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ON BOUNDEDNESS, SQUARE INTEGRABILITY AND UNIFORM STABILITY FOR NEUTRAL NON AUTONOMOUS THIRD ORDER DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. In the study of the solutions for a given class of neutral third order differential equations with delay, a suitable conditions are given based on the Lyapunov second method by considering convenable Lyapunov functional to guarantee the uniform asymptotic stability, the boundedness and the square integrability.

Keywords: asymptotic stability; square integrability; Lyapunov functional; non autonomous differential equation of third order with delay

MSC 2020: 34C11, 34C25, 34D20, 34D23

1. Introduction

As it is well known, the third-order differential equations are derived from many different areas of applied mathematics and physics, for instance, the deflection of buckling beam with a fixed or variable cross-section, the three-layer beam, the electromagnetic waves, the gravity-driven flows and many other applications, see [1], [5], [13] for more details. The investigation of the qualitative behavior of solutions such as the stability, the boundedness, the asymptotic behavior and the square integrability are very important aspects in the theory and applications of differential equations.

In the present paper, the boundedness, the square integrability and the uniform asymptotic stability are studied for the non autonomous neutral differential equation of the third order with delay of the form

(1.1)
$$[x'(t)+\varrho_1 x'(t-r)+h(x)+\varrho_1 h(x(t-r))]''+P(t)x''(t)+Q(t)x'(t) \\ +R(t)[g(x(t))+\varrho_2 g(x(t-\sigma))]=\psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t))$$

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for all $t \geqslant t_1 = t_0 + \max\{r, \sigma\}$, where $\max\{\varrho_1, \varrho_2\} = \varrho < 1$, r, σ, ϱ_1 and ϱ_2 are positive constants, and $P, Q, R \in \mathcal{C}^1(\mathbb{R}^+, (0, \infty))$, $\mathbb{R}^+ = [0, \infty)$, $g \in \mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $h \in \mathcal{C}^2(\mathbb{R}, \mathbb{R})$, $\psi \in \mathcal{C}(\mathbb{R}^+ \times \mathbb{R}^5, \mathbb{R})$ and $g(0) = 0$.

Many authors have investigated the stability and the boundedness of solutions for certain differential equations of the third order with delay [2]–[4], [7]–[9], [11], [12]. The achievement of the results in this paper for equation (1.1) is motivated by the results concerning the main equations investigated in the articles [2], [6], which can be considered as a special case of equation (1.1).

In particular, for the case of $h \equiv 0$, $\psi \equiv 0$ and $\varrho_1 = \varrho_2 = \varrho$, equation (1.1) is given by

$$[x'(t) + \varrho x'(t-r)]'' + P(t)x''(t) + Q(t)x'(t) + R(t)[g(x(t)) + \varrho g(x(t-r))] = 0,$$

which is equivalent to the equation considered in [6]

$$[x(t) + \beta x(t-r)]''' + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0,$$

and for the case of $\varrho_1 = \varrho_2 = 0$ and $\psi \equiv 0$, equation (1.1) is given by

$$[x' + h(x)]'' + P(t)x''(t) + Q(t)x'(t) + R(t)g(x(t)) = 0,$$

which is exactly the equation investigated in [2].

Before that, in the article [10], the author has studied the nonlinear differential equation of third order

$$x'''(t) + a(t)x''(t) + b(t)x'(t) + c(t)f(x(t-r)) = 0,$$

which can be obtained from equation (1.1) in the case of $\varrho_1 = \varrho_2 = 0$, $h \equiv 0$ and $\psi \equiv 0$.

In 1892, a fundamental method has been proposed by Lyapunov for studying the problem of stability of motion for a differential equation; this method called the Lyapunov second method works by constructing scalar functions known as Lyapunov functions. The second method of Lyapunov is widely used in literature in investigation of qualitative behaviors in large classes of several categories of differential equations, which makes it very useful in theory and application.

By solution of (1.1) we mean a continuous function $x\colon [t_x, \infty) \to \mathbb{R}$ for $t_x \geqslant t_1$ which satisfies equation (1.1) in $[t_x, \infty)$ and such that

$$x(t) + \varrho_1 x(t-r) \in \mathcal{C}^3([t_x, \infty), \mathbb{R}).$$

2. Assumptions and main results

The following hypotheses on the functions appearing in equation (1.1) will be useful in next subsequent sections. Assume that there are positive constants p_0, p_1, R_0, R_1, q_0, q_1, ξ_1, ξ_2, ξ_3 and γ such that the following conditions are satisfied:

(i) $0 < p_0 \leqslant P(t) \leqslant p_1$, $0 < q_0 \leqslant Q(t) \leqslant q_1$, $0 < R_0 \leqslant R(t) \leqslant R_1$,

$$Q'(t) \leqslant 0,\ R'(t) \leqslant 0 \quad \forall\, t \geqslant t_1;$$

(ii) $g(0) = 0$, $\xi_1 \leqslant g(x)/x \leqslant \xi_2$ $(x \neq 0)$ and $|g'(x)| \leqslant \xi_3$ for all $x \in \mathbb{R}$;

(iii) $\int_{t_1}^{t} (|P'(s)| - R'(s))\,\mathrm{d}s \leqslant \gamma$.

Equation (1.1) is equivalent to the system

$$\begin{cases} x'(t) = y(t) - h(x(t)), \\ y'(t) = z(t), \\ Z'(t) = -P(t)z(t) + P(t)h'(x)\vartheta(t) - Q(t)\vartheta(t) - (1+\varrho_2)R(t)g(x) \\ \qquad + \varrho_2 R(t)\displaystyle\int_{t-\sigma}^{t} (y(s) - h(x(s)))g'(x(s)\,\mathrm{d}s \\ \qquad + \psi(t, x(t), x(t-\sigma), x'(t), x'(t-\sigma), x''(t)), \end{cases} \tag{2.1}$$

where

$$y(t) - h(x(t)) = \vartheta(t), \quad y(t) = x'(t) + h(x(t)),$$
$$Z(t) = y'(t) + \varrho_1 y'(t-r) = z(t) + \varrho_1 z(t-r).$$

The first main result of this work is the following theorem, where $h \neq 0$ and $\psi \neq 0$.

Theorem 2.1. *In addition to assumptions* (i)–(iii), *assume that there are positive constants* δ, φ_1 *and* D_1 *such that the following conditions are satisfied:*

(H1) $|h'(u)| \leqslant \delta$ *for all* $u \in \mathbb{R}$ *and* $\delta < \min\{2kR_0\xi_1/(R_1\xi_2^2), (2p_0 - 3k)/(p_1 - k)\}$,

(H2) $|\psi(t, x(t), x(t-r), x'(t), x'(t-r), x''(t))| \leqslant \varphi(t) \leqslant \varphi_1$ *and* $\int_{t_1}^{t} \varphi(s)\,\mathrm{d}s \leqslant D_1$.

Then there exists a finite positive constant η *such that all the solutions* $x(\cdot)$ *of* (1.1) *and their derivatives* $x'(\cdot)$ *and* $x''(\cdot)$ *fulfill:*

(I) $|x(t)| \leqslant \eta$, $|x'(t)| \leqslant \eta$ *and* $|x''(t) + \varrho_1 x''(t-r)| \leqslant \eta$ *for all* $t \geqslant t_1$,

(II) $\int_{t_1}^{\infty} (x^2(s) + x'^2(s) + (x''(s) + \varrho_1 x''(s-r))^2)\,\mathrm{d}s < \infty$ *provided that*

$$\varrho < \min\Big\{\frac{2(k+\delta)q_0 - R_1(2\xi_3 + \delta) - k(1 + 2p_1) - \delta(p_1 - k)}{R_1\xi_3[(2+3k)\sigma + 2] + q_1 + k + R_1\delta}, \frac{2kR_0\xi_1 + \delta R_1\xi_2^2}{R_1(\xi_3\sigma k + \xi_2^2)}, \frac{2p_0 - 3k - \delta(p_1 - k)}{2R_1(\xi_3\sigma + 1) + 3k + 8p_1 + q_1}\Big\}, \tag{2.2}$$

where

(2.3)

$$k < \min\Big\{\frac{p_0}{2}, \frac{q_0^2}{4p_1^2}, \frac{q_0}{4}\Big\}, \quad q_0 > \max\Big\{\frac{R_1(2\xi_3+\delta)+k(1+2p_1)+\delta(p_1-k)}{2(k+\delta)}, \frac{4R_1\xi_3}{k}\Big\}.$$

The second main result is the following theorem, where $h \equiv 0$ and $\psi \equiv 0$.

Theorem 2.2. *Suppose that assumptions* (i)–(iii) *hold. Then every solution of* (1.1) *is uniformly asymptotically stable provided that*

$$\varrho < \min\Big\{\frac{2kq_0 - 2R_1\xi_3 - k(1+2p_1)}{R_1\xi_3[(2+3k)\sigma+2]+q_1+k}, \frac{2kR_0\xi_1}{R_1(\xi_3\sigma k+\xi_2^2)}, \frac{2p_0-3k}{2R_1(\xi_3\sigma+1)+3k+8p_1+q_1}\Big\}, \tag{2.4}$$

where

$$k < \min\Big\{\frac{p_0}{2}, \frac{q_0^2}{4p_1^2}, \frac{q_0}{4}\Big\}, \quad q_0 > \max\Big\{\frac{2R_1\xi_3+k(1+2p_1)}{2k}, \frac{4R_1\xi_3}{k}\Big\}. \tag{2.5}$$

P r o o f of Theorem 2.1. *Boundedness:* Our main tool is the continuously differentiable function $V = V(t;x;y;z)$ defined by

$$V = W\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big), \tag{2.6}$$

in which $\Delta(t) = |P'(t)| - R'(t)$, the function $W = W(t;x;y;z)$ is given by

$$W = \frac{1}{2}Z^2 + k\vartheta Z + \frac{k}{2}P(t)\vartheta^2 + (1+\varrho_2)kR(t)G(x) + (1+\varrho_2)R(t)\vartheta g(x) + \frac{Q(t)}{2}\vartheta^2 + kxZ + \frac{1}{2}kQ(t)x^2 + kP(t)x\vartheta + \mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t}\vartheta^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

with Γ, μ and λ being positive constants to be determined later in the proof and $G(x) = \int_0^x g(u)\,\mathrm{d}u$. Rewrite W as

$$W = W_1 + W_2 + W_3 + W_4 + \mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t}\vartheta^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

where

$$\begin{aligned} W_1 &= \frac{1}{4}Z^2 + k\vartheta Z + \frac{1}{2}kP(t)\vartheta^2,\\ W_2 &= (1+\varrho_2)kR(t)G(x) + (1+\varrho_2)R(t)\vartheta g(x) + \frac{1}{4}Q(t)\vartheta^2,\\ W_3 &= \frac{1}{4}kQ(t)x^2 + kP(t)x\vartheta + \frac{1}{4}Q(t)\vartheta^2,\\ W_4 &= \frac{1}{4}Z^2 + kxZ + \frac{1}{4}kQ(t)x^2. \end{aligned}$$

In view of conditions (i), (ii) and (2.3) we have

$$\begin{aligned}W_1 &= \frac{1}{4}Z^2 + k\vartheta Z + \frac{1}{2}kP(t)\vartheta^2 \\ &= \frac{1}{4}[(Z+2k\vartheta)^2 + 2k(P(t)-2k)\vartheta^2] \\ &\geqslant \frac{1}{2}k(p_0-2k)\vartheta^2 \geqslant k_1\vartheta^2,\end{aligned}$$

where

$$k_1 = \frac{1}{2}k(p_0-2k).$$

Rearranging W_2 we obtain the estimate

$$\begin{aligned}W_2 &= (1+\varrho_2)kR(t)G(x) + (1+\varrho_2)R(t)\vartheta g(x) + \frac{1}{4}Q(t)\vartheta^2 \\ &= (1+\varrho_2)kR(t)G(x) + \frac{Q(t)}{4}\Big[\vartheta^2 + \frac{4(1+\varrho_2)R(t)g(x)\vartheta}{Q(t)}\Big] \\ &= (1+\varrho_2)kR(t)\int_0^x g(u)\,\mathrm{d}u \\ &\quad + \frac{Q(t)}{4}\Big[\Big(\vartheta + \frac{2(1+\varrho_2)R(t)}{Q(t)}g(x)\Big)^2 - 4(1+\varrho_2)^2\frac{R^2(t)}{Q^2(t)}g^2(x)\Big] \\ &\geqslant (1+\varrho_2)kR(t)\int_0^x g(u)\,\mathrm{d}u - (1+\varrho_2)^2\frac{R^2(t)}{Q(t)}g^2(x) \\ &\geqslant (1+\varrho_2)kR(t)\Big[\int_0^x g(u)\,\mathrm{d}u - 2(1+\varrho_2)\frac{R(t)}{kQ(t)}\int_0^x g(u)g'(u)\,\mathrm{d}u\Big] \\ &\geqslant (1+\varrho_2)kR_0\int_0^x\Big(1-\frac{4R_1\xi_3}{kq_0}\Big)g(u)\,\mathrm{d}u = (1+\varrho_2)kR_0\Big(1-\frac{4R_1\xi_3}{kq_0}\Big)G(x).\end{aligned}$$

Note that by (ii) we have

$$\xi_1^2 \leqslant \frac{g^2(x)}{x^2},$$

which implies

$$\frac{\xi_1^2}{2\xi_3}x^2 \leqslant \frac{1}{2\xi_3}g^2(x) = \frac{1}{\xi_3}\int_0^x g(u)g'(u)\,\mathrm{d}u \leqslant G(x),$$

so

$$W_2 \geqslant k_2x^2,$$

where

$$k_2 = \frac{\xi_1^2}{2\xi_3}(1+\varrho_2)kR_0\Big(1-\frac{4R_1\xi_3}{kq_0}\Big).$$

We have also

$$
\begin{aligned}
W_3 &= \frac{1}{4}Q(t)\vartheta^2 + kP(t)x\vartheta + \frac{1}{4}kQ(t)x^2 \\
&= \frac{1}{4}Q(t)\Big[\Big(\vartheta + \frac{2kP(t)}{Q(t)}x\Big)^2 - \frac{4k^2P^2(t)}{Q^2(t)}x^2 + kx^2\Big] \\
&= \frac{1}{4}Q(t)\Big[\Big(\vartheta + \frac{2kP(t)}{Q(t)}x\Big)^2 + k[1 - \frac{4kP^2(t)}{Q^2(t)}]x^2\Big] \\
&\geqslant \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big]x^2 \geqslant k_3x^2,
\end{aligned}
$$

where

$$k_3 = \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big].$$

Rearranging W_4 we get

$$
\begin{aligned}
W_4 &= \frac{1}{4}kQ(t)x^2 + kxZ + \frac{1}{4}Z^2 = \frac{1}{4}kQ(t)\Big[x^2 + \frac{4}{Q(t)}xZ + \frac{1}{kQ(t)}Z^2\Big] \\
&= \frac{1}{4}kQ(t)\Big[\Big(x + \frac{2}{Q(t)}Z\Big)^2 + \frac{1}{Q(t)}\Big(\frac{1}{k} - \frac{4}{Q(t)}\Big)Z^2\Big] \\
&\geqslant \frac{1}{4}k\Big[\frac{1}{k} - \frac{4}{q_0}\Big]Z^2 \geqslant k_4Z^2,
\end{aligned}
$$

where

$$k_4 = k\Big[\frac{1}{k} - \frac{4}{q_0}\Big].$$

Since

$$\mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t}\vartheta^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s > 0,$$

it follows that

$$W \geqslant k_5(x^2 + \vartheta^2 + Z^2), \tag{2.7}$$

where

$$k_5 = \min\{k_1, k_2, k_3, k_4\}.$$

By (iii) we conclude that

$$1 \geqslant \exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big) \geqslant \exp\Big(\frac{-\gamma}{\Gamma}\Big). \tag{2.8}$$

Using (2.7) and (2.8) we obtain

$$V \geqslant k_6(x^2 + \vartheta^2 + Z^2),$$

where

$$k_6 = \exp\Big(\frac{-\gamma}{\Gamma}\Big)k_5.$$

For the time derivative of the function W along the trajectories of system (2.1), a straightforward calculation yields

$$\dot{W}_{(2.1)} = W_5 + W_6 + W_7 \tag{2.9}$$

such that

$$\begin{aligned} W_5 &= (k - P(t))z^2 + h'(x)\vartheta[(P(t) - k)Z - (1 + \varrho_2)R(t)g(x) - Q(t)\vartheta] \\ &\quad + \varrho_1 kzz(t - r) - \varrho_1 Q(t)\vartheta z(t - r) - \varrho_1 P(t)zz(t - r) \\ &\quad - \varrho_1(1 + \varrho_2)R(t)g(x)z(t - r) - kQ(t)\vartheta^2 + (1 + \varrho_2)R(t)g'(x)\vartheta^2 + k\vartheta Z \\ &\quad - k(1 + \varrho_2)R(t)g(x)x + kP(t)\vartheta^2 + \mu z^2 - \mu z^2(t - r) - \lambda \int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s \\ &\quad + \lambda\sigma\vartheta^2 + [k(x + \vartheta) + Z]\psi(t, x(t), x(t - \sigma), x'(t), x'(t - \sigma), x''(t)), \\ W_6 &= \varrho_2 R(t)[z + \varrho_1 z(t - r) + k\vartheta + kx] \int_{t-\sigma}^{t} g'(x(s))\vartheta(s)\,\mathrm{d}s \quad \text{and} \quad W_7 = \frac{\partial W}{\partial t}. \end{aligned}$$

By conditions (i), (ii) and by applying the estimate $2st \leqslant s^2 + t^2$,

$$\begin{aligned} W_6 &\leqslant \varrho_2 R_1 \Big[\frac{\xi_3\sigma}{2}z^2 + \frac{\xi_3}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s + \varrho_1\frac{\xi_3\sigma}{2}z^2(t - \sigma) + \varrho_1\frac{\xi_3}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s \\ &\quad + \frac{\xi_3\sigma}{2}k\vartheta^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s + \frac{\xi_3\sigma}{2}kx^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s\Big] \\ &\leqslant \varrho R_1 \Big[\frac{\xi_3\sigma}{2}z^2 + \frac{\xi_3}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s + \varrho\frac{\xi_3\sigma}{2}z^2(t - \sigma) + \varrho\frac{\xi_3}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s \\ &\quad + \frac{\xi_3\sigma}{2}k\vartheta^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s + (1 + \varrho)\frac{\xi_3\sigma}{2}kx^2 + \frac{\xi_3 k}{2}\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s\Big]. \end{aligned}$$

From conditions (i), (ii), (H1), (2.2), (2.3) and the estimate $u \leqslant |u| \leqslant u^2 + 1$,

$$\begin{aligned} W_5 &\leqslant (k - p_0)z^2 + \frac{\delta}{2}[(p_1 - k)(Z^2 + \vartheta^2) + (1 + \varrho_2)R_1(g^2(x) + \vartheta^2) - 2q_0\vartheta^2] \\ &\quad + \frac{\varrho_1 k}{2}z^2 + \frac{\varrho_1 k}{2}z^2(t - r) + \frac{\varrho_1}{2}q_1\vartheta^2 + \frac{\varrho_1}{2}q_1 z^2(t - r) + \frac{\varrho_1}{2}p_1 z^2 \\ &\quad + \frac{\varrho_1}{2}p_1 z^2(t - r) + \frac{\varrho_1}{2}(1 + \varrho_2)R_1\xi_2^2 x^2 + \frac{\varrho_1}{2}(1 + \varrho_2)R_1 z^2(t - r) \\ &\quad - kq_0\vartheta^2 + (1 + \varrho_2)R_1\xi_3\vartheta^2 + \frac{k}{2}\vartheta^2 + \frac{k}{2}z^2 + \frac{\varrho_1 k}{2}\vartheta^2 + \frac{\varrho_1 k}{2}z^2(t - r) \\ &\quad - k(1 + \varrho_2)R_0\xi_1 x^2 + kp_1\vartheta^2 + \mu z^2 - \mu z^2(t - r) + \lambda\sigma\vartheta^2 - \lambda\int_{t-\sigma}^{t} \vartheta^2(s)\,\mathrm{d}s \\ &\quad + [k(x^2 + \vartheta^2 + 2) + Z^2 + 1]|\psi(t, x(t), x(t - \sigma), x'(t), x'(t - \sigma), x''(t))| \\ &\quad - 2\varrho_1 P(t)|zz(t - r)| + 2\varrho_1 P(t)|zz(t - r)| + (\varrho_1 - \varrho_1^2)P(t)z^2(t - r) \end{aligned}$$

$$
\begin{aligned}
&\leqslant (k-p_0)z^2 + \frac{\delta}{2}[(p_1-k)(Z^2+\vartheta^2) + (1+\varrho_2)R_1(g^2(x)+\vartheta^2) - 2q_0\vartheta^2] \\
&\quad + \frac{\varrho_1 k}{2}z^2 + \frac{\varrho_1 k}{2}z^2(t-r) + \frac{\varrho_1}{2}q_1\vartheta^2 + \frac{\varrho_1}{2}q_1 z^2(t-r) + \frac{\varrho_1}{2}p_1 z^2 \\
&\quad + \frac{\varrho_1}{2}p_1 z^2(t-r) + \frac{\varrho_1}{2}(1+\varrho_2)R_1\xi_2^2 x^2 + \frac{\varrho_1}{2}(1+\varrho_2)R_1 z^2(t-r) \\
&\quad - kq_0\vartheta^2 + (1+\varrho_2)R_1\xi_3\vartheta^2 + \frac{k}{2}\vartheta^2 + \frac{k}{2}z^2 + \frac{\varrho_1 k}{2}\vartheta^2 + \frac{\varrho_1 k}{2}z^2(t-r) \\
&\quad - k(1+\varrho_2)R_0\xi_1 x^2 + kp_1\vartheta^2 + \mu z^2 - \mu z^2(t-r) + \lambda\sigma\vartheta^2 - \lambda\int_{t-\sigma}^{t}\vartheta^2(s)\,\mathrm{d}s \\
&\quad + [k(x^2+\vartheta^2+2) + Z^2 + 1]|\psi(t,x(t),x(t-\sigma),x'(t),x'(t-\sigma),x''(t))| \\
&\quad + \varrho_1 P(t)z^2 + 2\varrho_1 P(t)z^2(t-r) - 2\varrho_1 P(t)|zz(t-r)| - \varrho_1^2 P(t)z^2(t-r).
\end{aligned}
$$

Therefore

$$
\begin{aligned}
W_5 + W_6 \leqslant\; & -\Big[(k+\delta)q_0 - \frac{\varrho}{2}(R_1\xi_3\sigma k + q_1 + k) - (1+\varrho)R_1\Big(\xi_3 + \frac{\delta}{2}\Big) \\
& - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k) - \lambda\sigma\Big]\vartheta^2 \\
& - (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\varrho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2 \\
& - \Big[p_0 - \frac{\varrho}{2}(R_1\xi_3\sigma + k + 5p_1) - \frac{3}{2}k - \mu\Big]z^2 + \frac{\delta}{2}(p_1-k)Z^2 \\
& + \Big[\frac{\varrho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 3p_1) - \mu\Big]z^2(t-r) \\
& + [\varrho R_1\xi_3(1+k) - \lambda]\int_{t-\sigma}^{t}\vartheta^2(s)\,\mathrm{d}s + d(x^2+\vartheta^2+Z^2)\varphi(t) \\
& + 3d\varphi(t) - 2\varrho_1 p_0|zz(t-r)| - \varrho_1^2 p_0 z^2(t-r).
\end{aligned}
$$

By taking

$$
\lambda = \varrho R_1\xi_3(1+k), \quad \mu = \frac{\varrho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1), \quad d = \max\{k,1\}
$$

and by (2.7),

$$
\begin{aligned}
W_5 + W_6 \leqslant\; & -\Big[(k+\delta)q_0 - \frac{\varrho}{2}(R_1\xi_3[(2+3k)\sigma + 2] + q_1 + k + R_1\delta) - R_1\Big(\xi_3 + \frac{\delta}{2}\Big) \\
& - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k)\Big]\vartheta^2 \\
& - (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\varrho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2 + \frac{\delta}{2}(p_1-k)Z^2 \\
& - \Big[p_0 - \frac{\varrho}{2}(2R_1(\xi_3\sigma+1) + 3k + 8p_1 + q_1) - \frac{3k}{2}\Big] \\
& \times [z^2 + 2\varrho_1|zz(t-r)| + \varrho_1^2 z^2(t-r)] + d(x^2+\vartheta^2+Z^2)\varphi(t) + 3d\varphi(t)
\end{aligned}
$$

$$
\begin{aligned}
\leqslant\ & -\Big[(k+\delta)q_0 - \frac{\varrho}{2}(R_1\xi_3[(2+3k)\sigma+2]+q_1+k+R_1\delta) - R_1\Big(\xi_3+\frac{\delta}{2}\Big)\\
& - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k)\Big]\vartheta^2\\
& - (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\varrho R_1}{2}(\xi_3\sigma k+\xi_2^2)\Big]x^2\\
& - \Big[p_0 - \frac{\varrho}{2}(2R_1(\xi_3\sigma+1)+3k+8p_1+q_1) - \frac{3k}{2} - \frac{\delta}{2}(p_1-k)\Big]Z^2\\
& + d(x^2+\vartheta^2+Z^2)\varphi(t) + 3d\varphi(t)
\end{aligned}
$$

provided that

$$
\begin{aligned}
\varrho < \min\Big\{ & \frac{2(k+\delta)q_0 - R_1(2\xi_3+\delta) - k(1+2p_1) - \delta(p_1-k)}{R_1\xi_3[(2+3k)\sigma+2]+q_1+k+R_1\delta},\\
& \frac{2kR_0\xi_1+\delta R_1\xi_2^2}{R_1(\xi_3\sigma k+\xi_2^2)},\ \frac{2p_0-3k-\delta(p_1-k)}{2R_1(\xi_3\sigma+1)+3k+8p_1+q_1}\Big\}.
\end{aligned}
$$

Hence, there exists a positive constant S such that

$$
\begin{aligned}
W_5+W_6 &\leqslant -S[x^2+\vartheta^2+Z^2] + d(x^2+\vartheta^2+Z^2)\varphi(t) + 3d\varphi(t)\\
&\leqslant (d\varphi_1 - S)(x^2+\vartheta^2+Z^2) + 3d\varphi(t),
\end{aligned}
\tag{2.10}
$$

where $S > d\varphi_1$ and

$$
\begin{aligned}
S = \min\Big\{ & (k+\delta)q_0 - \frac{\varrho}{2}(R_1\xi_3[(2+3k)\sigma+2]+q_1+k+R_1\delta) - R_1\Big(\xi_3+\frac{\delta}{2}\Big)\\
& - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k), (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\varrho R_1}{2}(\xi_3\sigma k+\xi_2^2)\Big],\\
& p_0 - \frac{\varrho}{2}(2R_1(\xi_3\sigma+1)+3k+8p_1+q_1) - \frac{3k}{2} - \frac{\delta}{2}(p_1-k)\Big\},
\end{aligned}
$$

also,

$$
\begin{aligned}
W_7 &= \frac{1}{2}kP'(t)\vartheta^2 + (1+\varrho_2)R'(t)g(x)\vartheta + k(1+\varrho_2)R'(t)G(x)\\
&\quad + \frac{1}{2}Q'(t)\vartheta^2 + \frac{1}{2}kQ'(t)x^2 + kP'(t)x\vartheta\\
&= \frac{1}{2}kP'(t)\vartheta^2 + (1+\varrho_2)R'(t)[g(x)\vartheta + kG(x)] + \frac{1}{2}Q'(t)(\vartheta^2+kx^2) + kP'(t)x\vartheta\\
&\leqslant \frac{1}{2}k|P'(t)|\vartheta^2 + (1+\varrho)|R'(t)|\Big[\frac{1}{2}g^2(x)+\frac{1}{2}\vartheta^2\Big] + \frac{1}{2}k|P'(t)|(x^2+\vartheta^2)\\
&\leqslant \frac{1}{2}k\Delta(t)(x^2+2\vartheta^2) + \frac{1}{2}(1+\varrho)\Delta(t)[\xi_2^2x^2+\vartheta^2] \leqslant \omega\Delta(t)(x^2+\vartheta^2+Z^2),
\end{aligned}
\tag{2.11}
$$

where

$$
\omega = \frac{2k+(1+\varrho)(\xi_2^2+1)}{2}.
$$

By (2.7), (2.10), (2.11), expression (2.9) can be rewritten as

$$\dot{W}_{(2.1)} \leqslant -M(x^2+\vartheta^2+Z^2) + \frac{\omega}{k_5}\Delta(t)W + 3d\varphi(t),$$

where $M = S - d\varphi_1$.

The derivative of the functional V along the trajectories of system (2.1) is given by

$$\begin{aligned}\dot{V}_{(2.1)} &= \Big[\dot{W}_{(2.1)} - \frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big)\\ &\leqslant \Big[-M(x^2+\vartheta^2+Z^2) + \frac{\omega}{k_5}\Delta(t)W + 3d\varphi(t) - \frac{\Delta(t)}{\Gamma}W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big).\end{aligned}$$

Let $\Gamma^{-1} = \omega/k_5$, hence

$$\dot{V}_{(2.1)} \leqslant [-M(x^2+\vartheta^2+Z^2) + 3d\varphi(t)]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big).$$

By inequality (2.8)

$$\dot{V}_{(2.1)} \leqslant -N(x^2+\vartheta^2+Z^2) + 3d\varphi(t), \tag{2.12}$$

where $N = M\exp(-\gamma/\Gamma)$. By integrating (2.12) from t_1 to t, where $t \geqslant t_1$,

$$V(t) \leqslant V(t_1) + 3d\int_{t_1}^{t}\varphi(s)\,\mathrm{d}s \leqslant D_2, \tag{2.13}$$

where $D_2 = V(t_1) + 3dD_1$. By (2.6)

$$W = V\exp\Big(\frac{1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big)$$

and from (2.8) and (2.13)

$$W \leqslant D_2\exp\Big(\frac{\gamma}{\Gamma}\Big).$$

Due to the boundedness of W, there exists a positive constant η such that

$$|x(t)| \leqslant \eta,\ |\vartheta(t)| \leqslant \eta \text{ and } |Z(t)| \leqslant \eta. \tag{2.14}$$

We have

$$|x'(t)| = |y(t) - h(x(t))| = |\vartheta(t)| \leqslant \eta.$$

Thanks to the boundedness of $x(t)$ and (H1),

$$\begin{aligned}|x''(t) + \varrho_1 x''(t-r)| &= |Z(t) - h'(x(t))x'(t) - \varrho_1 h'(x(t-r))x'(t-r)|\\ &\leqslant |Z(t)| + |h'(x(t))||x'(t)| + \varrho_1|h'(x(t-r))||x'(t-r)| \leqslant \eta_1,\end{aligned}$$

where $\eta_1 = \eta(1 + (1+\varrho_1)\delta)$. Finally, x, x' and $x''(t) + \varrho_1 x''(t-r)$ are bounded.

Square integrability: We show that the solutions and their derivatives belong to L^2.We define the function

$$E(t) = V(t) + \alpha \int_{t_1}^{t} (x^2(s) + \vartheta^2(s) + Z^2(s))\, \mathrm{d}s \quad \forall\, t \geqslant t_1,\ \alpha > 0. \tag{2.15}$$

According to (2.12)

$$\dot{E}(t) \leqslant \dot{V}(t) + \alpha(x^2(t) + \vartheta^2(t) + Z^2(t)) \leqslant (\alpha - N)(x^2(t) + \vartheta^2(t) + Z^2(t)) + 3d\varphi(t).$$

If we take $\alpha < N$, then

$$\dot{E}(t) \leqslant 3d\varphi(t). \tag{2.16}$$

Integrating (2.16) from t_1 to t we obtain

$$E(t) \leqslant E(t_1) + 3d \int_{t_1}^{t} \varphi(s)\, \mathrm{d}s \leqslant E(t_1) + 3dD_1. \tag{2.17}$$

We use (2.15) and (2.17) to get

$$\begin{aligned} \alpha \int_{t_1}^{t} (x^2(s) + \vartheta^2(s) + Z^2(s))\, \mathrm{d}s &\leqslant V(t) + \alpha \int_{t_1}^{t} (x^2(s) + \vartheta^2(s) + Z^2(s))\, \mathrm{d}s \\ &\leqslant E(t_1) + 3dD_1, \end{aligned}$$

while from $E(t_1) = V(t_1)$ it follows that

$$\int_{t_1}^{t} (x^2(s) + \vartheta^2(s) + Z^2(s))\, \mathrm{d}s \leqslant \frac{V(t_1) + 3dD_1}{\alpha} = D_2.$$

Therefore,

$$\int_{t_1}^{t} x^2(s)\, \mathrm{d}s \leqslant D_2, \quad \int_{t_1}^{t} \vartheta^2(s)\, \mathrm{d}s \leqslant D_2 \quad \text{and} \quad \int_{t_1}^{t} Z^2(s)\, \mathrm{d}s \leqslant D_2 \tag{2.18}$$

and we have

$$\int_{t_1}^{t} x'^2(s)\, \mathrm{d}s = \int_{t_1}^{t} \vartheta^2(s)\, \mathrm{d}s \leqslant D_2.$$

By (2.18) and (H1) we have

$$\begin{aligned}\int_{t_1}^{t}&[x''(s)+\varrho_1 x''(s-r)]^2\,\mathrm{d}s\\ &=\int_{t_1}^{t}[Z(s)-h'(x(s))x'(s)-\varrho_1 h'(x(s-r))x'(s-r)]^2\,\mathrm{d}s\\ &=\int_{t_1}^{t}[Z^2(s)+h'^2(x(s))x'^2(s)+\varrho_1^2 h'^2(x(s-r))x'^2(s-r)\\ &\quad-2Zh'(x(s))x'(s)-2\varrho_1 Zh'(x(s-r))x'(s-r)\\ &\quad+2\varrho_1 h'(x(s))h'(x(s-r))x'(s)x'(s-r)]\,\mathrm{d}s\\ &\leqslant(1+\delta(1+\varrho_1))\int_{t_1}^{t}Z^2(s)\,\mathrm{d}s+(\delta^2(1+\varrho_1)+\delta)\int_{t_1}^{t}x'^2(s)\,\mathrm{d}s\\ &\quad+\varrho_1\delta(1+\delta(1+\varrho_1))\int_{t_1}^{t}x'^2(s-r)\,\mathrm{d}s\end{aligned}$$

and from (2.14)

$$\int_{t_1}^{t}x'^2(s-r)\,\mathrm{d}s=\int_{t_1-r}^{t-r}x'^2(u)\,\mathrm{d}u\leqslant\int_{t_1-r}^{t_1}x'^2(u)\,\mathrm{d}u+D_2\leqslant\eta^2 r+D_2.$$

Finally

$$\begin{aligned}\int_{t_1-r}^{t}[x''(s)+\varrho_1 x''(s-r)]^2\,\mathrm{d}s&\leqslant(1+\delta(1+\varrho_1))D_2+(\delta^2(1+\varrho_1)+\delta)D_2\\ &\quad+\varrho_1\delta(1+\delta(1+\varrho_1))(\eta^2 r+D_2).\end{aligned}$$

The proof of Theorem 2.1 is completed. □

P r o o f of Theorem 2.2. In this case, equation (1.1) becomes

(2.19) $$[x'(t)+\varrho_1 x'(t-r)]''+P(t)x''(t)+Q(t)x'(t)+R(t)[g(x(t))+\varrho_2 g(x(t-\sigma))]=0.$$

Equation (2.19) is equivalent to the system

(2.20) $$\begin{gathered}x'(t)=y(t),\quad y'(t)=z(t),\\ Z'(t)=-P(t)z-Q(t)y-(1+\varrho_2)R(t)g(x)+\varrho_2 R(t)\int_{t-\sigma}^{t}y(s)g'(x(s))\,\mathrm{d}s,\end{gathered}$$

where

$$Z(t)=y'(t)+\varrho_1 y'(t-r)=z(t)+\varrho_1 z(t-r).$$

The proof depends on some fundamental properties of a continuously differentiable functional $V=V(t,x(t),y(t),z(t))$ defined by

$$V=W\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big),$$

in which $\Delta(t) = |P'(t)| - R'(t)$, the function $W = W(t; x; y; z)$ is defined by

$$W = \frac{1}{2}Z^2 + kyZ + \frac{1}{2}kP(t)y^2 + (1+\varrho_2)kR(t)G(x) + (1+\varrho_2)R(t)yg(x) + \frac{Q(t)}{2}y^2 + kxZ + \frac{1}{2}kQ(t)x^2 + kP(t)xy + \mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s$$

with Γ, μ and λ being positive constants to be determined later in the proof. Rewriting W as

$$W = W_1 + W_2 + W_3 + W_4 + \mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s,$$

where

$$\begin{aligned}
W_1 &= \frac{1}{4}Z^2 + kyZ + \frac{1}{2}kP(t)y^2,\\
W_2 &= (1+\varrho_2)kR(t)G(x) + (1+\varrho_2)R(t)yg(x) + \frac{1}{4}Q(t)y^2,\\
W_3 &= \frac{1}{4}kQ(t)x^2 + kP(t)xy + \frac{1}{4}Q(t)y^2,\\
W_4 &= \frac{1}{4}Z^2 + kxZ + \frac{1}{4}kQ(t)x^2,
\end{aligned}$$

with a similar steps to the previous proof we obtain the next results.

Firstly let us show that W is positive definite.

$$W_1 \geqslant k_1y^2, \quad W_2 \geqslant k_2x^2, \quad W_3 \geqslant k_3x^2, \quad W_4 \geqslant k_4Z^2,$$

where

$$k_1 = \frac{1}{2}k(p_0 - 2k), \quad k_2 = \frac{\xi_1^2}{2\xi_3}(1+\varrho_2)kR_0\Big(1 - \frac{2R_1\xi_3}{kq_0}\Big),$$
$$k_3 = \frac{1}{4}q_0k\Big[1 - \frac{4kp_1^2}{q_0^2}\Big], \quad k_4 = k\Big[\frac{1}{k} - \frac{4}{q_0}\Big].$$

Since

$$\mu\int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda\int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s > 0,$$

it follows that

$$W \geqslant k_5(x^2 + y^2 + Z^2), \quad \text{where } k_5 = \min\{k_1, k_2, k_3, k_4\}. \tag{2.21}$$

By (iii) we conclude that

$$\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big) \geqslant \exp\Big(\frac{-\gamma}{\Gamma}\Big). \tag{2.22}$$

We use (2.21) and (2.22) to obtain

$$V \geqslant k_6(x^2 + y^2 + Z^2), \quad \text{where } k_6 = \exp\Big(\frac{-\gamma}{\Gamma}\Big)k_5.$$

Also, it is easy to see that there is a positive constant δ_1 such that

$$V \leqslant \delta_1(x^2 + y^2 + Z^2) + \mu \int_{t-r}^{t} z^2(s)\,\mathrm{d}s + \lambda \int_{-\sigma}^{0}\int_{t+s}^{t} y^2(\tau)\,\mathrm{d}\tau\,\mathrm{d}s \tag{2.23}$$

for all x, y and Z, and all $t \geqslant t_1$.

For the time derivative of the function W along the trajectories of system (2.20), a straightforward calculation yields

$$\dot{W}_{(2.20)} = W_5 + W_6 + W_7$$

such that

$$\begin{aligned}
W_5 &= (k - P(t))z^2 + \varrho_1 kzz(t-r) - \varrho_1 Q(t)yz(t-r) - \varrho_1 P(t)zz(t-r)\\
&\quad - \varrho_1(1+\varrho_2)R(t)g(x)z(t-r) - kQ(t)y^2 + (1+\varrho_2)R(t)g'(x)y^2 + kyZ\\
&\quad - k(1+\varrho_2)R(t)g(x)x + kP(t)y^2 + \mu z^2 - \mu z^2(t-r)\\
&\quad + \lambda\sigma y^2 - \lambda\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s,\\
W_6 &= \varrho_2 R(t)[z + \varrho z(t-r) + ky + kx]\int_{t-\sigma}^{t} y(s)g'(x(s))\,\mathrm{d}s \quad \text{and} \quad W_7 = \frac{\partial W}{\partial t}.
\end{aligned}$$

By conditions (i), (ii) , (iii), (2.4), (2.5) and by applying the estimate $2uv \leqslant u^2 + v^2$ we obtain

$$\begin{aligned}
W_5 + W_6 \leqslant\; & -\Big[kq_0 - \frac{\varrho}{2}(R_1\xi_3\sigma k + q_1 + k) - (1+\varrho)R_1\xi_3 - \frac{k}{2}(1 + 2p_1) - \lambda\sigma\Big]y^2\\
& - (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\varrho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big]x^2\\
& - \Big[p_0 - \frac{\varrho}{2}(R_1\xi_3\sigma + k + 3p_1) - \frac{3}{2}k - \mu\Big]z^2\\
& + \Big[\frac{\varrho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1) - \mu\Big]z^2(t-r)\\
& + [\varrho R_1\xi_3(1+k) - \lambda]\int_{t-\sigma}^{t} y^2(s)\,\mathrm{d}s\\
& - 2\varrho_1 p_0 zz(t-r) - \varrho_1^2 p_0 z^2(t-r).
\end{aligned}$$

Let

$$\lambda = \varrho R_1\xi_3(1+k), \quad \mu = \frac{\varrho}{2}(R_1\xi_3\sigma + q_1 + 2R_1 + 2k + 5p_1),$$

$$\varrho < \min\Big\{\frac{2kq_0 - 2R_1\xi_3 - k(1+2p_1)}{R_1\xi_3[(2+3k)\sigma + 2] + q_1 + k}, \frac{2kR_0\xi_1}{R_1(\xi_3\sigma k + \xi_2^2)}, \frac{2p_0 - 3k}{2R_1(\xi_3\sigma + 1) + 3k + 8p_1 + q_1}\Big\}.$$

The last inequality becomes

$$W_5 + W_6 \leqslant -S(x^2 + y^2 + Z^2),$$

where

$$\begin{aligned} S = \ &\min\Big\{kq_0 - \frac{\varrho}{2}(R_1\xi_3[(2+3k)\sigma + 2] + q_1 + k) - R_1\xi_3 - \frac{k}{2}(1+2p_1), p_0, (1+\varrho_2) \\ &\times \Big[kR_0\xi_1 - \frac{\varrho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big], p_0 - \frac{\varrho}{2}(2R_1(\xi_3\sigma + 1) + 3k + 8p_1 + q_1) - \frac{3k}{2}\Big\}, \\ W_7 = \ &\frac{1}{2}kP'(t)y^2 + (1+\varrho_2)R'(t)g(x)y + (1+\varrho_2)kR'(t)G(x) \\ &+ \frac{1}{2}Q'(t)y^2 + \frac{1}{2}kQ'(t)x^2 + kP'(t)xy \\ &\leqslant \omega\Delta(t)(x^2 + y^2 + Z^2) \end{aligned}$$

and $\omega = (2k + (1+\varrho)(\xi_2^2 + 1))/2$.

The above estimates lead to

$$\dot{W}_{(2.20)} \leqslant -S(x^2 + y^2 + Z^2) + \omega\Delta(t)(x^2 + y^2 + Z^2).$$

Finally, by (2.21) the derivative of the functional V along the trajectories of system (2.20) is given by

$$\begin{aligned} \dot{V}_{(2.20)} &= \Big[\dot{W}_{(2.20)} - \frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big) \\ &\leqslant \Big[-S(x^2 + y^2 + Z^2) + \frac{\omega}{k_5}\Delta(t)W - \frac{1}{\Gamma}\Delta(t)W\Big]\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big). \end{aligned}$$

Let $\Gamma^{-1} = \omega/k_5$, so

$$\dot{V}_{(2.20)} \leqslant -S(x^2 + y^2 + Z^2)\exp\Big(\frac{-1}{\Gamma}\int_{t_1}^{t}\Delta(s)\,\mathrm{d}s\Big). \tag{2.24}$$

From (2.8) and (2.24) we have

$$\dot{V}_{(2.20)} \leqslant -\delta_2(x^2 + y^2 + Z^2), \tag{2.25}$$

where $\delta_2 = S\exp(-\gamma/\Gamma)$.

We have established that the zero solution of (2.20) is uniformly asymptotically stable. This fact completes the proof of Theorem 2.2. □

3. Example

We consider the following third order non autonomous delay neutral diffential equation:

$$\begin{aligned}&\Big[x'(t)+0.02x'(t-0.5)+\frac{1}{5}\sin(x(t))+\frac{0.02}{5}\sin(x(t-0.5))\Big]''\\&\quad+\Big(5+\frac{2}{\pi}\arctan(t)\Big)x''(t)+\Big(17+\frac{1}{2+t^2}\Big)x'(t)+\Big(4+\frac{1}{\pi}\arctan(-t)\Big)\\&\quad\times\Big[\Big(\frac{3}{5}x(t)+\frac{x(t)}{1+x^2(t)}\Big)+0.03\Big(\frac{3}{5}x(t-0,2)+\frac{x(t-0,2)}{1+x^2(t-0,2)}\Big)\Big]\\&=\frac{\cos(t)\sin(x'(t-0.2))}{1+t^2+x^2(t)+x^2(t-0.2)}.\end{aligned}$$

For all $t\geqslant t_1=t_0+0.5$ it is easy to see that:

(i)
$$4=p_0\leqslant P(t)=5+\frac{2}{\pi}\arctan(t)\leqslant 6=p_1,$$
$$\frac{7}{2}=R_0\leqslant R(t)=4+\frac{1}{\pi}\arctan(-t)\leqslant\frac{9}{2}=R_1,$$
$$17=q_0\leqslant Q(t)=17+\frac{1}{2+t^2}\leqslant\frac{35}{2}=q_1,$$
$$Q'(t)=\frac{-2t}{(2+t^2)^2}\leqslant 0,\quad R'(t)=\frac{-1}{\pi(1+t^2)}\leqslant 0;$$

(ii)
$$\frac{3}{5}=\xi_1\leqslant\frac{g(x)}{x}=\frac{3}{5}+\frac{1}{1+x^2}\leqslant\frac{8}{5}=\xi_2,$$
$$|g'(x)|=\Big|\frac{3}{5}+\frac{1-x^2}{(1+x^2)^2}\Big|\leqslant\frac{8}{5}=\xi_3,\quad g(0)=0;$$

(iii)
$$\begin{aligned}\int_{t_1}^{t}(|P'(s)|-R'(s))\,\mathrm{d}s&=\int_{t_1}^{t}\Big(\frac{2}{\pi(1+s^2)}+\frac{1}{\pi(1+s^2)}\Big)\,\mathrm{d}s=\int_{t_1}^{t}\Big(\frac{3}{\pi(1+s^2)}\Big)\,\mathrm{d}s\\&\leqslant\int_{0}^{\infty}\Big(\frac{3}{\pi(1+s^2)}\Big)\,\mathrm{d}s\leqslant\frac{3}{2}<\infty;\end{aligned}$$

(H1)
$$|h'(x)|=\frac{1}{5}|\cos(x(t))|\leqslant\frac{1}{5}=\delta\quad\forall\,x\in\mathbb{R}$$

and

$$\frac{1}{5}=\delta<\min\Big\{\frac{2kR_0\xi_1}{R_1\xi_2^2};\frac{2p_0-3k}{p_1-k}\Big\}=\min\{0.57;0.78\};$$

(H2)
$$\begin{aligned}&|\psi(t,x(t),x(t-0.2),x'(t),x'(t-0.2),x''(t))|\\&\quad=\Big|\frac{\cos(t)\sin(x'(t-0.2))}{1+t^2+x^2(t)+x^2(t-0.2)}\Big|\leqslant\frac{|\cos(t)|}{1+t^2}=\varphi(t)\leqslant\varphi_1=1\end{aligned}$$

and

$$\int_{t_1}^{t}\varphi(s)\,\mathrm{d}s=\int_{t_1}^{t}\frac{|\cos(s)|}{1+s^2}\,\mathrm{d}s\leqslant\int_{t_1}^{\infty}\frac{1}{1+s^2}\,\mathrm{d}s\leqslant\int_{0}^{\infty}\frac{1}{1+s^2}\,\mathrm{d}s=\frac{\pi}{2}=D_1.$$

For d = 1.5 we obtain

$$\varrho = 0.03 < \min\Big\{\frac{2(k+\delta)q_0 - R_1(2\xi_3+\delta) - k(1+2p_1) - \delta(p_1-k)}{R_1\xi_3[(2+3k)\sigma+2] + q_1 + k + R_1\delta},$$
$$\frac{2kR_0\xi_1 - \delta R_1\xi_2^2}{R_1(\xi_3\sigma k + \xi_2^2)}; \frac{2p_0 - 3k - \delta(p_1-k)}{2R_1(\xi_3\sigma+1) + 3k + 8p_1 + q_1}\Big\}$$
$$= \min\{0.5; 0.29; 0.0317\},$$

where

$$k = 1.5 < \min\Big\{\frac{p_0}{2}; \frac{q_0^2}{4p_1^2}; \frac{q_0}{4}\Big\} = \min\{2; 2601; 4.25\},$$
$$q_0 = 17 > \max\Big\{\frac{R_1(2\xi_3+\delta) + k(1+2p_1) + \delta(p_1-k)}{2(k+\delta)}; \frac{2R_1\xi_3}{k}\Big\} = \max\{15.41; 16\}$$

and

$$\begin{aligned}S &= 1.8873\\ &= \min\Big\{(k+\delta)q_0 - \frac{\varrho}{2}(R_1\xi_3[(2+3k)\sigma+2] + q_1 + k + R_1\delta) - R_1\Big(\xi_3 + \frac{\delta}{2}\Big)\\ &\qquad - \frac{k}{2}(1+2p_1) - \frac{\delta}{2}(p_1-k), (1+\varrho_2)\Big[kR_0\xi_1 - \frac{\delta}{2}R_1\xi_2^2 - \frac{\varrho R_1}{2}(\xi_3\sigma k + \xi_2^2)\Big],\\ &\qquad p_0 - \frac{\varrho}{2}(2R_1(\xi_3\sigma+1) + 3k + 6p_1 + q_1) - \frac{3k}{2} - \frac{\delta}{2}(p_1-k)\Big\}\\ &= \min\{32.810, 1.8873, 59.122\} > 1.5 = d\varphi_1.\end{aligned}$$

The given constants in the example guarantee the existence of the parameters ϱ, k, q_0 and S which satisfy all the conditions of Theorem 2.1 and Theorem 2.2. Then all the solutions are bounded and square integrable and if $h \equiv 0$ and $\psi \equiv 0$, they are uniformly asymptotically stable.

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