

Gagandeep Singh; Gurcharanjit Singh

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*Archivum Mathematicum*, Vol. 61 (2025), No. 2, 63–72

Persistent URL: <http://dml.cz/dmlcz/153017>

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# CERTAIN SUBCLASS OF ALPHA-CONVEX BI-UNIVALENT FUNCTIONS DEFINED USING $q$ -DERIVATIVE OPERATOR

GAGANDEEP SINGH AND GURCHARANJIT SINGH

**ABSTRACT.** The present investigation deals with a new subclass of alpha-convex bi-univalent functions in the unit disc  $E = \{z: |z| < 1\}$  defined with  $q$ -derivative operator. Bounds for the first two coefficients and Fekete-Szegő inequality are established for this class. Many known results follow as consequences of the results derived here.

## 1. INTRODUCTION

Let us consider the analytic functions  $f$  which have the expansion of the form

$$(1) \quad f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

in the unit disc  $E = \{z: |z| < 1\}$  and normalized by  $f(0) = f'(0) - 1 = 0$ . The class of these functions is denoted by  $\mathcal{A}$ . Further, the class of functions  $f \in \mathcal{A}$  and which are univalent in  $E$ , is denoted by  $\mathcal{S}$ . The functions of the form  $u(z) = \sum_{k=1}^{\infty} c_k z^k$ , which are analytic in the unit disc  $E$  and satisfy the conditions  $u(0) = 0$  and  $|u(z)| < 1$ , are called Schwarz functions and the class of these functions is denoted by  $\mathcal{U}$ .

The classes  $\mathcal{S}^*$  of starlike functions and  $\mathcal{K}$  of convex functions are defined as follows:

$$\mathcal{S}^* = \left\{ f: f \in \mathcal{A}, \operatorname{Re} \left( \frac{zf'(z)}{f(z)} \right) > 0, z \in E \right\}$$

and

$$\mathcal{K} = \left\{ f: f \in \mathcal{A}, \operatorname{Re} \left( \frac{(zf'(z))'}{f'(z)} \right) > 0, z \in E \right\}.$$

For  $0 \leq \alpha \leq 1$ , Mocanu [17] introduced the class  $\mathcal{M}(\alpha)$ , which is a unification of the classes  $\mathcal{S}^*$  and  $\mathcal{K}$  and is defined as

$$\mathcal{M}(\alpha) = \left\{ f: f \in \mathcal{A}, \operatorname{Re} \left( (1-\alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{(zf'(z))'}{f'(z)} \right) > 0, z \in E \right\}.$$

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2020 *Mathematics Subject Classification*: primary 30C45; secondary 30C50.

*Key words and phrases*: analytic functions, bi-univalent functions, alpha-convex functions, coefficient bounds, Fekete-Szegő inequality,  $q$ -derivative, subordination.

Received April 20, 2024, revised February 2025. Editor M. Kolář.

DOI: 10.5817/AM2025-2-63

The functions in the class  $\mathcal{M}(\alpha)$  are known as alpha-convex functions. In particular,  $\mathcal{M}(0) \equiv \mathcal{S}^*$  and  $\mathcal{M}(1) \equiv \mathcal{K}$ .

If  $f$  and  $g$  are two analytic functions in  $E$ , then  $f$  is said to be subordinate to  $g$  (denoted as  $f \prec g$ ) if there exists a Schwarz function  $u \in \mathcal{U}$  such that  $f(z) = g(u(z))$ . By making use of a subordination theorem for analytic functions, many authors derived several subordination relationships between certain subclasses of analytic functions, for example, see [8, 16, 28]. Further, if  $g$  is univalent in  $E$ , then  $f \prec g$  implies  $f(0) = g(0)$  and  $f(E) \subset g(E)$ . For  $-1 \leq B < A \leq 1$  and  $0 \leq \eta < 1$ , Polatoglu et al. [19] defined the class  $\mathcal{P}(A, B; \eta)$  which consists of the functions  $p(z)$  such that  $p(z) \prec \frac{1 + [B + (A - B)(1 - \eta)]z}{1 + Bz}$ . For  $\eta = 0$ , the class  $\mathcal{P}(A, B; \eta)$  reduces to  $\mathcal{P}(A, B)$ , which is a subclass of  $\mathcal{A}$  introduced by Janowski [12].

Quantum calculus is ordinary classical calculus which introduces  $q$ -calculus, where  $q$  stands for quantum. Nowadays,  $q$ -calculus has attracted many researchers as it is widely useful in various branches of Mathematics and Physics. The application of  $q$ -calculus was initiated by Jackson [10, 11] and he developed  $q$ -integral and  $q$ -derivative in a systematic way. For  $0 < q < 1$ , Jackson [10] defined the  $q$ -derivative of a function  $f \in \mathcal{A}$  as

$$(2) \quad D_q f(z) = \begin{cases} \frac{f(z) - f(qz)}{(1-q)z} & \text{for } z \neq 0, \\ f'(0) & \text{for } z = 0, \end{cases}$$

where  $D_q^2 f(z) = D_q(D_q f(z))$ .

From (2), it is obvious that

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1},$$

where  $[k]_q = \frac{1-q^k}{1-q} = 1 + q + q^2 + \cdots + q^{k-1}$ . If  $q \rightarrow 1^-$ , then  $[k]_q \rightarrow k$ . Further  $D_q z^k = [k]_q z^{k-1}$  and  $\lim_{q \rightarrow 1^-} D_q f(z) = f'(z)$ .

Using  $q$ -derivative operator, Seoudy and Aouf [21] defined the subclasses of  $q$ -starlike and  $q$ -convex functions of order  $\alpha$  ( $0 \leq \alpha < 1$ ) as follows:

$$\mathcal{S}_q^*(\alpha) = \left\{ f : f \in \mathcal{A}, \operatorname{Re} \left( \frac{z D_q f(z)}{f(z)} \right) > \alpha, z \in E \right\}$$

and

$$\mathcal{K}_q(\alpha) = \left\{ f : f \in \mathcal{A}, \operatorname{Re} \left( \frac{D_q(z D_q f(z))}{D_q f(z)} \right) > \alpha, z \in E \right\}.$$

It is obvious that  $f \in \mathcal{K}_q(\alpha)$  if and only if  $f \in \mathcal{S}_q^*(\alpha)$ . For  $q \rightarrow 1^-$  and  $\alpha = 0$ , the classes  $\mathcal{S}_q^*(\alpha)$  and  $\mathcal{K}_q(\alpha)$  reduces to the classes  $\mathcal{S}^*$  and  $\mathcal{K}$ , respectively.

By Koebe one-quarter theorem [7], every function  $f \in \mathcal{S}$  has an inverse  $f^{-1}$ , defined by

$$f^{-1}(f(z)) = z (z \in E)$$

and

$$f(f^{-1}(w)) = w \left( |w| < r_0(f) : r_0(f) \geq \frac{1}{4} \right)$$

where

$$(3) \quad f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \dots$$

A function  $f \in \mathcal{A}$  for which both  $f$  and  $f^{-1}$  are univalent in  $E$ , is called a bi-univalent function. The class of functions of the form (1) and which are bi-univalent in  $E$ , is denoted by  $\Sigma$ . The functions  $\frac{z}{1-z}$ ,  $-\log(1-z)$ ,  $\frac{1}{2} \log \left( \frac{1+z}{1-z} \right)$ , are some of the examples of the functions in the class  $\Sigma$ . The well known Koebe function  $f(z) = \frac{z}{(1-z)^2}$  is not a member of the class  $\Sigma$ .

Lewin [13] was the first, who investigated the class  $\Sigma$  and proved that  $|a_2| < 1.51$ . Subsequently, bounds for the initial coefficients of numerous sub-classes of bi-univalent functions were studied by various authors in [3, 5, 9, 18, 22, 23, 24, 25, 26, 27]. Further several subclasses of bi-univalent functions defined with  $q$ -derivative operator were studied by various authors including [1, 2, 6, 14, 15, 20, 29, 30].

Using the notion of  $q$ -derivative, now we define a subclass of alpha-convex bi-univalent functions and establish the bounds of  $|a_2|$ ,  $|a_3|$  and Fekete-Szegő inequality for this class.

**Definition 1.1.** A function  $f \in \Sigma$  is said to be in the class  $\mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$  if the following conditions are satisfied:

$$(1 - \lambda) \frac{z D_q f(z)}{f(z)} + \lambda \frac{D_q(z D_q f(z))}{D_q f(z)} \prec \left( \frac{1 + [B + (A - B)(1 - \eta)]z}{1 + Bz} \right)^\alpha$$

and

$$(1 - \lambda) \frac{w D_q g(w)}{g(w)} + \lambda \frac{D_q(w D_q g(w))}{D_q g(w)} \prec \left( \frac{1 + [B + (A - B)(1 - \eta)]w}{1 + Bw} \right)^\alpha,$$

where  $g(w) = f^{-1}(w)$  as given in (3),  $-1 \leq B < A \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 < \alpha \leq 1$  and  $0 \leq \eta < 1$ .

The following observations are obvious:

- (i)  $\mathcal{M}\Sigma_q(1 - 2\beta, -1; 0; 1; \lambda) \equiv \mathcal{B}\Sigma_q(\beta, \lambda)$ .
- (ii)  $\mathcal{M}\Sigma_q(1, -1; 0; \alpha; \lambda) \equiv \mathcal{M}\Sigma_q(\alpha, \lambda)$ .
- (iii) For  $q \rightarrow 1^-$ ,  $\mathcal{M}\Sigma_q(1 - 2\beta, -1; 0; 1; \lambda) \equiv \mathcal{B}_\Sigma(\beta, \lambda)$ , the class studied by Li and Wang [14].
- (iv) For  $q \rightarrow 1^-$ ,  $\mathcal{M}\Sigma_q(1, -1; 0; \alpha; \lambda) \equiv \mathcal{M}_\Sigma(\alpha, \lambda)$ , the class introduced by Li and Wang [14].

$$(v) \mathcal{M}\Sigma_q(1, -1; 0; \alpha; \lambda) \equiv \mathcal{S}_{\Sigma_q}^*(\alpha, \lambda).$$

$$(vi) \mathcal{M}\Sigma_q(1 - 2\beta, -1; 0; 1; \lambda) \equiv \mathcal{K}_{\Sigma_q}(\beta, \lambda).$$

Throughout this paper, we make the assumptions that  $0 < \alpha \leq 1$ ,  $0 \leq \lambda \leq 1$ ,  $0 \leq \beta < 1$ ,  $0 \leq \eta < 1$ ,  $-1 \leq B < A \leq 1$ ,  $z \in E$ ,  $w \in E$  and  $g(w) = f^{-1}(w)$  as given in (3).

For deriving the main results, we use the following lemma:

**Lemma 1.1** ([4]). *If  $p(z) = \frac{1 + [B + (A - B)(1 - \eta)]u(z)}{1 + Bu(z)} = 1 + \sum_{k=1}^{\infty} p_k z^k$ ,  $u(z) \in \mathcal{U}$ , then*

$$|p_n| \leq (A - B)(1 - \eta), \quad n \geq 1.$$

## 2. THE CLASS $\mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$

**Theorem 2.1.** *If  $f \in \mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$ , then*

$$(4) \quad |a_2| \leq$$

$$\frac{\sqrt{2\alpha^2(A - B)(1 - \eta)}}{\sqrt{2\alpha[( [3]_q - [2]_q ) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3)] + (1 - \alpha)[([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}}$$

and

$$(5) \quad |a_3| \leq \frac{\alpha(A - B)(1 - \eta)}{([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} + \frac{\alpha^2(A - B)^2(1 - \eta)^2}{[( [2]_q - 1 ) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

**Proof.** From Definition 1.1, using the concept of subordination, we have

$$(6) \quad (1 - \lambda) \frac{zD_q f(z)}{f(z)} + \lambda \frac{D_q(zD_q f(z))}{D_q f(z)} = \left( \frac{1 + [B + (A - B)(1 - \eta)]u(z)}{1 + Bu(z)} \right)^\alpha = [p(z)]^\alpha, \quad u \in \mathcal{U}$$

and

$$(7) \quad (1 - \lambda) \frac{wD_q g(w)}{g(w)} + \lambda \frac{D_q(wD_q g(w))}{D_q g(w)} = \left( \frac{1 + [B + (A - B)(1 - \eta)]v(w)}{1 + Bv(w)} \right)^\alpha = [q(w)]^\alpha, \quad v \in \mathcal{U},$$

where  $p(z) = 1 + p_1 z + p_2 z^2 + \dots$  and  $q(w) = 1 + q_1 w + q_2 w^2 + \dots$

Expanding and equating the coefficients of  $z$  and  $z^2$  in (6) and of  $w$  and  $w^2$  in (7), we obtain

$$(8) \quad [([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]a_2 = \alpha p_1,$$

$$(9) \quad \begin{aligned} & [([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]a_3 \\ & + [(1 - [2]_q) + \lambda(-1 + [2]_q + [2]_q^2 - [2]_q^3)]a_2^2 = \alpha p_2 + \frac{\alpha(\alpha - 1)p_1^2}{2} \end{aligned}$$

and

$$(10) \quad - [([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]a_2 = \alpha q_1,$$

$$(11) \quad \begin{aligned} & [([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)](2a_2^2 - a_3) \\ & + [(1 - [2]_q) + \lambda(-1 + [2]_q + [2]_q^2 - [2]_q^3)]a_2^2 = \alpha q_2 + \frac{\alpha(\alpha - 1)q_1^2}{2}. \end{aligned}$$

(8) and (10) together gives

$$(12) \quad p_1 = -q_1$$

and

$$(13) \quad 2[( [2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2 a_2^2 = \alpha^2(p_1^2 + q_1^2).$$

Adding (9) and (11) and using (13), it yields

$$(14) \quad \begin{aligned} & 2\alpha [([3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3)]a_2^2 \\ & = \alpha^2(p_2 + q_2) + (\alpha - 1)[([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2 a_2^2, \end{aligned}$$

which gives

$$(15) \quad a_2^2 = \frac{\alpha^2(p_2 + q_2)}{2\alpha [([3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3)] + (1 - \alpha)[([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

Taking modulus, applying triangle inequality and using Lemma 1.1 in (15), we can easily obtain (4).

Now subtracting (11) from (9), we get

$$(16) \quad 2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)](a_3 - a_2^2) = \alpha(p_2 - q_2) + \frac{\alpha(\alpha - 1)}{2}(p_1^2 - q_1^2).$$

Using (12), (13) and (16), it gives

$$(17) \quad a_3 = \frac{\alpha(p_2 - q_2)}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} + \frac{\alpha^2(p_1^2 + q_1^2)}{2[( [2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

Using (12) and (13) in (17), and applying triangle inequality, we obtain

$$(18) \quad |a_3| \leq \frac{\alpha(|p_2| + |q_2|)}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} + \frac{\alpha^2(2|p_1|^2)}{2[( [2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

Using Lemma 1.1, the result (5) can be easily obtained from (18).  $\square$

On putting  $A = 1 - 2\beta$ ,  $B = -1$ ,  $\eta = 0$ ,  $\alpha = 1$ , Theorem 2.1 gives the following result.

**Corollary 2.1.** *If  $f \in \mathcal{B}\Sigma_q(\beta; \lambda)$ , then*

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{2\alpha[(3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3)}}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} + \frac{4(1-\beta)^2}{[(2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

For  $A = 1$ ,  $B = -1$ ,  $\eta = 0$ , Theorem 2.1 yields the following result.

**Corollary 2.2.** *If  $f \in \mathcal{M}\Sigma_q(\alpha; \lambda)$ , then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha[(3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3) + (1-\alpha)[([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}}}$$

and

$$|a_3| \leq \frac{2\alpha}{([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} + \frac{4\alpha^2}{[(2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

For  $q \rightarrow 1^-$  and on putting  $A = 1, B = -1, \eta = 0$ , Theorem 2.1 agrees with the following result due to Li and Wang [14].

**Corollary 2.3.** *If  $f \in \mathcal{M}\Sigma(\alpha; \lambda)$ , then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{(1+\lambda)(\alpha+1+\lambda-\alpha\lambda)}}$$

and

$$|a_3| \leq \frac{\alpha}{1+2\lambda} + \frac{4\alpha^2}{(1+\lambda)^2}.$$

For  $q \rightarrow 1^-$  and on substituting  $A = 1 - 2\beta, B = -1, \eta = 0, \alpha = 1$ , Theorem 2.1 coincides with the following result due to Li and Wang [14].

**Corollary 2.4.** *If  $f \in \mathcal{B}\Sigma(\beta; \lambda)$ , then*

$$|a_2| \leq \sqrt{\frac{2(1-\beta)}{1+\lambda}}$$

and

$$|a_3| \leq \frac{1-\beta}{1+2\lambda} + \frac{4(1-\beta)^2}{(1+\lambda)^2}.$$

On putting  $A = 1 - 2\beta, B = -1, \eta = 0, \alpha = 1, \lambda = 1$ , the following result can be easily obtained from Theorem 2.1.

**Corollary 2.5.** *If  $f \in \mathcal{K}_{\Sigma_q}(\beta)$ , then*

$$|a_2| \leq \frac{2\sqrt{(1-\beta)}}{\sqrt{2\alpha[-[3]_q + [3]_q^2 + [2]_q^2 - [2]_q^3]}}$$

and

$$|a_3| \leq \frac{2(1-\beta)}{-[3]_q + [3]_q^2} + \frac{4(1-\beta)^2}{[-[2]_q + [2]_q^2]^2}.$$

On putting  $A = 1$ ,  $B = -1$ ,  $\eta = 0$ ,  $\lambda = 0$  in Theorem 2.1, the following result is obvious.

**Corollary 2.6.** *If  $f \in \mathcal{S}_{\Sigma_q}^*(\alpha)$ , then*

$$|a_2| \leq \frac{2\alpha}{\sqrt{2\alpha([3]_q - [2]_q) + (1-\alpha)([2]_q - 1)^2}}$$

and

$$|a_3| \leq \frac{2\alpha}{[3]_q - 1} + \frac{4\alpha^2}{([2]_q - 1)^2}.$$

**Theorem 2.2.** *If  $f \in \mathcal{M}\Sigma_q(A, B; \eta; \alpha; \lambda)$ , then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{\alpha(1-\eta)(A-B)}{([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)}, & \text{if } 0 \leq |l(\mu)| < \frac{1}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)}, \\ 2\alpha(1-\eta)(A-B)|l(\mu)|, & \text{if } |l(\mu)| \geq \frac{1}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)}, \end{cases}$$

where

$$(19) \quad l(\mu) = \frac{\alpha(1-\mu)}{2\alpha[(3]_q - [2]_q) + \lambda(-2[3]_q + [3]_q^2 + [2]_q + [2]_q^2 - [2]_q^3) + (1-\alpha)([2]_q - 1) + \lambda(1 - 2[2]_q + [2]_q^2)]^2}.$$

**Proof.** Using (13) in (17), we have

$$(20) \quad a_3 = \frac{\alpha(p_2 - q_2)}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} + a_2^2.$$

Making use of (20), it yields

$$(21) \quad a_3 - \mu a_2^2 = \frac{\alpha(p_2 - q_2)}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} + (1-\mu).$$

Further (21) can be expressed as

$$(22) \quad a_3 - \mu a_2^2 = \alpha \left[ \left( l(\mu) + \frac{1}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} \right) p_2 + \left( l(\mu) - \frac{1}{2[(3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)} \right) q_2 \right],$$



where  $l(\mu)$  is defined in (19).

Taking modulus, applying triangle inequality and using Lemma 1.1, (22) yields

$$(23) \quad |a_3 - \mu a_2^2| \leq \alpha(1 - \eta)(A - B) \left| \left( l(\mu) + \frac{1}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} \right) + \left( l(\mu) - \frac{1}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]} \right) \right|.$$

For  $0 \leq |l(\mu)| < \frac{1}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]}$ ,

$$(24) \quad |a_3 - \mu a_2^2| \leq \frac{\alpha(1 - \eta)(A - B)}{([3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)}.$$

For  $|l(\mu)| \geq \frac{1}{2[( [3]_q - 1) + \lambda(1 - 2[3]_q + [3]_q^2)]}$ ,

$$(25) \quad |a_3 - \mu a_2^2| \leq 2\alpha(1 - \eta)(A - B)|l(\mu)|.$$

The proof of Theorem 2.2 is obvious from (24) and (25). □

## CONCLUSION

This paper is concerned with the study of a new and generalized class of alpha-convex bi-univalent functions using  $q$ -derivative operator. The class is defined using the concept of subordination. Some earlier known results follow as special cases of the results proved here. This paper will work as a motivation to the other researchers to study some more relevant subclasses of bi-univalent functions using subordination.

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DEPARTMENT OF MATHEMATICS, KHALSA COLLEGE,  
AMRITSAR, PUNJAB, INDIA  
*E-mail:* kamboj.gagandeep@yahoo.in

DEPARTMENT OF MATHEMATICS, GNDU COLLEGE,  
CHUNGH(TARN-TARAN), PUNJAB, INDIA  
*E-mail:* dhillongs82@yahoo.com