) + S(r, h),\end{aligned}$$

a contradiction.

Subcase 2.2.2.1.2. Suppose n is odd. Again $b_1^2/(4a_1c_1) = 1 = b_3^2/(4a_3c_3)$ and we have

$$\begin{aligned}(3.26)\qquad \frac{f_1^{n-2m}}{g_1^{n-2m}} &= \frac{a_3g_1^{2m}+b_3g_1^m+c_3}{a_1f_1^{2m}+b_1f_1^m+c_1} = \frac{a_3(g_1^m+b_3/2a_3)^2}{a_1(f_1^m+b_1/2a_1)^2},\\ \Rightarrow h^{n-2m} &= a_*\frac{(g_1^m+b_3/2a_3)^2}{(f_1^m+b_1/2a_1)^2} \Rightarrow h = a_*\frac{(g_1^m+b_3/2a_3)^2}{h^{n-2m-1}(f_1^m+b_1/2a_1)^2}.\end{aligned}$$

Since n is odd, it follows that $n-(2m+1)$ is even and we can have a meromorphic function σ such that

$$h = a_*\frac{(g_1^m+b_3/2a_3)^2}{h^{n-2m-1}(f_1^m+b_1/2a_1)^2} = \sigma^2.$$

As f_1, g_1 and h are meromorphic, so also is σ. Putting $h=\sigma^2$ in (3.24) we have

$$\left(g_1^m + \frac{b_1\sigma^{2n-2m}-b_3}{2a_1(\sigma^{2n}-a_*)}\right)^2 = \frac{\sigma^{2(n-2m)}c_3(\sigma^{2m}-\beta)^2}{a_1(\sigma^{2n}-a_*)^2}.$$

So,

$$(3.27)\qquad g_1^m + \frac{b_1\sigma^{2n-2m}-b_3}{2a_1(\sigma^{2n}-a_*)} = \pm\left(\frac{c_3}{a_1}\right)^{1/2}\frac{\sigma^{n-2m}(\sigma^{2m}-\beta)}{\sigma^{2n}-a_*}.$$

From (3.27) we have

$$(3.28)\qquad g_1^m = -\frac{b_1\sigma^{2n-2m}-b_3}{2a_1(\sigma^{2n}-a_*)} \pm\left(\frac{c_3}{a_1}\right)^{1/2}\frac{\sigma^{n-2m}(\sigma^{2m}-\beta)}{\sigma^{2n}-a_*}.$$

Now for all cases in (3.28), if the numerator $-(b_1\sigma^{2n-2m}-b_3)\pm 2(c_3/a_1)^{1/2}a_1\times \sigma^{n-2m}(\sigma^{2m}-\beta)$ and the denominator $\sigma^{2n}-a_*$ have any common factor $\sigma-x$, then x will be a zero of $-(b_1a_*-b_3z^{2m})\pm 2(c_3/a_1)^{1/2}a_1^2z^n(z^{2m}-\beta)$. Hence, the numerator and denominator can have at most $n+2m$ common factors. Hence, $\sigma^{2n}-a_*$ has at least $2n-(n+2m)=n-2m$ factors, say $\sigma-\mu_i$, $(i=1,2,\dots .n-2m)$, which are not factors of numerator. Again,

$$\begin{aligned}(n-2m-2)T(r,\sigma) &\leqslant \sum_{i=1}^{n-2m}\overline{N}(r,\mu_i;\sigma)+S(r,\sigma)\\ &\leqslant \frac{1}{m(k+1)}\sum_{i=1}^{n-2m}N(r,\mu_i;\sigma)+S(r,\sigma)\\ &\leqslant \frac{n-2m}{m(k+1)}T(r,\sigma)+S(r,\sigma) \leqslant \frac{n-2m}{2}T(r,\sigma)+S(r,\sigma),\end{aligned}$$

since σ is not rational, it implies a contradiction for $n>2m+4$.

Case 2.2.2.2. If h is a constant, then we get $f_1 \equiv hg_1$. □

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