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THE NARROW RECURRENCE OF RANDOM DYNAMICAL SYSTEMS

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Abstract. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where $\mathcal{F}$ is countably generated, and X be a Polish space. Let φ be a random dynamical system with time $\mathbb{T}$ on X. The skew product flow $\{\Theta_t,\ t \in \mathbb{T}\}$ induced by φ is a family of continuous operators acting on $\mathrm{Pr}_\Omega(X)$, the set of all probability measures on $X \times \Omega$ with marginal $\mathbb{P}$, which is a Polish space equipped with the narrow topology. In this work, we introduce and study the notion of narrow recurrence of the flow $\{\Theta_t,\ t \in \mathbb{T}\}$ on $\mathrm{Pr}_\Omega(X)$ and we give some results, which can be considered as an initiation of applications of properties of topological dynamics on stochastic process theory and random dynamical systems.

Keywords: hypercyclicity; transitivity; recurrence; the narrow topology; random dynamical system

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1. Introduction

Let X be a complex Banach space. In the following, by an operator, we mean a linear and continuous map acting on X.

A very central notion in topological dynamics that has a long story is that of recurrence, which goes back to Poincaré (see [12]), and it refers to the existence of points in the space for which parts of their orbits under a continuous map return to themselves, in other words, a vector $x \in X$ is called a *recurrent vector* for an operator T acting on X if there exists a strictly increasing sequence (n_k) of positive integers such that

$$T^{n_k} x \to x.$$

The purpose of this note is the study of the notion of recurrence, together with its variations, in the context of topological dynamics. Some examples and characteri-

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zations of recurrence for special classes of operators have appeared in [6], [7], [11] and a systematic study of this notion goes back to the works of Furstenberg [9], and Gottschalk and Hedlund [10].

Instead of the norm topology, by taking density in the weak topology, we can consider the notions of weak hypercyclicity, weak recurrence and weak orbits in general.

The study of weak orbits was began in 1996 by Van Neerven in [14]. Important contribution to weak hypercyclicity is due to Bès, Chan and Sanders (see [3], [4], [5], [13]). Moreover, the notion of weak recurrence was studied in [1] by Amouch et al.

In [15], [16], we studied the weak recurrence also known as narrow recurrence of a new class of operators and semi groups, which are the Markov kernels and transition functions, respectively, along with their characteristics. Our aim was to utilize this concept to study the distributional stability of Markov chains on general state spaces.

Now, throughout in the sequel, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, where $\mathcal{F}$ is assumed countably generated, and X is a Polish space equipped with a complete metric d. The σ-algebra of Borel sets of X is denoted by $\mathcal{B}$. The product space $X \times \Omega$ is understood to be a measurable space with the product σ-algebra $\mathcal{B} \otimes \mathcal{F}$, which is the smallest σ-algebra on $X \times \Omega$ with respect to which both the canonical projections $\pi_X \colon X \times \Omega \to X$ and $\pi_\Omega \colon X \times \Omega \to \Omega$ are measurable. Let $\mathbb{T}$ be either $\mathbb{N}$ or $\mathbb{R}^+$ understood with their respective natural topologies (and Borel σ-algebras). The set of all subsets of X is denoted by $\mathcal{P}(X)$, the set of all probability measures on X is denoted by $\mathrm{Pr}(X)$ and the set of all probability measures on $X \times \Omega$ with the marginal $\mathbb{P}$ is denoted by $\mathrm{Pr}_{\mathbb{P}}(X \times \Omega)$.

In this paper, following a similar approach as in the articles [15] and [16], we study some properties of recurrence within the context of topological dynamics for random dynamical systems. More precisely, the contribution of the paper is two-fold. First, we examine recurrence within the framework of topological dynamics for a novel class of operator families, namely the skew product flows induced by random dynamical systems, and study their characteristics. Second, we exploit the concept of recurrence to study the stability of random dynamical systems.

The paper is organized as follows. In Section 1, we give a summary of some notions and results concerning random dynamical systems that we will need in the next paragraphs. In Section 2, we introduce the notion of narrow recurrence for random dynamical systems and provide a simple example. In Section 3, we give some properties that characterize the narrow recurrence of random dynamical systems. In particular, we characterize the narrow recurrence of random dynamical systems in terms of the convergence of measures of random sets. We also present results that characterize the narrow recurrence of random dynamical systems, based on the existence of other topologies on $\mathrm{Pr}_\Omega(X)$ that are equivalent to the narrow topology.

2. The basic set-up

In this part, we give a summary of some notions and results that we will need in the next paragraph. For a comprehensive exposition on this subject see [2] and [8].

Definition 1. Let $v\colon \mathbb{T}\times\Omega\to\Omega$, $(t,\omega)\mapsto v_t\omega$ be a measurable map. We say that v or $(v_t)_{t\in\mathbb{T}}$ is a flow of measure preserving transformations on $(\Omega,\mathcal{F},\mathbb{P})$ if it satisfies the following:
(1) v_t preserves the probability measure $\mathbb{P}$ for any $t\in\mathbb{T}$,
(2) $v_{t+s}=v_t\circ v_s$ for all $s,t\in\mathbb{T}$, and $v_0=\mathrm{id}_\Omega$.

Definition 2 (Random dynamical system)**.** Let $(v_t)_{t\in\mathbb{T}}$ be a flow of measure preserving transformations on $(\Omega,\mathcal{F},\mathbb{P})$. A random dynamical system with time $\mathbb{T}$ on X over $(\Omega,\mathcal{F},\mathbb{P},(v_t)_{t\in\mathbb{T}})$ is a measurable map

$$\varphi\colon\ \mathbb{T}\times X\times\Omega\to X,\ (t,x,\omega)\mapsto\varphi(t,\omega)x$$

with $\varphi(0,\omega)=\mathrm{id}_X$, and for every $s\in\mathbb{T}$ there exists $N_s\in\mathcal{F}$, $\mathbb{P}(N_s)=0$ such that

$$\varphi(t+s,\omega)=\varphi(t,v_s\omega)\circ\varphi(s,\omega)$$

for all $t\in\mathbb{T}$ and for all $\omega\in N_s^C$.

Remark 1. A random dynamical system φ is said to be continuous if the map

$$\varphi(t,\omega)\colon\ X\to X,\ x\mapsto\varphi(t,\omega)x$$

is continuous for all $t\in\mathbb{T}$ and for all ω outside a $\mathbb{P}$-nullset not depending on t.

Remark 2.

(1) We can relate any random dynamical system φ with a measurable map

$$\Theta\colon\ \mathbb{T}\times X\times\Omega\to X\times\Omega,\ (t,x,\omega)\mapsto(\varphi(t,\omega)x,v_t\omega).$$

(2) The family $\{\Theta_t;\ t\in\mathbb{T}\}:=\{\Theta(t,\cdot,\cdot);\ t\in\mathbb{T}\}$ is called the skew product flow induced by φ, and satisfies $\Theta_{t+s}=\Theta_t\circ\Theta_s$ for all $t,s\in\mathbb{T}$ and $\Theta_0=\mathrm{id}$ on $X\times\Omega$.
(3) For every $t\in\mathbb{T}$, the skew product map Θ_t acts on functions on $X\times\Omega$ in the usual way,

$$\Theta f((x,\omega))=f\circ\Theta_t(x,\omega)=f(\varphi(t,\omega)x,v_t\omega)\quad\text{for any function } f \text{ on } X\times\Omega.$$

(4) For every $t\in\mathbb{T}$, the skew product map Θ_t acts also on measures on $X\times\Omega$ by the relationship

$$\Theta_t\mu(A)=\mu(\Theta_t^{-1}(A))\quad\forall\,A\in\mathcal{B}\otimes\mathcal{F},\ \forall\,t\in\mathbb{T}.$$

(5) The actions of Θ_t on measures and on functions are related by

$$\int_{X\times\Omega} \Theta_t h \,\mathrm{d}\mu = \int_{X\times\Omega} hd(\Theta\mu)$$

for all measures μ on $\mathcal{B}\otimes\mathcal{F}$ and for all measurable functions $h\colon\ X\times\Omega\to\mathbb{R}$ which are μ-integrable.

We recall the definition of a closed random set.

Definition 3 (Closed random set)**.** A random set

$$A\colon\ \Omega\to\mathcal{P}(X),\ \omega\mapsto A(\omega)$$

is said to be closed, if A takes values in the closed subsets of X, and for any $x\in X$ the map $\omega\mapsto d(x,A(\omega))$ is measurable.

Remark 3. A random set

$$U\colon\ \Omega\to\mathcal{P}(X),\ \omega\mapsto U(\omega)$$

is said to be an open random set if if its complement

$$U^{\mathrm{c}}\colon\ \Omega\to\mathcal{P}(X),\ \omega\mapsto U^{\mathrm{c}}(\omega)$$

is a closed random set.

We recall the definition of a random probability measure.

Definition 4 (Random measure)**.** A random probability measure or Markov kernel on X is a map

$$\mu\colon\ \mathcal{B}\times\Omega\to[0,1],\ (B,\omega)\mapsto\mu_\omega(B)$$

satisfying

(1) for every $B\in\mathcal{B}$, $\omega\mapsto\mu_\omega(B)$ is measurable,
(2) for $\mathbb{P}$-almost every $\omega\in\Omega$, $B\mapsto\mu_\omega(B)$ is a probability measure on X.

The measure μ is denoted by $\omega\mapsto\mu_\omega$.

Remark 4. We identify two random measures $\omega\mapsto\mu_\omega$ and $\omega\mapsto\nu_\omega$ if $\mu=\nu$ $\mathbb{P}$-a.s.; that is if $\mu_\omega=\nu_\omega$ for $\mathbb{P}$-almost all ω. We denote the set of all random measures on $\mathcal{B}\times X$ by

$$\mathrm{Pr}_\Omega(X)=\{\mu\colon\ \mathcal{B}\times X\to[0,1]\colon\ \omega\mapsto\mu_\omega \text{ random measure}\}$$

with two random measures identified if they coincide $\mathbb{P}$-a.s.

Proposition 1. *Let $\omega \mapsto \mu_\omega$ be a random measure on X. Then:*

▷ *For all measurable $h\colon X \times \Omega \to X$ with h bounded or nonnegative the map*

$$\omega \mapsto \int_X h(x,\omega)\,\mathrm{d}\mu_\omega$$

is measurable.

▷ *The map*

$$\mu\colon A \mapsto \int_\Omega \int_X 1_A(x,\omega)\,\mathrm{d}\mu_\omega\,\mathrm{d}\mathbb{P}(\omega) \quad \text{with } A \in \mathcal{B}\otimes\mathcal{F}$$

defines a probability measure on $X \times \Omega$, which is the marginal of μ on Ω is $\mathbb{P}$; that is $\pi_\Omega\mu = \mathbb{P}$.

▷ *For every probability measure μ on $X \times \Omega$ with $\pi_\Omega\mu = \mathbb{P}$, there exists a unique $\mathbb{P}$-a.s. random measure $\omega \mapsto \mu_\omega$ such that*

$$\int_\Omega \int_X h(x,\omega)\,\mathrm{d}\mu(x,\omega) = \int_\Omega \int_X h(x,\omega)\,\mathrm{d}\mu_\omega\,\mathrm{d}\mathbb{P}(\omega)$$

for every bounded measurable $h\colon X \times \Omega \to \mathbb{R}$.

Remark 5. Let $\mathrm{Pr}(X \times \Omega)$ the set of all probability measures on $X \times \Omega$, and $\mathrm{Pr}_{\mathbb{P}}(X \times \Omega)$ the set of all probability measures on $X \times \Omega$ with the marginal $\mathbb{P}$; that is,

$$\mathrm{Pr}_{\mathbb{P}}(X \times \Omega) = \{\mu \in \mathrm{Pr}(X \times \Omega)\colon \pi_\Omega\mu = \mathbb{P}\}.$$

The map

$$A \mapsto \int_\Omega \int_X 1_A(x,\omega)\,\mathrm{d}\mu_\omega\,\mathrm{d}\mathbb{P}(\omega) \quad \forall\, A \in \mathcal{B}\otimes\mathcal{F}$$

defines an isomorphism between $\mathrm{Pr}_\Omega(X)$ and $\mathrm{Pr}_{\mathbb{P}}(X \times \Omega)$. Henceforth we only speak of the random measures.

We recall the definition of a random continuous function.

Definition 5 (Random continuous function)**.** A random continuous function is a function $f\colon X \times \Omega \to \mathbb{R}$ such that:

(1) for all $x \in X$, the x-section $\omega \mapsto f(x,\omega)$ is measurable,

(2) for all $\omega \in \Omega$, the ω-section $x \mapsto f(x,\omega)$ is continuous and bounded,

(3) $\omega \mapsto \sup\{|f(x,\omega)|\colon x \in X\}$ is integrable with respect to $\mathbb{P}$.

Remark 6 (See [2]).

(1) Two random continuous functions are identified if they coincide $\mathbb{P}$-a.s.

(2) Every random continuous function is jointly measurable.

(4) The set of all random contiuous functions is a linear space denoted by $C_\Omega(X)$.

(4) The map $|\cdot|_\infty\colon f \mapsto \int_\Omega \sup\limits_{x\in X} |f(x,\omega)|\,\mathrm{d}\mathbb{P}(\omega)$ defines a norm on $C_\Omega(X)$.

In the following, for every $\mu \in \mathrm{Pr}_\Omega(X)$ and for every $h \in C_\Omega$, we use the notation:

$$\mu(h) = \int_\Omega \int_X h(x,\omega)\,\mathrm{d}\mu_\omega\,\mathrm{d}\mathbb{P}(\omega) = \int_\Omega \int_X h(x,\omega)\,\mathrm{d}\mu(x,\omega).$$

Proposition 2 ([2]). *For any random measure μ and $f, g \in C_\Omega(X)$, we have:*

$$|\mu(f) - \mu(g)| \leqslant |f - g|_\infty.$$

We finish basic results by the key results in this work.

Definition 6. The narrow topology is the coarsest topology on $\mathrm{Pr}_\Omega(X)$ such that

$$\mu \mapsto \mu(f)$$

is continuous for all $f \in C_\Omega(X)$.

Theorem 1. *Let φ be a continuous random dynamical system on X and $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ. For any $t \in \mathbb{T}$, Φ_t maps $\mathrm{Pr}_\Omega(X)$ to itself, and Φ_t is continuous on $\mathrm{Pr}_\Omega(X)$ with respect to the narrow topology.*

3. The narrow recurrence of random dynamical systems

In this section, we introduce the notion of narrow recurrence for random dynamical systems.

Definition 7. Let φ be a random dynamical system on X and $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ. The orbit of a random probability measure $\mu \in \mathrm{Pr}_\Omega(X)$ under φ is defined as

$$\mathrm{Orb}(\varphi, \mu) := \{\Phi_t \mu \colon\ t \in \mathbb{T}\}.$$

Definition 8. Let φ be a random dynamical system on X and $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ. A random probability measure μ is said to be a *narrow recurrent probability measure* of φ if there exists a strictly increasing sequence (n_k) of positive integers and a sequence of time points (t_{n_k}) from $\mathbb{T}$ such that

$$\Phi_{t_{n_k}} \mu \xrightarrow[k \to \infty]{N} \mu.$$

We denote by N-Rec(φ) the set of all narrow recurrent random probability measures for φ.

R e m a r k 7. Let φ be a random dynamical system on X and $\{\Phi_t, t \in \mathbb{T}\}$ its skew product on $\Omega \times X$. Suppose that φ admits an invariant probability measure $\mu \in \Pr_\Omega(X)$, that is $\Phi_t\mu = \mu$ for all $t \in \mathbb{T}$. Then μ is a narrow recurrent measure of φ, which means that

$$\mathrm{Inv}(\varphi) \subset \text{N-Rec}(\varphi),$$

where $\mathrm{Inv}(\varphi)$ is the set of all invariant probability measures under φ.

4. Characterization of the narrow recurrence of random dynamical systems

In this section, we provide some properties that characterize the narrow recurrence of random dynamical systems, which can be considered as extensions of the narrow recurrence characterizations for Markov chains with discrete and continuous time proven in [15] and [16], respectively, to random dynamical systems.

The following result is based on the Portmanteau theorem for random dynamical systems, see [8], and provides useful conditions equivalent to the narrow recurrence of random dynamical systems. In particular, it characterizes the narrow recurrence of random dynamical systems in terms of the convergence of measures of random sets.

Theorem 2. *Let φ be a random dynamical system on X, $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ and $\mu \in \Pr_\Omega(X)$. The following statements are equivalent:*

(1) $\mu \in \text{N-Rec}(\varphi)$.

(2) *There exists a strictly increasing sequence (n_k) of positive integers such that*

$$\overline{\lim_k}\, \Phi_{t_{n_k}}\mu(F) \leqslant \mu(F) \quad \text{for every closed random set } F.$$

(3) *There exists a strictly increasing sequence (n_k) of positive integers such that*

$$\underline{\lim_k}\, \Phi_{t_{n_k}}\mu(U) \geqslant \mu(U) \quad \text{for every open random set } U.$$

P r o o f. (1) $\Rightarrow$ (2): Suppose that $\mu \in \text{W-Rec}(\varphi)$, then there exists a strictly increasing sequence (n_k) of positive integers such that

$$\Phi_{t_{n_k}}\mu \xrightarrow[k\to\infty]{N} \mu.$$

Let C be a closed random set. For $p \in \mathbb{N}$, we put

$$f_p(x,\omega) = 1 - \min\{pd(x, C(\omega)), 1\}.$$

Then $f_p \in C_\Omega(X)$, $f_p \geqslant 1_C$, and f_p decreases monotonically to 1_C with $k \to \infty$. For every fixed p,

$$\overline{\lim_k}\, \Phi_{t_{n_k}} \mu(C) \leqslant \lim \Phi_{t_{n_k}} \mu(f_p) = \mu(f_p),$$

hence

$$\overline{\lim_k}\, \Phi_{t_{n_k}} \mu(C) \leqslant \inf_p \mu(f_p) = \mu(C).$$

(2) $\Rightarrow$ (3): A simple complementation argument proves this equivalence.

(3) $\Rightarrow$ (1): Suppose that there exists a strictly increasing sequence (n_k) of positive integers such that

$$\underline{\lim_k}\, \Phi_{t_{n_k}} \mu(U) \geqslant \mu(U) \quad \text{for every open random set } U.$$

It suffices to prove that $\lim\limits_k \Phi_{t_{n_k}} \mu(f) = \mu(f)$ for every random continuous function f with $0 \leqslant f \leqslant 1$. We first establish

$$\overline{\lim_k}\, \Phi_{t_{n_k}} \mu(f) \leqslant \mu(f).$$

Fix $m \in \mathbb{N}$ and define $C_p \subset X \times \Omega$, $0 \leqslant p \leqslant m$ by

$$C_p(\omega) = \Big\{ x \in X\colon\ f(x,\omega) \geqslant \frac{p}{m} \Big\},$$

then C_p is a closed random set. We get from (3)

$$\mu(f) \geqslant \frac{1}{n} \sum_{p=1}^{m} \overline{\lim_k}\, \Phi_{t_{n_k}} \mu(C_p) \geqslant \overline{\lim_k}\, \frac{1}{m} \sum_{p=1}^{m} \Phi_{t_{n_k}} \mu(C_p) \geqslant \overline{\lim_k}\, \Phi_{t_{n_k}} \mu(f) - \frac{1}{m}.$$

Since m is arbitrary we obtain

$$\lim_k \Phi_{t_{n_k}} \mu(f) = \mu(f)$$

for every random continuous function f with $0 \leqslant f \leqslant 1$. $\square$

The following results are based on the existence of topologies on $\mathrm{Pr}_\Omega(X)$ that are equal to the narrow topology. Let f be a bounded function on X. The function f is Lipschitz if

$$\|f\|_L = \sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty.$$

We put $\|f\|_{BL} = \|f\|_L + \sup\limits_{x \in X} |f(x)|$, and we define the Banach space $BL_\Omega(X)$ of random Lipschitz functions on X,

$$BL_\Omega(X) := \{ f \in C_\Omega(X)\colon\ \omega \mapsto \|f(\cdot,\omega)\|_{BL} < C \text{ for some } C\ \mathbb{P}\text{-a.s.} \}.$$

Theorem 3. *Let φ be a random dynamical system on X, $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ and $\mu \in \mathrm{Pr}_\Omega(X)$. Then $\mu \in \text{N-Rec}(\varphi)$ if and only if there exists a strictly increasing sequence (n_k) of positive integers such that*

$$\Phi_{t_{n_k}}\mu(g) \underset{k\to\infty}{\longrightarrow} \mu(g)$$

for every $g \in BL_\Omega(X)$ with $0 \leqslant g \leqslant 1$ and $\|g(\cdot,\omega)\|_{BL} \leqslant 1$ $\mathbb{P}$-a.s.

Proof. Let $\mathcal{T}$ be the coarsest topology on $\mathrm{Pr}_\Omega(X)$ such that $\mu \mapsto \mu(g)$, is continuous for all $g \in BL_\Omega(X)$ with $0 \leqslant g \leqslant 1$ and $\|g(\cdot,\omega)\|_{BL} \leqslant 1$ $\mathbb{P}$-a.s. The topology $\mathcal{T}$ and the narrow topology on $\mathrm{Pr}_\Omega(X)$ are equal $\mathbb{P}$-a.s., see [8], Proposition 4.9. Then $\mu \in \text{N-Rec}(\varphi)$ if and only if there exists a strictly increasing sequence (n_k) of positive integers such that

$$\Phi_{t_{n_k}}\mu(g) \underset{k\to\infty}{\longrightarrow} \mu(g)$$

for every $g \in BL_\Omega(X)$ with $0 \leqslant g \leqslant 1$ and $\|g(\cdot,\omega)\|_{BL} \leqslant 1$ $\mathbb{P}$-a.s. □

Corollary 1. *Let φ be a random dynamical system on X, $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ and $\mu \in \mathrm{Pr}_\Omega(X)$. Then $\mu \in \text{N-Rec}(\varphi)$ if and only if there exists a strictly increasing sequence (n_k) of positive integers such that*

$$\Phi_{t_{n_k}}\mu(g) \underset{k\to\infty}{\longrightarrow} \mu(g)$$

for every $g \in BL_\Omega(X)$.

Proof. Since

$$\{g \in BL_\Omega(X)\colon\ g \geqslant 0 \text{ and } \|g(\cdot,\omega)\|_{BL} \leqslant 1 \ \mathbb{P}\text{-a.s.}\} \subset BL_\Omega(X) \subset C_\Omega(X),$$

it follows that that the narrow topology on $\mathrm{Pr}_\Omega(X)$ is generated by $BL_\Omega(X)$, hence $\mu \in \text{N-Rec}(\varphi)$ if and only if there exists a strictly increasing sequence (n_k) of positive integers such that

$$\Phi_{t_{n_k}}\mu(g) \underset{k\to\infty}{\longrightarrow} \mu(g)$$

for every $g \in BL_\Omega(X)$. □

Since the σ-algebra $\mathcal{F}$ is assumed countably generated, then there exists a countable algebra $\mathcal{A} = \{A_n\colon n \in \mathbb{N}\}$ generating $\mathcal{F}$. Hence we can define a metric ζ on $\mathrm{Pr}_\Omega(X)$ as

$$\zeta(\mu,\nu) = \sum_{n\in\mathbb{N}} \frac{1}{2^n} \sup\biggl\{\int_{A_n} (\mu_\omega(g) - \nu_\omega(g))\,\mathrm{d}\mathbb{P}(\omega)\colon\ g \in BL(X),\ 0 \leqslant g \leqslant 1,\ \|g\|_L \leqslant 1\biggr\}$$

for any $\mu,\nu \in \mathrm{Pr}_\Omega(X)$.

Theorem 4. *Let φ be a random dynamical system on X, and $\{\Phi_t, t \in \mathbb{T}\}$ be the skew product flow induced by φ and $\mu \in \mathrm{Pr}_\Omega(X)$. Then $\mu \in$ N-Rec(φ) if and only if there exists a strictly increasing sequence (n_k) of positive integers such that*

$$\lim_k \zeta(\Phi_{t_{n_k}}\mu, \mu) = 0.$$

P r o o f. The topology on $\mathrm{Pr}_\Omega(X)$ induced by ζ metrises the narrow topology, [8], Theorem 4.16. Then $\mu \in$ N-Rec(φ) if and only if there exists a strictly increasing sequence (n_k) of positive integers such that:

$$\lim_k \zeta(\Phi_{t_{n_k}}\mu, \mu) = 0.$$

$\square$

Theorem 5. *Let φ be a random dynamical system on X, and $\{\Phi_t, t \in \mathbb{T}\}$ the skew product flow induced by φ and $\mu \in \mathrm{Pr}_\Omega(X)$. Assume that there exists a strictly increasing sequence (n_k) of positive integers such that for every $f \in C_\Omega(X)$ with $\mathbb{P}\{\omega \in \Omega\colon\ 0 \leqslant f(\cdot,\omega) \leqslant 1\} = 1$, we have $\Phi_{t_{n_k}}\mu(f) \underset{k\to\infty}{\longrightarrow} \mu(f)$, then*

$$\mu \in \text{N-Rec}(\varphi).$$

P r o o f. Let $\mathcal{T}$ be the coarsest topology on $\mathrm{Pr}_\Omega(X)$ such that $\mu \mapsto \mu(f)$ is continuous for all $f \in C_\Omega(X)$ with $\mathbb{P}\{\omega \in \Omega\colon\ 0 \leqslant f(\cdot,\omega) \leqslant 1\} = 1$. We show that $\mathcal{T}$ and the narrow topology on $\mathrm{Pr}_\Omega(X)$ are equal $\mathbb{P}$-a.s.

Pick $f \in C_\Omega(X)$ with $f \geqslant 0$ $\mathbb{P}$-a.s. and let $\varepsilon > 0$. Since $\omega \mapsto \sup\limits_{x\in X} f(x,\omega)$ is integrable, there exists $N \in \mathbb{N}$ such that

$$\int_{F_N} \sup_{x\in X} f(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) < \frac{\varepsilon}{3},$$

where $F_N = \Big\{\omega \in \Omega\colon\ \sup\limits_{x\in X} f(x,\omega) \geqslant N\Big\}$. Thus

$$|f - f\wedge N|_\infty \leqslant \int_{F_N} \sup_{x\in X} f(x,\omega)\,\mathrm{d}\mathbb{P}(\omega) < \frac{\varepsilon}{3},$$

where $f \wedge N = \min\{f, N\}$. Fix $\nu \in \mathrm{Pr}_\Omega(X)$, and put

$$\begin{aligned} U &= \Big\{\sigma \in \mathrm{Pr}_\Omega(X)\colon\ |\nu(f\wedge N) - \sigma(f\wedge N)| < \frac{\varepsilon}{3}\Big\} \\ &= \Big\{\sigma \in \mathrm{Pr}_\Omega(X)\colon\ \Big|\frac{\nu(f\wedge N)}{N} - \frac{\sigma(f\wedge N)}{N}\Big| < \frac{\varepsilon}{3N}\Big\}. \end{aligned}$$

Since

$$0 \leqslant \frac{(f\wedge N)}{N} \leqslant 1,$$

U is an open neighborhood of ν in the topology under consideration. We get for any $\sigma \in U$,

$$|\nu(f) - \sigma(f)| \leqslant |\nu(f) - \nu(f \wedge N)| + |\nu(f \wedge N) - \sigma(f \wedge N)| + |\sigma(f) - \sigma(f \wedge N)| < \varepsilon.$$

This holds for $\nu \in \mathrm{Pr}_\Omega(X)$ arbitrary. Consequently $\nu \mapsto \nu(f)$ is continuous for every $f \in C_\Omega(X)$ with $f \geqslant 0$. Finally, $\nu \mapsto \nu(f) = \nu(f^+) - \nu(f^-)$ for every $f \in C_\Omega(X)$. Hence $\mathcal{T}$ coincides with the narrow topology $\mathbb{P}$-a.s. $\square$

In the following, we consider any random measure $\mu \in \mathrm{Pr}_\Omega(X)$ as a random variable $\mu\colon \Omega \to \mathrm{Pr}(X)$ with values in the Polish space $\mathrm{Pr}(X)$, equipped with the Borel σ-algebra of the narrow topology, see [8]. We choose a metric d on $\mathrm{Pr}(X)$ and we define on $\mathrm{Pr}_\Omega(X)$ the metric α:

$$\alpha(\mu, \nu) = \inf\{\varepsilon > 0\colon\ \mathbb{P}(\{\omega \in \Omega\colon\ d(\mu_\omega, \nu_\omega) > \varepsilon\}) \leqslant \varepsilon\} \quad \forall\, \mu, \nu \in \mathrm{Pr}_\Omega(X).$$

Theorem 6. *Let φ be a random dynamical system on X, $\{\Phi_t, t \in \mathbb{T}\}$ be the skew product flow induced by φ and $\mu \in \mathrm{Pr}_\Omega(X)$. Assume that there exists a strictly increasing sequence (n_k) of positive integers such that $\lim\limits_k \alpha(\Phi_{t_{n_k}}\mu, \mu) = 0$, then*

$$\mu \in \text{N-Rec}(\varphi).$$

P r o o f. The topology induced by the metric α is stronger than the narrow topology on $\mathrm{Pr}_\Omega(X)$, see [8], Propostion 5.4. Then, if there exists a strictly increasing sequence (n_k) of positive integers such that

$$\lim_k \alpha(\Phi_{t_{n_k}}\mu, \mu) = 0, \quad \text{then} \quad \Phi_{t_{n_k}}\mu \xrightarrow[k\to\infty]{N} \mu,$$

hence $\mu \in \text{N-Rec}(\varphi)$. $\square$

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