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MULTI-ARGUMENT SPECIALIZATION SEMILATTICES

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Abstract. If X is a closure space with closure K, we consider the semilattice $(\mathcal{P}(X), \cup)$ endowed with a further relation $x \sqsubseteq \{y_1, y_2, \dots, y_n\}$ between elements of $\mathcal{P}(X)$ and finite subsets of $\mathcal{P}(X)$, whose interpretation is $x \subseteq Ky_1 \cup Ky_2 \cup \ldots \cup Ky_n$.

We present axioms for such multi-argument specialization semilattices and show that this list of axioms is sound and complete for substructures of closure spaces, namely, a model satisfies the axioms if and only if it can be embedded into the structure associated to a closure space as in the previous sentence. As a main tool for the proof, we provide a canonical embedding of a multi-argument specialization semilattice into (the structure associated to) a closure semilattice.

Keywords: multi-argument specialization semilattice; closure semilattice; closure space; universal extension

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1. Introduction

The notion of *closure* is pervasive in mathematics, both in the topological sense and in the sense of *hull*, *generated by*. The general notion of a closure space which can be abstracted from the above two cases has been dealt with or foreshadowed by such mathematicians as Schröder, Dedekind, Cantor, Riesz, Hausdorff, Moore, Čech, Kuratowski, Sierpiński, Tarski, Birkhoff and Ore, as listed in Erné [7], with applications, among others, to ordered sets, lattice theory, logic, algebra, topology, computer science and connections with category theory. Due to the importance of the notion, it is interesting to study variations and weakenings. See, e.g., [2], [3],

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[4], [6], [7], [8], [9], [11], [17], [18], [19], [21], [23], [24], [25] and further references and details in the quoted sources.

In detail, a *closure space* is a set X together with a unary operation (an *operator*) K on $\mathcal{P}(X)$ such that $x \subseteq Kx$ (K is *extensive*), $KKx = Kx$ (*idempotent*) and $x \subseteq y$ implies $Kx \subseteq Ky$ (*isotone*), for all subsets x, y of X. In particular, by Kuratowski characterization [20], Chapter II, Section 3, a topological space is a closure space satisfying $K\emptyset = \emptyset$ and $K(x \cup y) = Kx \cup Ky$ for all subsets x, y of X. Similarly to the case of topological spaces, closure spaces have an equivalent characterization in terms of a family of closed subsets. See [7] for further details.

A asymmetry in the otherwise clean correspondence [18] between topology and its algebraic counterpart suggested by the Kuratowski axioms is the fact that continuous functions among topological spaces do not preserve closures. Indeed, if φ is a function between topological spaces and $\varphi^{\rightarrow}$ denotes the direct image function associated to φ, then φ is continuous if and only if $\varphi^{\rightarrow}(Kz) \subseteq K\varphi^{\rightarrow}(z)$, and this is not the same as $\varphi^{\rightarrow}(Kz) = K\varphi^{\rightarrow}(z)$. Hence, in [13], [15], [17] we started the study of *specialization semilattices*, that is, join semilattices endowed with a further coarser preorder $\sqsubseteq$ satisfying the condition

$$a \sqsubseteq b \;\&\; a_1 \sqsubseteq b \Rightarrow a \vee a_1 \sqsubseteq b, \tag{1.1}$$

referred to as (S3) in the quoted sources.

If X is a topological space, then $(\mathcal{P}(X), \cup, \sqsubseteq)$ is a specialization semilattice, where $\sqsubseteq$ is the binary relation on $\mathcal{P}(X)$ defined by $a \sqsubseteq b$ if $a \subseteq Kb$, for $a, b \subseteq X$, and where K is a topological closure. In [17], Theorem 5.7, we showed that every specialization semilattice can be represented as a subsemilattice of a "topological" specialization semilattice as above. Moreover, in this situation, continuous functions correspond exactly to homomorphisms of specialization semilattices (more details below).

In passing, let us mention that there are nontopological ways of constructing specialization semilattices. Any semilattice homomorphism $\varphi\colon \mathbf{S} \to \mathbf{T}$ induces a specialization on $\mathbf{S}$ by setting $a \sqsubseteq_{\mathbf{S}} b$ if $\varphi(a) \leqslant_{\mathbf{T}} \varphi(b)$. Moreover, every specialization semilattice can be represented in this way. See Theorem 5.6 below. Thus, specialization semilattices are at the same time both "substructures" of topological spaces and quotients of semilattices. As an example, if we consider the quotient of $\mathcal{P}(X)$ modulo the ideal of finite subsets of X and the corresponding quotient function, the corresponding "specialization" is inclusion modulo finite [1]. Notice that in this case no underlying notion of "closure" is present. Similar quotient constructions appear in many disparate settings, in various fields and with a wide range of applications [2], [8].

Other examples of specialization semilattices or, more generally, specialization posets, appear in computer science as *Complete Implicational Systems* [3], Sec-

tion 7.4, in theoretical studies related to measure theory, in algebraic logic and even in theoretical physics [12]. See [13], [15], [17] for further examples, details and references, in particular, [17], Section 4. More generally, see [5] for other applications of semilattices, possibly with further structure.

Let us now return to the representation of a specialization semilattice as a substructure of $(\mathcal{P}(X), \cup, \sqsubseteq)$. As hinted above, under this interpretation, homomorphisms of specialization semilattices are indeed "functorial" in the sense that if X and Y are topological spaces and $\varphi\colon X \to Y$ is a function, then φ is continuous if and only if the image function $\varphi^{\to}$ is a homomorphism of the associated specialization semilattices. See [17], Proposition 2.4 for details.

All the above arguments apply also in the more general situation of closure spaces, thus specialization semilattices can be thought of at the same time as subreducts of topological spaces and as subreducts of closure spaces. Namely, (the first-order universal theory of) specialization semilattices does not distinguish between the two kinds of structures [17], Corollary 5.8. In order to distinguish the two cases it is appropriate to introduce a ternary relation R interpreted by

$$R(a, b, c) \quad \text{if} \quad a \subseteq Kb \cup Kc. \tag{1.2}$$

We will frequently write $a \sqsubseteq b,\ c$ in place of $R(a, b, c)$, and similarly for relations involving more elements. Topological spaces satisfy $R(a, b, c) \Leftrightarrow R(a, b \vee c, b \vee c)$, an equivalence which is generally false in closure spaces[1]. As above, continuous functions correspond to homomorphisms which respect R, and similarly for the $n+1$-ary relations we are going to introduce. See [17], Remark 5.9 for further comments.

Hence, it is interesting to introduce n-ary generalizations of (1.2). In this note we study and axiomatize *multi-argument specialization semilattices*, that is, semilattices endowed with further relations whose intended interpretation is given by a generalization of (1.2) above. We show that any multi-argument specialization semilattice can be embedded into a closure space. See Theorem 5.2 below. As a useful tool, when dealing with embeddings into closure semilattices (a much more comprehensive notion than closure spaces), we get a refined result: every multi-argument specialization semilattice has a canonical free extension into a regular principal multi-argument specialization semilattice. See Theorem 4.5. Here "principal" means that closures always exist (namely, we are in a closure semilattice, but in the signature of specialization semilattices) and "regular" means that the multi-argument relation is

[1] By the way, $R(a, b \vee c, b \vee c)$ is also equivalent to $a \sqsubseteq b \vee c$, and a similar remark applies to relations with more arguments. Thus, when dealing with substructures of topological spaces, n-ary relations, for $n \geqslant 2$, can be dispensed with. In other words, the theory presented here is suited for closure spaces and reduces to the theory of specialization semilattices in the case of topological spaces. See Remark 4.7 (c) for more details.

canonically expressible in terms of join and of the binary specialization. See Definitions 2.1 (c) and 3.3.

Theorem 4.5 is parallel to similar results proved in [13], [15] for specialization semilattices.

2. Preliminaries

Definition 2.1. (a) A *closure operation* on a partially ordered set (henceforth, *poset*, for short) is a unary operation K which is extensive, idempotent and isotone. The expression *operator* is frequently used for a unary operation on the power set $\mathcal{P}(X)$ of a set X, where the partial order is generally assumed to be inclusion. In particular, the convention applies to closure spaces. A *closure semilattice* is a join-semilattice endowed with a closure operation. See [7], Section 3 for a detailed study of closure posets and semilattices, with many applications. As custom in order theory, we do not include additivity in the definition of a closure operation, namely we do not require $K(a \vee b) = Ka \vee Kb$, an identity however holding in topological spaces. Recall from the introduction that a *specialization semilattice* is a join semilattice endowed with a further coarser preorder $\sqsubseteq$ such that $a \vee a_1 \sqsubseteq b$, whenever $a \sqsubseteq b$ and $a_1 \sqsubseteq b$.

(b) Similarly to the case of closure and topological spaces, a closure operation on a poset induces a *specialization* $\sqsubseteq$ as follows: $x \sqsubseteq y$ if $x \leqslant Ky$. Thus, if $(P, \vee, K_P)$ is a closure semilattice, then $(P, \vee, \sqsubseteq)$ is a specialization semilattice. From the latter structure we can retrieve the original closure K_P: indeed, $K_P x$ is the $\leqslant$-largest element y of $\mathbf{P}$ such that $y \sqsubseteq x$. However, such a largest element does not necessarily exist in an arbitrary specialization semilattice: consider, for example, the specialization given by inclusion modulo finite, as briefly recalled in the introduction. See [13], [15], [17] for more details.

(c) If a specialization semilattice $\mathbf{S}$ is such that for every $x \in S$ there exists the $\leqslant$-largest element $y \in S$ such that $y \sqsubseteq x$, then $\mathbf{S}$ is said to be *principal*. Such a y will be denoted by Kx, as well. If the structure on $\mathbf{S}$ is induced by a closure semilattice $(P, \vee, K_P)$, as above, then K, as defined in the previous sentence, turns out to be equal to K_P, hence, the overlapping notation causes no ambiguity.

(d) Given a semilattice $(P, \vee)$, we have seen in (b) above that, to a closure operation K on $\mathcal{P}$, there is associated a specialization $\sqsubseteq$ which makes $(P, \vee, \sqsubseteq)$ a principal specialization semilattice. Conversely, if $(P, \vee, \sqsubseteq)$ is a principal specialization semilattice, then K, as introduced in (c), is a closure operation on $(P, \vee)$. See [7], Proposition 3.9.

The above correspondences are one the inverse of the other: see [7], Section 3.1, in particular, [7], Proposition 3.9 for details. Thus, there is a bijective correspondence between closure semilattices and principal specialization semilattices.

R e m a r k 2.2. If a and b are elements of a specialization semilattice and both Ka and Kb exist, then $Ka \leqslant Kb$ if and only if $a \sqsubseteq b$. See [13], Remark 2.1 (c) for a proof.

Definition 2.3. (a) A *homomorphism* for the relation $\sqsubseteq$ is a function φ such that $a \sqsubseteq b$ implies $\varphi(a) \sqsubseteq \varphi(b)$ for all a, b in the domain. A homomorphism (embedding) of closure semilattices is a semilattice homomorphism (embedding) satisfying $\varphi(Ka) = K\varphi(a)$ for all a.

(b) Notice that homomorphisms of specialization semilattices do not necessarily preserve closures; see [13], Remark 2.2 or Remark 3.8 below. In other words, a homomorphism between two principal specialization semilattices is not necessarily a homomorphism between the associated closure semilattices.

(c) If φ is a homomorphism between two *principal* specialization semilattices, then φ is a *K-homomorphism* if $\varphi(Ka) = K\varphi(a)$ for every a. In this case K-homomorphisms are actually homomorphisms for the associated closure semilattices.

(d) Notice that if φ is a semilattice homomorphism between two principal specialization semilattices and φ satisfies $\varphi(Ka) = K\varphi(a)$, then φ is automatically a homomorphism of specialization semilattices, namely, φ preserves also $\sqsubseteq$.

See [7], [13], [15], [17] for further details about the above notions.

3. Multi-argument specialization semilattices

N o t a t i o n 3.1. We will consider semilattices $\mathbf{M}$ endowed with a further "multi-argument" relation $\sqsubseteq$ between elements of M and finite nonempty subsets of M. For notational simplicity, we will write $a \sqsubseteq b_1, \ldots, b_n$ in place of $a \sqsubseteq \{b_1, \ldots, b_n\}$, but it should be remarked that the order in which $b_1, \ldots, b_n$ appear will never be relevant. As mentioned, the intended interpretation of $a \sqsubseteq b_1, \ldots, b_n$ is given by $a \subseteq Kb_1 \cup \ldots \cup Kb_n$ in a closure space, or, more generally, $a \leqslant Kb_1 \vee \ldots \vee Kb_n$ in a closure semilattice. Notice that the binary relation $a \sqsubseteq b$, when a singleton appears on the right, corresponds to the case of the "specializations" we have dealt with in the above discussions.

A *homomorphism* for the relation $\sqsubseteq$ is a function φ such that $a \sqsubseteq b_1, \ldots, b_n$ implies $\varphi(a) \sqsubseteq \varphi(b_1), \ldots, \varphi(b_n)$ for all $a, b_1, \ldots$ in the domain. An *embedding* is an injective homomorphism such that also the converse holds, namely, $\varphi(a) \sqsubseteq \varphi(b_1), \ldots, \varphi(b_n)$ implies $a \sqsubseteq b_1, \ldots, b_n$.

Definition 3.2. A *multi-argument specialization semilattice* is a join semilattice $\mathbf{M}$ together with a further relation $\sqsubseteq$ between elements of M and finite nonempty subsets of M such that the following conditions hold.

(M1) $a \sqsubseteq a$,

(M2) $a \sqsubseteq b_1, b_2, \ldots, b_n \;\&\; b_1 \sqsubseteq c \Rightarrow a \sqsubseteq c, b_2, \ldots, b_n$,

(M3) $a \leqslant b \;\&\; b \sqsubseteq c_1, \ldots, c_m \Rightarrow a \sqsubseteq c_1, \ldots, c_m$,

(M4) $a \sqsubseteq b_1, \ldots, b_n \Rightarrow a \sqsubseteq b_1, \ldots, b_n, b_{n+1}$,

(M5) $a \sqsubseteq b_1, \ldots, b_n \;\&\; a_1 \sqsubseteq b_1, \ldots, b_n \Rightarrow a \vee a_1 \sqsubseteq b_1, \ldots, b_n$

for every $a, a_1, b, b_1, \ldots, c,\ c_1, \ldots \in M$ and where, as mentioned, $a \sqsubseteq b_1, \ldots, b_n$ is a shorthand for $a \sqsubseteq \{b_1, \ldots, b_n\}$.

Notice that the order relation appears only in (M3) and the semilattice structure on $\mathbf{M}$ is used only in condition (M5). A poset with a relation $\sqsubseteq$ satisfying (M1)–(M4) will be called a *multi-argument specialization poset*.

We now derive some consequences from (M1)–(M5).

From (M1) and (M3) with $m = 1$ and $b = c_1$ we get

(M1*) $$a \leqslant b \Rightarrow a \sqsubseteq b.$$

By taking $n = 1$ in (M2) we get

(M2−) $$a \sqsubseteq b \;\&\; b \sqsubseteq c \Rightarrow a \sqsubseteq c,$$

a relation which involves only the binary $\sqsubseteq$ and which we have called (S2) when dealing with specialization semilattices. Then using (M1*) we get

(M2*) $$a \sqsubseteq b \;\&\; b \leqslant c \Rightarrow a \sqsubseteq c.$$

By iterating (M2), recalling that the order in which the b_is are enumerated is not relevant, we get

(M2+) $$a \sqsubseteq b_1, b_2, \ldots, b_n \;\&\; b_1 \sqsubseteq c_1 \;\&\; \ldots \;\&\; b_n \sqsubseteq c_n \Rightarrow a \sqsubseteq c_1, c_2, \ldots, c_n,$$

hence, by (M1*)

(M2*+) $$a \sqsubseteq b_1, b_2, \ldots, b_n \;\&\; b_1 \leqslant c_1 \;\&\; \ldots \;\&\; b_n \leqslant c_n \Rightarrow a \sqsubseteq c_1, c_2, \ldots, c_n.$$

By iterating (M4) we get

(M4+) $$a \sqsubseteq B \;\&\; B \subseteq B_1 \Rightarrow a \sqsubseteq B_1$$

for all finite subsets B, B_1 of M.

So far, we have not used (M5), hence the above conditions hold in multi-argument specialization posets, as well.

From $c_i \sqsubseteq c_i$ given by (M1) and iterating (M4), we get $c_i \sqsubseteq c_1, \ldots, c_m$ for every $i \leqslant m$. Then by iterating (M5) we get

(M1+) $$c_1 \vee \ldots \vee c_m \sqsubseteq c_1, \ldots, c_m.$$

If $a \leqslant c_1 \vee \ldots \vee c_m$, then from (M1+) and (M3) with $b = c_1 \vee \ldots \vee c_m$ we get $a \sqsubseteq c_1, \ldots, c_m$, hence

(M6) $$a \leqslant c_1 \vee \ldots \vee c_m \Rightarrow a \sqsubseteq c_1, \ldots, c_m.$$

Since $c_i \sqsubseteq c_1 \vee \ldots \vee c_m$ for every $i \leqslant m$, by (M1*), if $a \sqsubseteq c_1, \ldots, c_m$, then we can repeatedly apply (M2) in order to get $a \sqsubseteq c_1 \vee \ldots \vee c_m$. Thus

(M7) $$a \sqsubseteq c_1, \ldots, c_m \Rightarrow a \sqsubseteq c_1 \vee \ldots \vee c_m.$$

We will see in Example 3.4 (b) below that the converse implication in (M7) does not necessarily hold.

Finally, assume that $\{b_1, \ldots, b_h\}$ and $\{d_1, \ldots, d_k\}$ are nonempty sets. Then

(M8) $$\text{if for every } i \leqslant h \text{ there is } j \leqslant k \text{ such that } b_i \sqsubseteq d_j,$$
$$\text{then } b_1 \vee \ldots \vee b_h \sqsubseteq d_1 \vee \ldots \vee d_k.$$

Indeed, under the assumption, $b_i \sqsubseteq d_1 \vee \ldots \vee d_k$ for every $i \leqslant h$, by (M2*). Then $b_1 \vee \ldots \vee b_h \sqsubseteq d_1 \vee \ldots \vee d_k$ by repeated applications of (M5).

Recall from the introduction that a specialization semilattice is a join semilattice endowed with a further coarser preorder $\sqsubseteq$ satisfying (1.1). By (M1), (M1*), (M2−) and the case $n = 1$ of (M5), if $\mathbf{M}$ is a multi-argument specialization semilattice, then $(M, \vee, \sqsubseteq)$ is a specialization semilattice, the *specialization reduct* of $\mathbf{M}$. Here $\sqsubseteq$ is the restriction of the multi-argument specialization $\sqsubseteq$ to the case when singletons appear on the right.

Definition 3.3. Recall the definition of a principal specialization semilattice from Section 2. A *principal multi-argument specialization semilattice* is a multi-argument specialization semilattice whose specialization reduct is a principal specialization semilattice. The notion of being principal involves only the binary relation $\sqsubseteq$. On the other hand, the following definition involves the full multi-argument specialization relation $\sqsubseteq$.

A principal multi-argument specialization semilattice is *regular* if

$$x \sqsubseteq y_1, \dots, y_n \text{ holds if and only if } x \leqslant Ky_1 \vee \dots \vee Ky_n. \tag{3.1}$$

Notice that the definition of regularity applies only to principal multi-argument specialization semilattices, since otherwise K is not defined (in any case, K is not assumed to belong to the language of multi-argument specialization semilattices). Notice also that by (M7) and the definition of K, the "only if" condition in (3.1) holds in every principal multi-argument specialization semilattice.

E x a m p l e 3.4. (a) There are nonregular principal multi-argument specialization semilattices.

Consider the four-element semilattice with elements $c_1, c_2 < b < a$, with c_1, c_2 incomparable, and consider the closure operation K given by $Kc_1 = c_1$, $Kc_2 = c_2$, $Kb = Ka = a$. Let $x \sqsubseteq y$ if $x \leqslant Ky$, set $a \sqsubseteq c_1, c_2$ and add all the other $\sqsubseteq$ relations necessary to have the axioms of multi-argument specialization semilattices satisfied. In detail, $\sqsubseteq$ in $\mathbf{M}$ is defined as in (3.1), with the only exception that $x \sqsubseteq y_1, \dots, y_n$ holds also when $x = a$ and both c_1 and c_2 appear in the set $\{y_1, \dots, y_n\}$. Thus, we get a multi-argument specialization semilattice $\mathbf{M}$. The definition of $\sqsubseteq$ in the binary case implies that $\mathbf{M}$ is a principal multi-argument specialization semilattice with closure K, but the relation $a \sqsubseteq c_1, c_2$ implies that (3.1) fails.

(b) The same semilattice (with a different multi-argument specialization) can be used to show that we cannot replace $a \leqslant b$ with $a \sqsubseteq b$ in (M3), namely, that

$$a \sqsubseteq b \;\&\; b \sqsubseteq c_1, \dots, c_m \;\Rightarrow\; a \sqsubseteq c_1, \dots, c_m \tag{3.2}$$

does not necessarily hold in a multi-argument specialization semilattice.

Let $\mathbf{M}'$ be defined on the same semilattice as above, and this time let $\sqsubseteq$ be given by (3.1). Then $a \sqsubseteq b$ and $b \sqsubseteq c_1, c_2$, but not $a \sqsubseteq c_1, c_2$. Since $\mathbf{M}'$ is regular by construction, (3.2) may fail also in regular multi-argument specialization semilattices.

Moreover, $a \sqsubseteq b = c_1 \vee c_2$, but not $a \sqsubseteq c_1, c_2$, hence the converse implication in (M7) does not necessarily hold, even in regular multi-argument specialization semilattices.

(c) If (3.2) holds, then the multi-argument specialization structure is determined by the specialization structure, namely, by $\vee$ and the binary $\sqsubseteq$.

Indeed, by taking $c_1 \vee \ldots \vee c_m$ in place of b in (3.2), we get that $a \sqsubseteq c_1 \vee \ldots \vee c_m$ implies $a \sqsubseteq c_1, \ldots, c_m$, by (M1+). Thus by (M7), if (3.2) holds, then $a \sqsubseteq c_1 \vee \ldots \vee c_m$ is equivalent to $a \sqsubseteq c_1, \ldots, c_m$. This means that, assuming (3.2), $\sqsubseteq$ is determined by $\vee$ and the binary $\sqsubseteq$.

(d) By the very definition, the multi-argument specialization structure is determined by the specialization structure in every regular principal multi-argument specialization semilattice. In fact, in a regular principal multi-argument specialization semilattice

$$x \sqsubseteq y_1, \ldots, y_n \text{ if and only if there are } z_1 \sqsubseteq y_1, \ldots, z_n \sqsubseteq y_n \text{ such that } x \leqslant z_1 \vee \ldots \vee z_n. \tag{3.3}$$

(e) Let us say that a multi-argument specialization semilattice $\mathbf{M}$ is *pre-regular* if $\mathbf{M}$ satisfies (3.3). Thus, a *principal* multi-argument specialization semilattice is regular if and only if it is pre-regular. On the other hand, closures do not appear explicitly in (3.3), hence the definition of pre-regularity applies to any multi-argument specialization semilattice $\mathbf{M}$, regardless whether $\mathbf{M}$ is principal or not. However, (3.3) involves an existential assumption, while in this note we are generally concerned with universal properties.

(f) If $\mathbf{S}$ is a specialization semilattice, then condition (3.3) can be used in order to define a pre-regular multi-argument specialization on S, by interpreting $z_1 \sqsubseteq y_1, \ldots$ in (3.3) in the sense of the binary specialization from $\mathbf{S}$.

If $\mathbf{M}(\mathbf{S})$ is the resulting multi-argument specialization semilattice, then the specialization reduct of $\mathbf{M}(\mathbf{S})$ is $\mathbf{S}$. Conversely, if $\mathbf{M}$ is a pre-regular multi-argument specialization semilattice and $\mathbf{S}$ is the specialization reduct of $\mathbf{M}$, then, due to (3.3), $\mathbf{M} = \mathbf{M}(\mathbf{S})$. Moreover, if $\mathbf{T}$ is another specialization semilattice, then a function $\varphi\colon S \to T$ is a homomorphism from $\mathbf{S}$ to $\mathbf{T}$ if and only if φ is a homomorphism form $\mathbf{M}(\mathbf{S})$ to $\mathbf{M}(\mathbf{T})$.

This means that the categories of specialization semilattices and of pre-regular multi-argument specialization semilattices are isomorphic as concrete categories.

Definition 3.5. (a) A *homomorphism* of multi-argument specialization semilattices is a semilattice homomorphism which is also a homomorphism for $\sqsubseteq$. Extending the comment in Definition 2.3 (d), if φ is a semilattice homomorphism between two principal regular multi-argument specialization semilattices and φ satisfies $\varphi(Ka) = K\varphi(a)$ for every a, then φ is a homomorphism of multi-argument specialization semilattices.

(b) Regularity is necessary in the above statement. If ι is the identity function on the set in Example 3.4 (a) (b) above, then ι is both a semilattice and

a K-homomorphism from $\mathbf{M}$ to $\mathbf{M}'$; in fact, the semilattice structure and the function K are the same. On the other hand, ι is not a homomorphism of multi-argument specialization semilattices, since $a \sqsubseteq c_1, c_2$ holds in $\mathbf{M}$ but fails in $\mathbf{M}'$. This is the reason why in many statements below we need to deal with regular multi-argument specialization semilattices.

Clauses (1) and (4) in the following proposition are straightforward. They assert that the axioms (M1)–(M5) are true in the intended model and, moreover, the notion of homomorphism is preserved. Clauses (2) and (3) are essentially a reformulation of [7], Proposition 3.9 and of the definition of regularity.

Definition 3.6. If $\mathbf{S}$ is a closure semilattice, set $a \sqsubseteq b_1, \ldots, b_n$ if $a \leqslant Kb_1 \vee \ldots \vee Kb_n$. The structure $(S, \vee, \sqsubseteq)$ will be called the *associated*, or the *multi-argument specialization reduct* of $\mathbf{S}$.

Proposition 3.7.

(1) *If* $\mathbf{S}$ *is a closure semilattice, then* $(S, \vee, \sqsubseteq)$, *as defined in* 3.6, *is a principal regular multi-argument specialization semilattice. In particular, the above statement applies when* $\mathbf{S}$ *is a closure space.*

(2) *Conversely, if* $\mathbf{S}$ *is a principal regular multi-argument specialization semilattice and we define* K *as in Definition* 2.1 (c), *then* S *acquires the structure of a closure semilattice.*

(3) *The constructions in* (1) *and* (2) *are one the inverse of the other. In detail, if* $\mathbf{S}$ *is a semilattice, the correspondence which assigns to a closure operation on* $\mathbf{S}$ *the relation* $\sqsubseteq$ *from Definition* 3.6 *is a bijection from the set of closure operations on* $\mathbf{S}$ *to the set of multirelations* $\sqsubseteq$ *on* S, *which make* $\mathbf{S}$ *a principal regular multi-argument specialization semilattice.*

(4) *If* $\mathbf{S}$, $\mathbf{T}$ *are closure semilattices and* $\varphi\colon \mathbf{S} \to \mathbf{T}$ *is a homomorphism of closure semilattices, then* φ *is a homomorphism between the associated multi-argument specialization semilattices as given by* (1).

P r o o f. As we mentioned, (1) and (4) are straightforward. In fact, (4) is just a reformulation of the comment in Definition 3.5 (a). Clause (2) follows from the quoted [7], Proposition 3.9 since the definition of K involves only the binary specialization relation (indeed, the assumption that $\mathbf{S}$ is principal regular will be used only in (3)). Compare Definition 2.1 (c) (d).

The part concerning the binary specialization relation in (3) follows again from [7], Proposition 3.9. But this is enough since in both cases the full multi-argument relation $\sqsubseteq$ is determined in the same way by the semilattice operation and the closure operation. In each case, $\sqsubseteq$ is given by (3.1), by definition, in (1) and since we assume that $\mathbf{S}$ is principal regular in (2). □

If in S in Definition 3.6 we consider only the multi-argument specialization and the order structure (rather than the semilattice structure), we will call the resulting structure the *multi-argument specialization poset associated to* $\mathbf{S}$.

R e m a r k 3.8. As we mentioned, a homomorphism of principal specialization semilattices is not necessarily a homomorphism of closure semilattices [13], Remark 2.2. The same remark applies to multi-argument specialization semilattices, as can be already seen in the basic example of topological spaces. Indeed, as mentioned in the introduction, a function φ between two topological spaces is continuous if and only if the corresponding image function $\varphi^{\rightarrow}$ is a homomorphism between the associated specialization semilattices, if and only if $\varphi^{\rightarrow}$ is a homomorphism between the associated multi-argument specialization semilattices. On the other hand, if φ is a function between two topological spaces, then $\varphi^{\rightarrow}$ is a homomorphism for the associated closure semilattices, i.e., $\varphi^{\rightarrow}$ respects closure if and only if φ is continuous and closed. Not every continuous function is closed.

R e m a r k 3.9. Our convention in 3.1 about $\sqsubseteq$ simplifies notation, but is not suitable, as it stands, to be interpreted in a standard model-theoretical setting [10]. We will not use model theory here, but we mention that a model-theoretical interpretation is easily accomplished. Just consider $a \sqsubseteq b_1, \ldots, b_n$ as a shorthand for an atomic formula $R_n(a, b_1, \ldots, b_n)$, where R_n is an $n+1$-ary relation symbol, thus we have one relation for each $n \geqslant 1$. In this sense, a multi-argument specialization semilattice is intended as a model of the form $(\mathbf{M}, \vee, R_n)_{n \geqslant 1}$. Under the above conventions, the notions of homomorphism and embedding, as introduced above, correspond exactly to the classical model-theoretical notions [10]. In the above sense, satisfiability is not necessarily invariant with respect to permutations of *the sequence* $(b_1, \ldots, b_n)$, hence the following axioms should be added:

$$\text{(MT1)} \qquad R_n(a, b_1, \ldots, b_n) \Rightarrow R_n(a, b_{\sigma 1}, \ldots, b_{\sigma n}) \quad \text{for every permutation } \sigma \text{ of } \{1, 2, \ldots, n\},$$

$$\text{(MT2)} \qquad R_{n+1}(a, b_1, \ldots, b_{n-1}, b_n, b_n) \Leftrightarrow R_n(a, b_1, \ldots, b_{n-1}, b_n).$$

E x a m p l e 3.10. If $(P, \leqslant)$ is a poset and we set $a \sqsubseteq b_1, \ldots, b_n$ if $a \leqslant b_i$, for some $i \leqslant n$, then $(P, \leqslant, \sqsubseteq)$ is a principal multi-argument specialization poset; in fact, its binary reduct $\sqsubseteq$ is $\leqslant$.

The above construction is the order-dual of the construction presented in the second paragraph of [22], Example 3. This suggests that possible connections between multi-argument specialization posets and multi-posets in the sense of [22] are worth to be investigated. The connection also suggests the possibility of considering "non commutative" multi-argument specialization posets and semilattices, namely,

to work in the framework of Remark 3.9 without assuming the axioms (MT1) and (MT2) and, of course, rephrasing (M2), (M4) in such a way that the first, or last indexes assume no special role.

4. Free principal regular extensions

In this section we state and prove some results similar to [13], [15]. The statements are essentially the same, while the proofs differ in many places. We will provide full details when the arguments are different, while we sometimes refer to [15] for those proofs which are very similar. As in [15], the existence of the "universal" extensions and morphisms which we are going to construct follows from abstract categorical arguments; see, e.g., [13], Lemma 4.1. Needless to say, an explicit description of such universal objects is sometimes very hard to find.

R e m a r k 4.1. First, we informally describe the ideas in our construction.

(a) Similarly to [15], we work on the "free" semilattice extension $\widetilde{\mathbf{M}}$ of a multi-argument specialization semilattice $\mathbf{M}$. The extension $\widetilde{\mathbf{M}}$ is generated by M together with a set of new elements $\{\tilde{a};\ a \in M\}$. In the explicit construction we will present in Definition 4.2 the element $\tilde{a}$ corresponds to $(a, \{a\})$. It will turn out that $\tilde{a}$ is the closure of a in $\widetilde{\mathbf{M}}$ when endowed with the appropriate specialization structure, hence, in the present informal remark we will write Ka for $\tilde{a}$.

The required conditions here are $a \leqslant Ka$ and

$$a \leqslant Kb_1 \vee \ldots \vee Kb_n \text{ if and only if } a \sqsubseteq_{\mathbf{M}} b_1, \ldots, b_n.$$

Compare Definition 3.3. We will denote the "old" relations and operations with the subscript of the parent structure; the "new" ones will be unsubscripted. We want to define $\sqsubseteq$ in $\widetilde{\mathbf{M}}$ in such a way that the new element Ka is actually the closure of a and, moreover, we want the resulting multi-argument specialization semilattice to be regular. If we want that the above conditions hold, then we necessarily need to have

- ▷ $a \leqslant c \vee Kd_1 \vee \ldots \vee Kd_k$ whenever there is an element $d \sqsubseteq_{\mathbf{M}} d_1, \ldots, d_k$ in $\mathbf{M}$ such that $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$, and
- ▷ $Kb \leqslant c \vee Kd_1 \vee \ldots \vee Kd_k$ if there is $j \leqslant k$ such that $b \sqsubseteq_{\mathbf{M}} d_j$.

Such considerations justify clauses (a1)–(a2) in Definition 4.2 below. Compare Remark 4.3 below.

(b) The construction hinted above adds a new closure of a in the extension, even when a has already a closure in $\mathbf{M}$. This is necessary, in general, since we want to construct a *regular* principal multi-argument specialization semilattice $\widetilde{\mathbf{M}}$, while, say, $\mathbf{M}$ might be already principal—thus closures already exist in $\mathbf{M}$—though $\mathbf{M}$

might not be regular, as in Example 3.4 (a). This means that existing closures in $\mathbf{M}$ cannot do the appropriate job. In a parallel situation, in [15], Section 4, we have constructed an extension which preserves a specified set of closures, but, by the above comment, here an analogue construction can be performed only under some appropriate assumptions. In the present note we will not pursue the issue further.

If M is any set, let $\mathbf{M}^{<\omega}$ be the semilattice of the finite subsets of M, with the operation of union.

Definition 4.2. Assume that $\mathbf{M} = (\mathbf{M}, \vee_{\mathbf{M}}, \sqsubseteq_{\mathbf{M}})$ is a multi-argument specialization semilattice. On the product $M \times M^{<\omega}$ define the following relations:

(a) $(a, \{b_1, b_2, \dots, b_h\}) \precsim (c, \{d_1, d_2, \dots, d_k\})$ if

(a1) there is an element $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$ in M such that $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$ (if $k = 0$, that is, if $\{d_1, d_2, \dots, d_k\}$ is empty, the clause simply reads $a \leqslant_{\mathbf{M}} c$), and

(a2) for every $i \leqslant h$, there is $j \leqslant k$ such that $b_i \sqsubseteq_{\mathbf{M}} d_j$.

(b) $(a, \{b_1, b_2, \dots, b_h\}) \sim (c, \{d_1, d_2, \dots, d_k\})$ if both $(a, \{b_1, b_2, \dots, b_h\}) \precsim (c, \{d_1, d_2, \dots, d_k\})$ and $(c, \{d_1, d_2, \dots, d_k\}) \precsim (a, \{b_1, b_2, \dots, b_h\})$.

We will soon show in Lemma 4.4 that $\sim$ is an equivalence relation, thus the following parts of the definition are justified.

Let $\widetilde{M}$ be the quotient of $M \times M^{<\omega}$ under the equivalence relation $\sim$ and define $K\colon \widetilde{M} \to \widetilde{M}$ by

$$K[a, \{b_1, \dots, b_h\}] = [a, \{a \vee_{\mathbf{M}} b_1 \vee_{\mathbf{M}} \dots \vee_{\mathbf{M}} b_h\}],$$

where $[a, \{b_1, \dots, b_h\}]$ denotes the $\sim$-equivalence class of the pair $(a, \{b_1, \dots, b_h\})$.

We will prove in Lemma 4.4 (ii) (iii) that K is well-defined and that $\widetilde{M}$ naturally inherits a semilattice operation $\vee$ from the semilattice product $(M, \vee_{\mathbf{M}}) \times \mathbf{M}^{<\omega}$.

For $n \geqslant 1$, define $\sqsubseteq$ on $\widetilde{M}$ by

$$[a, \{b_1, \dots, b_h\}] \sqsubseteq [c_1, \{d_{1,1}, \dots, d_{1,k_1}\}], \dots, [c_n, \{d_{n,1}, \dots, d_{n,k_n}\}]$$

if

$$[a, \{b_1, \dots, b_h\}] \leqslant K[c_1, \{d_{1,1}, \dots, d_{1,k_1}\}] \vee \dots \vee K[c_n, \{d_{n,1}, \dots, d_{n,k_n}\}],$$

where $\leqslant$ is the order induced by $\vee$ on $\widetilde{M}$.

Let $\widetilde{\mathbf{M}} = (\widetilde{M}, \vee, \sqsubseteq)$, $\widetilde{\mathbf{M}}' = (\widetilde{M}, \vee, K)$. Finally, define $\upsilon_{\mathbf{M}}\colon M \to \widetilde{M}$ by $\upsilon_{\mathbf{M}}(a) = [a, \emptyset]$.

R e m a r k 4.3. We think of $[a, \{b_1, \dots, b_h\}]$ as $a \vee Kb_1 \vee \dots \vee Kb_h$, where $Kb_1, \dots, Kb_h$ are the "new" closures we need to introduce. In particular, $[a, \emptyset]$ corresponds to a and $[b, \{b\}]$ corresponds to a new element Kb. Compare Remark 4.1 (a).

Lemma 4.4. *Under the notation and the definitions in* 4.2*:*

(i) *The relation* $\precsim$ *from Definition* 4.2 (a) *is reflexive and transitive on* $M \times M^{<\omega}$. *Thus, the relation* $\sim$ *from* 4.2 (b) *is an equivalence relation.*

(ii) *The operation* K *from Definition* 4.2 *is well-defined on the* $\sim$*-equivalence classes.*

(iii) *The relation* $\sim$ *is a semilattice congruence on the semilattice* $(M, \vee) \times \mathbf{M}^{<\omega}$, *hence, the quotient* $\widetilde{M}$ *is a semilattice when endowed with the operation* $\vee$ *defined by*

$$[a, \{b_1, \dots, b_h\}] \vee [c, \{d_1, \dots, d_k\}] = [a \vee_{\mathbf{M}} c, \{b_1, \dots, b_h, d_1, \dots, d_k\}] \tag{4.1}$$

on the equivalence classes. Moreover, the following holds:

$$[a, \{b_1, \dots, b_h\}] \leqslant [c, \{d_1, \dots, d_k\}] \quad \text{if and only if } (a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\}). \tag{4.2}$$

(iv) $\widetilde{\mathbf{M}}' = (\widetilde{M}, \vee, K)$ *is a closure semilattice.*

P r o o f. (i) We first check that $\precsim$ is reflexive. Indeed, if $a = c$, then condition (a1) is verified by an arbitrary choice of d, say, $d = d_1$, by (M1) and (M4). Notice that (a1) is verified by definition if $a = c$ and $\{d_1, d_2, \dots, d_k\}$ is empty.

If $\{b_1, b_2, \dots, b_h\} = \{d_1, d_2, \dots, d_k\}$, then condition (a2) is verified by (M1).

In order to prove transitivity of $\precsim$, suppose that

$$(a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\}) \precsim (e, \{f_1, \dots, f_l\}).$$

By (a1), $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$ for some d such that $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$, and $c \leqslant_{\mathbf{M}} e \vee_{\mathbf{M}} f$ for some f such that $f \sqsubseteq_{\mathbf{M}} f_1, \dots, f_l$. Thus, $a \leqslant_{\mathbf{M}} e \vee_{\mathbf{M}} d \vee_{\mathbf{M}} f$. By (a2), for every $j \leqslant k$ there is $m \leqslant l$ such that $d_j \sqsubseteq_{\mathbf{M}} f_m$, hence, from $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$ we get $d \sqsubseteq_{\mathbf{M}} f_1, \dots, f_l$, by (M2+) and, possibly, (M4). Finally, by (M5), we get $d \vee_{\mathbf{M}} f \sqsubseteq_{\mathbf{M}} f_1, \dots, f_l$, thus, the element $d \vee_{\mathbf{M}} f$ witnesses condition (a1) for the relation $(a, \{b_1, \dots, b_h\}) \precsim (e, \{f_1, \dots, f_l\})$. So far, we have assumed that k and l are nonzero. If $l \neq 0$ and $k = 0$, that is, $\{d_1, d_2, \dots, d_k\}$ is empty, then $a \leqslant_{\mathbf{M}} c$, hence, $a \leqslant_{\mathbf{M}} e \vee_{\mathbf{M}} f$ witnesses (a1) for the desired relation. If $l = 0$, then $k = 0$ by (a2) applied to the relation $(c, \{d_1, \dots, d_k\}) \precsim (e, \{f_1, \dots, f_l\})$. The assumptions then read $a \leqslant_{\mathbf{M}} c \leqslant_{\mathbf{M}} e$, thus $a \leqslant_{\mathbf{M}} e$. The proof of (a1) for $(a, \{b_1, \dots, b_h\}) \precsim (e, \{f_1, \dots, f_l\})$ is complete.

Condition (a2) holds since for every $i \leqslant h$ there is $j \leqslant k$ such that $b_i \sqsubseteq_{\mathbf{M}} d_j$ and for every $j \leqslant k$ there is $m \leqslant l$ such that $d_j \sqsubseteq_{\mathbf{M}} f_m$, hence, we get $b_i \sqsubseteq_{\mathbf{M}} f_m$ by (M2−).

Since $\precsim$ is reflexive and transitive, then so is $\sim$; hence, $\sim$ is an equivalence relation, being symmetric by definition.

In order to prove (ii) it is enough to show that

$$\text{if } (a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\}), \text{ then } (a, \{\bar{b}\}) \precsim (c, \{\bar{d}\}), \tag{4.3}$$

where we have set $\bar{b} = a \vee_{\mathbf{M}} b_1 \vee_{\mathbf{M}} \dots \vee_{\mathbf{M}} b_h$ and $\bar{d} = c \vee_{\mathbf{M}} d_1 \vee_{\mathbf{M}} \dots \vee_{\mathbf{M}} d_k$.

If $(a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\})$, then by (a1) there is $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$ such that $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$. By (M7) and (M2*) we get $d \sqsubseteq_{\mathbf{M}} \bar{d}$, hence, clause (a1) is satisfied witnessing $(a, \{\bar{b}\}) \precsim (c, \{\bar{d}\})$.

Since $c \sqsubseteq \bar{d}$ by (M1*), we get $a \sqsubseteq_{\mathbf{M}} c \vee_{\mathbf{M}} d \sqsubseteq_{\mathbf{M}} \bar{d}$ by (M1*) again and (M5) using the already proved $d \sqsubseteq_{\mathbf{M}} \bar{d}$. Hence, $a \sqsubseteq_{\mathbf{M}} \bar{d}$ by (M2−). Since for every $i \leqslant h$ there is $j \leqslant k$ such that $b_i \sqsubseteq_{\mathbf{M}} d_j$ by (a2), $b_1 \vee \dots \vee b_h \sqsubseteq_{\mathbf{M}} \bar{d}$ by (M8). Since we have also proved $a \sqsubseteq_{\mathbf{M}} \bar{d}$, then $\bar{b} \sqsubseteq_{\mathbf{M}} \bar{d}$ by (M5). Thus (a2) is witnessed. The proof of (ii) is complete.

(iii) As in [15], by symmetry, it is enough to show that if

$$(a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\}), \tag{4.4}$$

then

$$(a, \{b_1, \dots, b_h\}) \vee (e, \{f_1, \dots, f_l\}) \precsim (c, \{d_1, \dots, d_k\}) \vee (e, \{f_1, \dots, f_l\}),$$

that is,

$$(a \vee_{\mathbf{M}} e, \{b_1, \dots, b_h, f_1, \dots, f_l\})) \precsim (c \vee_{\mathbf{M}} e, \{d_1, \dots, d_k, f_1, \dots, f_l\}). \tag{4.5}$$

Clause (a1) for (4.4) is witnessed by $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$ for some $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$. By iterating (M4), we have $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k, f_1, \dots, f_l$ and this gives (a1) for (4.5). Clause (a2) for (4.5) follows from clause (a2) for (4.4) and (M1).

Condition (4.2) is proved exactly as the corresponding condition (3.2) in [15], Lemma 3.6.

(iv) By (iii), $(\widetilde{M}, \vee)$ is a semilattice; the fact that K is a closure operation follows from (4.2) and (4.3). See the proof of [15], Claim 3.7 for details. □

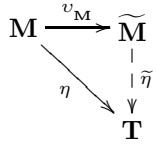
Theorem 4.5. *Suppose that* $\mathbf{M}$ *is a multi-argument specialization semilattice and let* $\widetilde{\mathbf{M}}$ *and* $\upsilon_{\mathbf{M}}$ *be as in Definition* 4.2. *Then the following statements hold.*

(1) $\widetilde{\mathbf{M}}$ *is a principal regular multi-argument specialization semilattice. If* M *is finite, then* $\widetilde{M}$ *is finite.*

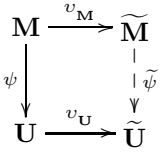
(2) $\upsilon_{\mathbf{M}}$ *is an embedding of* $\mathbf{M}$ *into* $\widetilde{\mathbf{M}}$.

(3) *The pair* $(\widetilde{\mathbf{M}}, \upsilon_{\mathbf{M}})$ *has the following universal property.*

For every principal regular multi-argument specialization semilattice $\mathbf{T}$ *and every homomorphism* $\eta\colon \mathbf{M} \to \mathbf{T}$*, there is a unique* K*-homomorphism* $\widetilde{\eta}\colon \widetilde{\mathbf{M}} \to \mathbf{T}$ *such that* $\eta = \upsilon_{\mathbf{M}} \circ \widetilde{\eta}$*.*



(4) *Suppose that* $\mathbf{U}$ *is another multi-argument specialization semilattice and* $\psi\colon \mathbf{M} \to \mathbf{U}$ *is a homomorphism. Then there is a unique* K*-homomorphism* $\widetilde{\psi}\colon \widetilde{\mathbf{M}} \to \widetilde{\mathbf{U}}$ *making the following diagram commute:*



P r o o f. Clause (1) follows from Lemma 4.4 (iv), the definition of $\sqsubseteq$ and Proposition 3.7 (1). Notice that if $\mathbf{M}$ is finite, then $\mathbf{M}^{<\omega}$, the family of finite subsets of M, is finite as well, hence, $\widetilde{M}$ is finite, being the quotient of a finite set.

(2) It follows from clause (a1) in Definition 4.2 and from (4.1) that $\upsilon_{\mathbf{M}}$ is a semilattice embedding; see the corresponding case in [15], Theorem 3.6 for details. In order to complete the proof of (2) we need to show that $\upsilon_{\mathbf{M}}$ is a $\sqsubseteq$-embedding. If $a \sqsubseteq_{\mathbf{M}} d_1, \ldots, d_n$, then $(a, \emptyset) \precsim (d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n, \{d_1, \ldots, d_n\})$, by taking $d = a$ in (a1), thus

$$\begin{aligned}\upsilon_{\mathbf{M}}(a) = [a, \emptyset] &\overset{(4.2)}{\leqslant} [d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n, \{d_1, \ldots, d_n\}] \overset{(4.1)}{=} [d_1, \{d_1\}] \vee \ldots \vee [d_n, \{d_n\}] \\ &= K[d_1, \{\emptyset\}] \vee \ldots \vee K[d_n, \{\emptyset\}] = K\upsilon_{\mathbf{M}}(d_1) \vee \ldots \vee K\upsilon_{\mathbf{M}}(d_n),\end{aligned}$$

that is, $\upsilon_{\mathbf{M}}(a) \sqsubseteq \upsilon_{\mathbf{M}}(d_1), \ldots, \upsilon_{\mathbf{M}}(d_1)$ according to the definition of $\sqsubseteq$ on $\widetilde{M}$ in Definition 4.2. Conversely, from $\upsilon_{\mathbf{M}}(a) \sqsubseteq \upsilon_{\mathbf{M}}(d_1), \ldots, \upsilon_{\mathbf{M}}(d_n)$, that is, $[a, \emptyset] \leqslant K\upsilon_{\mathbf{M}}(d_1) \vee \ldots \vee K\upsilon_{\mathbf{M}}(d_n) = [d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n, \{d_1, \ldots, d_n\}]$, that is, by (4.2), $(a, \emptyset) \precsim (d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n, \{d_1, \ldots, d_n\})$, we get $a \leqslant_{\mathbf{M}} d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n \vee_{\mathbf{M}} d$, for some d such that $d \sqsubseteq_{\mathbf{M}} d_1, \ldots, d_n$, by (a1). By (M1+) and (M5), $d_1 \vee_{\mathbf{M}} \ldots \vee_{\mathbf{M}} d_n \vee_{\mathbf{M}} d \sqsubseteq_{\mathbf{M}} d_1, \ldots, d_n$, hence, $a \sqsubseteq_{\mathbf{M}} d_1, \ldots, d_n$ by (M3). We have shown that $\upsilon_{\mathbf{M}}$ is an embedding.

We now prove (3). Since by assumption $\mathbf{T}$ is principal, then the specialization on $\mathbf{T}$ induces a closure operation $K_{\mathbf{T}}$ on T. Compare Definition 2.1 (c) (d). If $\eta\colon \mathbf{M} \to \mathbf{T}$ is

a homomorphism and there exists $\widetilde{\eta}$ such that $\eta = \upsilon_{\mathbf{M}} \circ \widetilde{\eta}$, then necessarily $\widetilde{\eta}([a, \emptyset]) = \widetilde{\eta}(\upsilon_{\mathbf{M}}(a)) = \eta(a)$ for every $a \in M$. If $\widetilde{\eta}$ is a K-homomorphism, then also $\widetilde{\eta}([b, \{b\}]) = \widetilde{\eta}(K[b, \emptyset]) = \widetilde{\eta}(K\upsilon_{\mathbf{M}}(b)) \overset{K\text{-hom}}{=} K_{\mathbf{T}}\widetilde{\eta}(\upsilon_{\mathbf{M}}(b)) = K_{\mathbf{T}}\eta(b)$, hence, necessarily

$$\widetilde{\eta}([a, \{b_1, \dots, b_h\}]) = \eta(a) \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(b_1) \vee_{\mathbf{T}} \dots \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(b_h), \tag{4.6}$$

noticing that $(a, \{b_1, \dots, b_h\}) \sim (a \vee_{\mathbf{M}} b_1 \vee_{\mathbf{M}} \dots \vee_{\mathbf{M}} b_h, \{b_1, \dots, b_h\})$ by (M1+) and Definition 4.2 (a) (b), and then using (4.1) and the assumption that $\widetilde{\eta}$ is a semilattice homomorphism. See [15], eq. (3.10) for full details. In particular, if $\widetilde{\eta}$ exists, it is unique.

We need to show that condition (4.6) determines a K-homomorphism $\widetilde{\eta}$ from $\widetilde{\mathbf{M}}$ to $\mathbf{T}$. First, we have to check that $\widetilde{\eta}$ is well-defined. By symmetry, it is enough to show that $(a, \{b_1, \dots, b_h\}) \precsim (c, \{d_1, \dots, d_k\})$ implies

$$\eta(a) \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(b_1) \vee_{\mathbf{T}} \dots \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(b_h) \leqslant \eta(c) \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(d_1) \vee_{\mathbf{T}} \dots \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(d_k). \tag{4.7}$$

Assume that clauses (a1) and (a2) in Definition 4.2 hold. From $d \sqsubseteq_{\mathbf{M}} d_1, \dots, d_k$ we get $\eta(d) \sqsubseteq_{\mathbf{T}} \eta(d_1), \dots, \eta(d_k)$ since η is a homomorphism, hence, $\eta(d) \leqslant_{\mathbf{T}} K_{\mathbf{T}}\eta(d_1) \vee_{\mathbf{T}} \dots \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(d_k)$ since $\mathbf{T}$ is assumed to be regular. From $a \leqslant_{\mathbf{M}} c \vee_{\mathbf{M}} d$ given by (a1), we get $\eta(a) \leqslant_{\mathbf{T}} \eta(c) \vee_{\mathbf{T}} \eta(d) \leqslant_{\mathbf{T}} \eta(c) \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(d_1) \vee_{\mathbf{T}} \dots \vee_{\mathbf{T}} K_{\mathbf{T}}\eta(d_k)$. By 4.2 (a2), for every $i \leqslant h$ there is $j \leqslant k$ such that $b_i \sqsubseteq_{\mathbf{M}} d_j$, hence $\eta(b_i) \sqsubseteq_{\mathbf{T}} \eta(d_j)$. By Remark 2.2 applied to the specialization reduct of $\mathbf{T}$, we get $K_{\mathbf{T}}\eta(b_i) \leqslant_{\mathbf{T}} K_{\mathbf{T}}\eta(d_j)$, hence (4.7) holds.

The fact that $\widetilde{\eta}$ is a semilattice homomorphism and a K-homomorphism follows from (4.1), the definitions of K and $\widetilde{\eta}$, the assumption that η is a semilattice homomorphism and the fact that $K_{\mathbf{T}}$ is a closure operation, by (d) in Definition 2.1 and since, by assumption, $\mathbf{T}$ is principal. See the displayed formulas at the bottom of page 2171 in [15] for details.

As in [13], [15], clause (4) follows from clause (3) applied to $\eta = \psi \circ \upsilon_{\mathbf{U}}$. □

Corollary 4.6. *Every multi-argument specialization semilattice can be embedded into the multi-argument specialization reduct of a closure semilattice.*

P r o o f. Immediate from Theorem 4.5 and Proposition 3.7 (2) (3). □

R e m a r k 4.7. (a) The assumption that $\widetilde{\eta}$ is a K-homomorphism is necessary in Theorem 4.5 (3); compare a parallel observation shortly before Remark 3.4 in [13]. On the other hand, as already remarked, $\upsilon_{\mathbf{M}}$, η and ψ in Theorem 4.5 are not required to preserve existing closures.

(b) If $\mathbf{S}$ is a finite specialization semilattice, then $\mathbf{S}$ is necessarily principal, by (1.1) and since every finite join of elements exists in $\mathbf{S}$. Thus, a result similar to Theorem 4.5 for *finite* specialization semilattices has a straightforward proof: it is enough to take $\widetilde{\mathbf{M}} = \mathbf{M}$.

On the other hand, the finiteness conclusion in Theorem 4.5 (1) is not immediate since, though any finite multi-argument specialization semilattice $\mathbf{M}$ is principal, it is not necessarily the case that $\mathbf{M}$ is regular. Compare Example 3.4 (a) and Remark 4.1 (b).

(c) Recall that a principal specialization semilattice is additive if $K(a \vee b) = Ka \vee Kb$ holds for all a and b. If $\mathbf{M}$ is a regular principal multi-argument specialization semilattice and the specialization reduct of $\mathbf{M}$ is additive, then $a \sqsubseteq b_1, \ldots, b_n$ is equivalent to $a \sqsubseteq b_1 \vee \ldots \vee b_n$ by additivity and the definition of regularity.

Hence, if a multi-argument specialization semilattice can be embedded into an additive regular principal multi-argument specialization semilattice, then $\sqsubseteq$ can be defined in terms of the semilattice operation and of the binary specialization $\sqsubseteq$ (as in Remark 3.4 (c), for related but different reasons. Notice that here no existential assumption is necessary; compare Remark 3.4 (d) (e)).

It follows that the structure theory of multi-argument specialization semilattices embeddable into *additive* regular principal multi-argument specialization semilattices is the same as the structure theory of specialization semilattices, which is presented in [13], [17]. In particular, in multi-argument specialization semilattices associated to topological spaces, the "multi"-structure is determined by $\sqsubseteq$ and $\vee$. Compare footnote 1 in the introduction.

(d) As mentioned, a result analogue to Theorem 4.5 for specialization semilattices has been proved in [15], Theorem 3.6. In view of the final sentence in Example 3.4 (f), Theorem 3.6 in [15] can be obtained as a corollary of Theorem 4.5 here; just consider only pre-regular multi-argument specialization semilattices $\mathbf{M}$ in the assumptions of Theorem 4.5, noticing that Definition 4.2 and [15], Definition 3.3, match and that $\mathbf{T}$ in Theorem 4.5 is assumed to be principal regular, in particular, pre-regular.

(e) Remark (d) above together with the fact that the proofs here and in [13] are quite simpler than the proofs in [15], confirm the idea that specialization semilattices (multi-argument specialization semilattices, respectively) are the "right" framework for dealing with subreducts of topological spaces (subreducts of closure spaces, respectively). On the other hand, while it is still true that every specialization semilattice is a subreduct of a closure space, the fact that the proofs in [15] are slightly more involved suggests that the correspondence between specialization semilattices and closure spaces is less natural. We have provided arguments in favor of this thesis in [17], Remark 5.9.

(f) It must be remarked, however, that the compactness theorem together with (b) above, can be used in order to provide a simple (but not constructive) proof that every specialization semilattice can be embedded into a principal specialization semilattice, thus the specialization reduct of a closure semilattice, by the result mentioned in Definition 2.1 (d). Indeed, the statement that a model is a principal specialization

semilattice can be expressed as a first-order sentence σ. Hence, if $\mathbf{S}$ is a specialization semilattice, it is enough to show that σ is consistent with the diagram $\mathrm{Diag}(\mathbf{S})$, by [10], Lemma 1.4.2. By the compactness theorem [10], Theorem 6.1.1, $\sigma \cup \mathrm{Diag}(\mathbf{S})$ has a model if and only if each finite subset has a model. But a finite subset σ_F of $\sigma \cup \mathrm{Diag}(\mathbf{S})$ involves only a finite number of (names for) elements of $\mathbf{S}$, hence, the subsemilattice $\mathbf{S_F}$ of $\mathbf{S}$ generated by the elements of S which have a name in σ_F is a finite submodel of $\mathbf{S}$ since a finitely generated semilattice is finite. Thus, $\mathbf{S_F}$ is a model for σ_F since $\mathbf{S_F}$ is principal, by (b) above.

The argument cannot be used as it stands in order to provide a proof of Corollary 4.6 since, as mentioned in (b), a finite multi-argument specialization semilattice is principal, but not necessarily regular. Similarly, the argument does not show that a specialization semilattice can be embedded into a principal *additive* specialization semilattice since principal does not imply additive. In any case an analogue of Theorem 4.5 holds in such a situation, as proved in [13], Theorem 3.2.

5. Embedding into a closure space and purely order-theoretical characterizations

In the previous section we have shown that every multi-argument specialization semilattice can be embedded into the multi-argument specialization reduct of a closure semilattice. Classical results show that every closure semilattice can be embedded into a closure space, hence, we get the corresponding result for multi-argument specialization semilattices. The next proposition is folklore, but we do not know a specific reference. In any case, we sketch a proof for the reader's convenience. To be strictly formal, in the next proposition a closure space is considered a structure of the form $(\mathcal{P}(X), \cup, K)$.

Proposition 5.1. *Every closure semilattice can be embedded into a closure space.*

Proof. If $\mathbf{S} = (\mathbf{S}, \vee_{\mathbf{s}}, \mathbf{K}_{\mathbf{s}})$ is a closure semilattice, let $\varphi\colon S \to \mathcal{P}(S)$ be the function defined by $\varphi(a) = \not\uparrow a = \{b \in S;\ a \not\leqslant_{\mathbf{s}} b\}$. Then $(\mathcal{P}(S), \cup, K)$ is a closure space, where for $x \subseteq S$, $Kx = \bigcap\{\not\uparrow a;\ a \in S, K_{\mathbf{s}} a = a, x \subseteq \not\uparrow a\}$. With the above definitions, φ is an embedding of closure semilattices. Full details appear in [17], Proposition 5.6 and, in a more general context, in [14], Corollary 3.5 (3). □

Theorem 5.2. *Every multi-argument specialization semilattice can be embedded into the multi-argument specialization semilattice associated to a closure space.*

P r o o f. By Proposition 3.7 (1) and the comment in Definition 3.5 (a), the embedding given by Proposition 5.1 is an embedding between the associated multi-argument specialization semilattices. The conclusion is then immediate from Corollary 4.6. □

The analogue of Theorem 5.2 holds for multi-argument specialization posets.

Proposition 5.3. *Every multi-argument specialization poset* $\mathbf{P}$ *can be embedded into the order-reduct of a multi-argument specialization semilattice (actually, a complete and atomic Boolean algebra), in such a way that all existing meets in* $\mathbf{P}$ *are preserved.*

Hence, by Theorem 5.2*, every multi-argument specialization poset* $\mathbf{P}$ *can be embedded into the multi-argument specialization poset associated to a closure space.*

P r o o f. Let $\mathbf{P} = (\mathbf{P}, \leqslant_{\mathbf{P}}, \sqsubseteq_{\mathbf{P}})$ be a multi-argument specialization poset. For every $a \in P$, let $\downarrow a = \{b \in P;\ b \leqslant_P a\}$ and let $S = \mathcal{P}(P)$. It is a classical fact [9], Chapter 1, Theorem 9.9 that the function ι which assigns to $a \in P$ the set $\downarrow a \in S$ is an order-embedding from $(P, \leqslant_P)$ to $(S, \subseteq)$ and ι preserves existing (possibly infinitary) meets. Since inclusion is the ordering associated to the join semilattice operation $\cup$, then ι is an order-embedding into the order-reduct of $(S, \cup)$.

We now want to give S the structure of a multi-argument specialization semilattice in such a way that ι is also an embedding for the multi-argument specialization. For every $X, Y_1, \ldots, Y_n \subseteq P$, let $X \sqsubseteq Y_1, \ldots, Y_n$ if, for every $c \in X$, there are $d_1 \in Y_1, \ldots, d_n \in Y_n$ such that $c \sqsubseteq_P d_1, \ldots, d_n$. We first check that $\mathbf{S} = (\mathcal{P}(\mathbf{P}), \cup, \sqsubseteq)$ is a multi-argument specialization semilattice. Indeed, properties (M1)–(M4) follow from the corresponding property in $\mathbf{P}$ in a straightforward way. As for (M5), if $X \sqsubseteq Y_1, \ldots, Y_n$, $X_1 \sqsubseteq Y_1, \ldots, Y_n$ and $c \in X \cup X_1$, then either $c \in X$ or $c \in X_1$, hence, in any case, by definition, $c \sqsubseteq_P d_1, \ldots, d_n$ for certain $d_1 \in Y_1, \ldots, d_n \in Y_n$. This applies to every $c \in X \cup X_1$, hence $X \cup X_1 \sqsubseteq Y_1, \ldots, Y_n$.

It remains to check that ι is an embedding for the multi-argument specialization. If $a \sqsubseteq_P b_1, \ldots, b_n$ and $c \in \downarrow a$, then $c \leqslant a$, hence $c \sqsubseteq_P b_1, \ldots, b_n$, by (M3). Since $b_1 \in \downarrow b_1, \ldots, b_n \in \downarrow b_n$, we get $\downarrow a \sqsubseteq \downarrow b_1, \ldots, \downarrow b_n$. Conversely, if $\downarrow a \sqsubseteq \downarrow b_1, \ldots, \downarrow b_n$, then since $a \in \downarrow a$, by definition there are $d_1 \in \downarrow b_1, \ldots, d_n \in \downarrow b_n$, that is, $d_1 \leqslant b_1, \ldots, d_n \leqslant b_n$ such that $a \sqsubseteq_P d_1, \ldots, d_n$. Then $a \sqsubseteq_P b_1, \ldots, b_n$ by (M2*+).

The second statement is immediate from Theorem 5.2 by composing the above embedding with the embedding given by Theorem 5.2. □

Turning to a different kind of representation, Theorem 4.5 can be used to provide a purely order-theoretical characterization of multi-argument specialization semilattices.

R e m a r k 5.4. Suppose that $\mathbf{M}$ is a poset, $\mathbf{S}$ is a semilattice, $\varphi, \psi\colon M \to S$ are functions such that φ is order preserving and

$$\varphi(a) \leqslant \psi(a) \tag{5.1}$$

and

$$\varphi(b) \leqslant \psi(c) \text{ implies } \psi(b) \leqslant \psi(c) \tag{5.2}$$

for every $a, b, c \in M$. If we set $a \sqsubseteq b_1, \ldots, b_n$ if $\varphi(a) \leqslant \psi(b_1) \vee \ldots \vee \psi(b_n)$, then $(M, \leqslant, \sqsubseteq)$ is a multi-argument specialization poset (clause (5.2) is used in order to get (M2)).

If in addition $\mathbf{M}$ is a semilattice and φ is a semilattice homomorphism, then $(M, \vee, \sqsubseteq)$ is a multi-argument specialization semilattice.

Corollary 5.5. *Every multi-argument specialization poset (semilattice)* $\mathbf{M}$ *can be represented in the fashion given by Remark* 5.4.

P r o o f. For multi-argument specialization semilattices use Theorem 4.5 (1) (2), letting $\mathbf{S} = \widetilde{\mathbf{M}}$, $\varphi = \upsilon_{\mathbf{M}}$, $\psi(a) = K\varphi(a)$ and using the conclusions that $\upsilon_{\mathbf{M}}$ is an embedding and that $\widetilde{\mathbf{M}}$ is principal regular. For posets do the same, after applying Proposition 5.3. □

A result similar to (and simpler than) Corollary 5.5 applies to specialization semilattices. The result has been stated without proof in [13], [17].

Theorem 5.6. *If* $(S, \vee)$ *and* $(T, \vee_{\mathbf{T}})$ *are semilattices,* $\varphi\colon S \to T$ *is a semilattice homomorphism and we set*

$$a \sqsubseteq_\varphi b \quad \text{if} \quad \varphi(a) \leqslant_{\mathbf{T}} \varphi(b) \tag{5.3}$$

for all $a, b \in S$, *then* $(S, \vee, \sqsubseteq_\varphi)$ *is a specialization semilattice.*

Conversely, if $(S, \vee, \sqsubseteq)$ *is a specialization semilattice, then there are a semilattice* $(T, \vee_{\mathbf{T}})$ *and a semilattice homomorphism* $\varphi\colon S \to T$ *such that* $\sqsubseteq$ *is equal to* $\sqsubseteq_\varphi$.

P r o o f. The first statement is elementary. To prove the second statement, assume that $(S, \vee, \sqsubseteq)$ is a specialization semilattice. In [13], [17] we have proved that every specialization semilattice $\mathbf{S}$ can be embedded into an additive principal specialization semilattice $\widetilde{\mathbf{S}}$.

Moreover, the embedding $\upsilon_{\mathbf{S}}\colon \mathbf{S} \to \widetilde{\mathbf{S}}$ in [13], Theorem 3.2 has the property that for every $a, b \in S$, $a \sqsubseteq b$ in $\mathbf{S}$ if and only if $K\upsilon_{\mathbf{S}}(a) \leqslant K\upsilon_{\mathbf{S}}(b)$ in $\widetilde{\mathbf{S}}$. This follows from

the definitions of $\sim$, K and υ_{s} in [13], Definition 3.1; see also Remark 3.4 therein. By the way, the same property is shared by the embedding κ constructed in the proof of [17], Theorem 5.5.

Set $\varphi(a) = K\upsilon_{\mathrm{s}}(a)$ for $a \in S$. Since $\widetilde{\mathbf{S}}$ is additive, then φ is a semilattice homomorphism. The property of υ_{s} mentioned in the previous paragraph means exactly that $a \sqsubseteq b$ if and only if $a \sqsubseteq_\varphi b$ for all $a, b \in S$. □

The arguments in the above theorem can be refined in order to show that the category of specialization semilattices is isomorphic to the category of semilattices with a congruence, and equivalent to the category $\mathcal{S}\mathfrak{Epi}$ of semilattice epimorphisms. Morphisms of $\mathcal{S}\mathfrak{Epi}$ are commuting squares of semilattice morphisms. See [16] for details.

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