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COLORING OF GRAPH OF RING WITH RESPECT TO IDEMPOTENTS

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Abstract. Let R be a ring with nonzero identity. A graph $G_{\rm Id}(R)$ of R with respect to idempotents of R has elements of R as vertices and distinct vertices x, y are adjacent if and only if $x+y$ is an idempotent of R. In this paper, we prove that $G_{\rm Id}(R)$ is weakly perfect and provide a condition for the perfectness of the same. Further, we characterize finite abelian rings for which the complement of $G_{\rm Id}(R)$ is connected.

Keywords: idempotent graph; weak perfect graph; zero-divisor graph

MSC 2020: 05C15, 05C17, 05C25

1. Introduction

The rings in this paper are associative and having unity; and all graphs are simple. An element h of a ring R such that $h^2=h$ is an *idempotent*. Clearly, in any ring with unity, 0 and 1 are idempotents, called trivial idempotents. Two idempotents h and k are *orthogonal* if $hk=kh=0$. Let $\mathrm{Id}(R)$ be the set of idempotents in R. For $a,b\in R$, we use the notation $a\leqslant b$ if $a=ab=ba$. Observe that this relation is *transitive* and *antisymmetric* on R, and it is reflexive only when restricted to $\mathrm{Id}(R)$. Thus, $(\mathrm{Id}(R),\leqslant)$ is a poset and $h'=1-h$ is the complementary idempotent of h in $\mathrm{Id}(R)$. Abian [1] extended this partial order to all the elements of a reduced ring and Anderson et al. [4] used Abian's partial order to determine the annihilator classes in a reduced commutative ring.

We color the vertices of a simple graph G such that any two adjacent vertices have distinct colors. The minimum number of colors required to color the vertices of G is the *chromatic number* $\chi(G)$ of G. A graph with every pair of distinct vertices adjacent is a *complete graph*. A *clique* in G is a complete subgraph of G, and the size of the

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maximal clique in G is the *clique number* $\omega(G)$ of G. Clearly $\chi(G) \geqslant \omega(G)$. A graph is *weakly perfect* if its chromatic number and clique number are the same. In 1988, Beck [6] introduced the zero-divisor graph for commutative rings. The *zero-divisor graph* $\Gamma(R)$ of a commutative ring R is a graph with elements of R as vertices and two vertices x and y are adjacent if $xy = 0$. Beck conjectured that $\Gamma(R)$ is weakly perfect whenever $\omega(\Gamma(R)) < \infty$. He proved that reduced rings and principal ideal rings are the classes of rings for which the conjecture is true. However, Anderson et al. [5] gave a counterexample of a commutative local ring for which the conjecture is not true.

Cvetko-Vah et al. [8] assigned a simple graph $G(R)$ to R whose vertex set is $\mathrm{Id}(R)$, and two vertices e and f are adjacent if and only if (1) $ef = fe = 0$, and (2) $eRf \neq 0$ or $fRe \neq 0$. It is evident from the second condition that if the idempotents of R are central, then $G(R)$ has no edges. Anderson et al. [3] defined and studied the zero-divisor graph $\Gamma(\mathrm{Idem}(R))$ of the idempotents of a commutative ring R. Later Akbari et al. [2] introduced the *idempotent graph* $I(R)$ of a ring R as the graph whose vertices are the nontrivial idempotents of R, and two distinct vertices h and k are adjacent if and only if $hk = kh = 0$. Observe that $h + k$ is an idempotent of R whenever h and k are orthogonal idempotents, which is a notable algebraic property. Clearly, for a commutative ring, $I(R)$ is a subgraph of $\Gamma(R)$. The interplay between the algebraic properties of R and graph-theoretic properties of $I(R)$ has been studied in [2], [8]. Patil et al. [10] studied the weak perfectness of the idempotent graph of a ring. Recently, Razaghi et al. [12] defined a graph $G_{\mathrm{Id}}(R)$ of R with respect to idempotents of R. The graph $G_{\mathrm{Id}}(R)$ has the elements of R as vertices and distinct vertices x, y are adjacent if and only if $x + y$ is an idempotent of R. Some basic properties such as connectedness, diameter and girth of $G_{\mathrm{Id}}(R)$ were studied in [12].

In the present paper, we study $G_{\mathrm{Id}}(R)$ in view of the coloring. We prove that $G_{\mathrm{Id}}(R)$ is weakly perfect. A graph is perfect if each of its induced subgraphs is weakly perfect. We provide a condition for the $G_{\mathrm{Id}}(R)$ to be perfect. For a graph G, the complement G^c is a graph with vertex set the same as G and two vertices are adjacent in G^c if and only if they are nonadjacent in G. We conclude by characterizing finite abelian rings for which the complement of $G_{\mathrm{Id}}(R)$ is connected. Examples are provided to delimit the results. We use $x \sim y$ to denote that the vertices x and y are adjacent.

2. Weak perfectness and some properties of $G_{\mathrm{Id}}(R)$

Recall that the *idempotent graph* $I(R)$ of a ring R is the graph whose vertices are the nontrivial idempotents of R, and two distinct vertices h and k are adjacent if and only if $hk = kh = 0$ (see Akbari et al. [2]).

Lemma 2.1. *The idempotent graph of R is a subgraph of $G_{\mathrm{Id}}(R)$.*

Proof. For the nontrivial idempotents $e, f \in R$, $e+f$ is an idempotent whenever $ef = fe = 0$. Hence, e and f are adjacent in $G_{\mathrm{Id}}(R)$ whenever they are adjacent in $I(R)$. Therefore the idempotent graph of R is a subgraph of $G_{\mathrm{Id}}(R)$. □

Corollary 2.2. *Let R be a ring with unity. If $\omega(G_{\mathrm{Id}}(R)) < \infty$, then the poset $(\mathrm{Id}(R), \leqslant)$ does not contain an infinite chain.*

Proof. Suppose that $\omega(G_{\mathrm{Id}}(R)) < \infty$. Since $I(R)$ is a subgraph of $G_{\mathrm{Id}}(R)$, $\omega(I(R)) < \infty$. Then the result follows from Patil et al. [10], Lemma 4. □

An idempotent in a ring is *primitive* if it cannot be written as a sum of two nonzero orthogonal idempotents. An element p of a poset P is an *atom* if 0 is the only element below p.

Corollary 2.3. *Let R be a ring with unity such that $\omega(G_{\mathrm{Id}}(R)) < \infty$. Then*

(1) *every set of pairwise orthogonal idempotents in R is finite,*
(2) *every nonzero idempotent in R contains a primitive idempotent.*

Proof. It follows from Corollary 2.2 and from Patil et al. [10], Corollary 1. □

Lemma 2.4. *Let R be a ring with unity such that $\omega(G_{\mathrm{Id}}(R)) < \infty$. Then every maximal set S of pairwise orthogonal idempotents in R has the following properties.*

(1) *S is finite and contains zero.*
(2) $\sum\limits_{x\in S} x = 1$.
(3) *For any nonzero $x \in R$ there exists $s \in S$ such that $xs \neq 0$.*
(4) *If S is a maximal set of pairwise orthogonal idempotents in R with largest cardinality, then every nonzero element of S is a primitive idempotent.*

Proof. (1) Since $\omega(G_{\mathrm{Id}}(R)) < \infty$, by Corollary 2.3, R contains only finitely many pairwise orthogonal idempotents; hence S is finite. Also, the maximality of S clearly gives $0 \in S$.

(2) Let $S = \{0, s_1, s_2, \ldots, s_n\}$. We claim that $\sum\limits_{i=1}^{n} s_i = 1$. Let $e = \sum\limits_{i=1}^{n} s_i$. On the contrary, if $e \neq 1$, then $1 - e \neq 0$. By Corollary 2.3, there exists a primitive idempotent, say f, such that $f \leqslant 1 - e$. Observe that $fs_i = s_i f = 0$. Then $S \cup \{f\}$ is a set of pairwise orthogonal idempotents which contains S, a contradiction to the maximality of S. Therefore $\sum\limits_{i=1}^{n} s_i = 1$.

(3) For any nonzero element $x \in R$ we have $x = \sum\limits_{i=1}^{n} s_i x$. Hence, there exists an i such that $s_i x \neq 0$.

(4) Suppose S is a maximal set of pairwise orthogonal idempotents in R with largest cardinality. Let $g \leqslant s_i$ for a nonzero $s_i \in S$. If $g \neq 0$, then $\{g, s_i - g\} \cup (S \setminus \{s_i\})$ is a set of pairwise orthogonal idempotents in R having cardinality greater than S, a contradiction to choice of S. Hence $g = 0$, consequently, s_i is an atom in the poset $(\mathrm{Id}(R), \leqslant)$. Then by Patil et al. [10], Lemma 3, s_i is a primitive idempotent. □

Remark 2.5. Han et al. [9], Theorem 2.3 and Remark 1 essentially proved that $ch(R) = 2$ if and only if $1 + e \in \mathrm{Id}(R)$ for all nonzero $e \in \mathrm{Id}(R)$.

Lemma 2.6. *Let R be a ring with unity and $ch(R) \neq 2$, where $ch(R)$ is the characteristic of the ring R. Then there exists a primitive idempotent $e \in R$ such that $1 + e \notin \mathrm{Id}(R)$.*

Proof. Since $ch(R) \neq 2$, by Remark 2.5, there is a nonzero $e \in \mathrm{Id}(R)$ such that $1 + e \notin \mathrm{Id}(R)$. We assume that e is minimal such, i.e., for any idempotent $f \leqslant e$, $1 + f \in \mathrm{Id}(R)$. We claim that e is an atom in the poset $\mathrm{Id}(R)$. If not, there exists nonzero $e_1 < e$, i.e., $e_1 = ee_1 = e_1e$. Then $e - e_1 < e$, i.e., $e - e_1 = (e - e_1)e = e(e - e_1)$. By assumption, both $1 + e_1$ and $1 + e - e_1$ are in $\mathrm{Id}(R)$. Observe that $(1+e_1)(1+e-e_1) = 1+e-e_1+e_1+e_1e-e_1^2 = 1+e-e_1+e_1+e_1-e_1 = 1+e$ and $(1+e-e_1)(1+e_1) = 1+e$, i.e., $1+e_1$ and $1+e-e_1$ commute with each other. Hence, $(1+e)^2 = [(1+e_1)(1+e-e_1)]^2 = (1+e_1)^2(1+e-e_1)^2 = (1+e_1)(1+e-e_1) = 1+e$, i.e., $1 + e \in \mathrm{Id}(R)$, a contradiction. Therefore e must be an atom in the poset $\mathrm{Id}(R)$. By Patil et al. [10], Lemma 3, e is a primitive idempotent. □

A ring is said to be *abelian* if its every idempotent is central. Now we prove that the graph of an abelian ring with respect to idempotents is weakly perfect.

Theorem 2.7. *Let R be an abelian ring with unity such that $\omega(G_{\mathrm{Id}}(R)) < \infty$. Then $G_{\mathrm{Id}}(R)$ is weakly perfect.*

Proof. Since $\omega(G_{\mathrm{Id}}(R)) < \infty$, by Lemma 2.4, every maximal set of pairwise orthogonal idempotents in R is finite, contains zero and the sum of all the elements of that set is equal to 1. If R contains no nontrivial idempotent, then by Razaghi et al. [12], Theorem 3.2, $G_{\mathrm{Id}}(R)$ is a bipartite graph, hence $\chi(G_{\mathrm{Id}}(R)) = \omega(G_{\mathrm{Id}}(R)) = 2$; and we are through. Suppose that R contains at least one nontrivial idempotent. If R is a Boolean ring, then $G_{\mathrm{Id}}(R)$ is a complete graph, which yields $\chi(G_{\mathrm{Id}}(R)) = \omega(G_{\mathrm{Id}}(R)) = |R|$, and we are done again. Now suppose that R is not a Boolean ring, i.e., there exists $x \in R$ such that $x^2 \neq x$. Let $S = \{s_1, s_2, \ldots, s_n\}$ be a maximal set of pairwise orthogonal idempotents in R of largest cardinality. Clearly $1 \notin S$. By Lemma 2.4, $\sum\limits_{i=1}^{n} s_i = 1$ and for any nonzero element $x \in R$, there exists an i such that $s_i x \neq 0$. Since R is abelian, $s_i + s_j$ is an idempotent in R for any i, j with $i \neq j$. Consequently, the elements of S form a clique in $G_{\mathrm{Id}}(R)$, hence $n \leqslant \omega(G_{\mathrm{Id}}(R))$.

Now let $x \in R \setminus (S \cup \{1\})$. If x is nonidempotent, then 0 is nonadjacent to x. If x is idempotent, then by maximality of S, x is nonadjacent to a nonzero element in S. Thus, any $x \in R \setminus (S \cup \{1\})$ is nonadjacent to an element of S.

Next, we assign n different colors to $s_1, s_2, \ldots, s_n$. Let x and y be any two adjacent vertices in $G_{\mathrm{Id}}(R)$ that are not in S. From the above paragraph there is an s_i nonadjacent to x. Without loss of generality, assume that s_1 is nonadjacent to x. Let s_j be nonadjacent to y. We claim that $j \neq 1$, i.e., there are distinct elements (among the s_i's) nonadjacent to x and y. On the contrary assume that s_1 is the only (among the s_i's) nonadjacent to both x and y, i.e., $x + s_j$ and $y + s_j$ are idempotents for $j = 2, 3, \ldots, n$. If $s_1 = 0$, then x and y are nonidempotents (because 0 is adjacent to idempotent). Then $s_j \neq 0$ for $j \in \{2, \ldots, n\}$. Let s_k be such that $xs_k \neq 0$. Then $(x + s_j)s_k$ is an idempotent $\leqslant s_k$, which yields $(x + s_j)s_k = 0$ or $(x + s_j)s_k = s_k$. Since $xs_k \neq 0$, we have $xs_k = s_k$. Now x is adjacent to s_k, $x + s_k$ is an idempotent, i.e., $(x + s_k)^2 = x + s_k$, which gives $x^2 + 2s_k = x$. Then multiplication by s_k gives $x^2 s_k + 2s_k = xs_k$, which yields (using $xs_k = s_k$) $2s_k = 0$. Therefore $x^2 + 2s_k = x$ gives $x^2 = x$, giving x and s_1 adjacent, a contradiction. Therefore $s_1 \neq 0$. Hence $s_j = 0$ for some $j \in \{2, \ldots, n\}$. Without loss of generality, suppose that $s_2 = 0$. Then $s_j \neq 0$ for each $j \in \{3, \ldots n\}$. Since x and y are adjacent to $s_2 = 0$, both x and y are idempotents. Also, s_1 is a primitive idempotent such that $xs_1 \leqslant s_1$. If $xs_1 = 0$, then $x + s_1$ is an idempotent making x and s_1 adjacent, a contradiction. Hence $xs_1 \neq 0$. Therefore $xs_1 = s_1$. Similarly $ys_1 = s_1$. Then $s_1 = xs_1 = xys_1$, hence $xy \neq 0$. Since x and y are adjacent, $x + y$ is an idempotent which gives $2xy = 0$. From Lemma 2.4 we have $\sum\limits_{j=1}^{n} s_j = 1$. Hence

$$x = x\left(\sum_{j=1}^{n} s_j\right) = xs_1 + \sum_{\substack{j \neq 1, \\ xs_j \neq 0}}^{n} xs_j,$$

but $xs_j = s_j$, whenever $xs_j \neq 0$ (s_j being primitive idempotent). Thus, we get

$$x = s_1 + \sum_{\substack{j \neq 1, \\ xs_j \neq 0}}^{n} s_j.$$

Let

$$t = \sum_{\substack{j \neq 1, \\ xs_j \neq 0}}^{n} s_j.$$

Hence $x = s_1 + t$. Recall that $x + s_j$ is idempotent for each $j \neq 1$. Consequently, $(x + s_j)^2 = x + s_j$, i.e., $x + xs_j + xs_j + s_j = x + s_j$, which gives $2s_j = 0$ (since

$xs_j = s_j$), hence $s_j = -s_j$ whenever $xs_j \neq 0$. This also gives $t = -t$, which leads to $x + t = s_1$. Then adding y to both sides we get $y + x + t = y + s_1$. Now $(y+x+t)^2 = (y+x+t)(y+x+t) = y^2 + xy + yt + xy + x^2 + xt + yt + xt + t^2 = y + 2xy + 2yt + 2xt + t$. Using $x^2 = x$, $y^2 = y$, $2t = 0$ and $t^2 = t$ (since s_j's are pairwise orthogonal), we get $(y + x + t)^2 = (y + x + t)$, i.e., $y + s_1$ is an idempotent, which makes y and s_1 adjacent, a contradiction.

Thus, there exists s_j ($j \neq 1$) such that y is nonadjacent to s_j. Hence, we can assign the color of s_1 to x and the color of s_j to y.

Now there are two cases.

Case I. Suppose that $ch(R) \neq 2$. Then by Lemma 2.6, there exists a primitive idempotent s_k such that $1 + s_k \notin \mathrm{Id}(R)$, i.e., 1 is not adjacent to s_k in $G_{\mathrm{Id}}(R)$. Hence, we can assign the color of s_k to 1. Thus, $\chi(G_{\mathrm{Id}}(R)) \leqslant n$. Therefore $n \leqslant \omega(G_{\mathrm{Id}}(R)) \leqslant \chi(G_{\mathrm{Id}}(R)) \leqslant n$. Thus $\omega(G_{\mathrm{Id}}(R)) = \chi(G_{\mathrm{Id}}(R)) = n$.

Case II. Suppose that $ch(R) = 2$. Then by Remark 2.5, $1 + e \in \mathrm{Id}(R)$, for any nonzero $e \in \mathrm{Id}(R)$. Recall that we are assuming R is non-Boolean ring containing a nontrivial idempotent. Let $S_1 = S \cup \{1\}$. Then the elements of S_1 form a clique in $G_{\mathrm{Id}}(R)$. We assign different colors to elements of S_1. Observe that for any two adjacent vertices x and y there exist different elements of S that are non adjacent to x and y, respectively. Consequently, $|S_1| \leqslant \omega(G_{\mathrm{Id}}(R)) \leqslant \chi(G_{\mathrm{Id}}(R)) \leqslant |S_1|$. Therefore $\omega(G_{\mathrm{Id}}(R)) = \chi(G_{\mathrm{Id}}(R)) = |S_1|$.

Thus in any case, $G_{\mathrm{Id}}(R)$ is weakly perfect. □

We illustrate Theorem 2.7 with the following example.

Example 2.8. Let $R_1 = \mathbb{Z}_{10}$. Then $\mathrm{Id}(R_1) = \{0, 1, 5, 6\}$, the maximal set of pairwise orthogonal idempotents in R_1 is $S = \{0, 5, 6\}$ and $G_{\mathrm{Id}}(R_1)$ is depicted in Figure 1. Observe that $\omega(G_{\mathrm{Id}}(R_1)) = \chi(G_{\mathrm{Id}}(R_1)) = 3$.

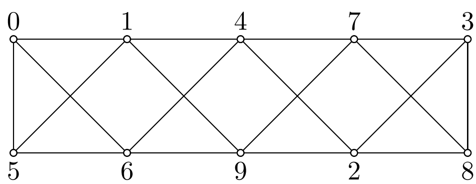


Figure 1. $G_{\mathrm{Id}}(\mathbb{Z}_{10})$.

Let $R_2 = (\mathbb{Z}_2 \times \mathbb{Z}_2)[x]/\langle x^2 \rangle = \{e_0 = (0,0), e_1 = (1,0), e_2 = (0,1), e_3 = (1,1), e_1x, e_2x, e_3x, e_1x + e_1, e_1x + e_2, e_1x + e_3, e_2x + e_1, e_2x + e_2, e_2x + e_3, e_3x + e_1, e_3x + e_2, e_3x + e_3\}$. Then $S = \{e_0, e_1, e_2\}$ is a maximal set of pairwise idempotents in R_2, $S_1 = S \cup \{e_3\}$ and $G_{\mathrm{Id}}(R_2)$ is depicted in Figure 2. Observe that $\omega(G_{\mathrm{Id}}(R_2)) = \chi(G_{\mathrm{Id}}(R_2)) = 4$.

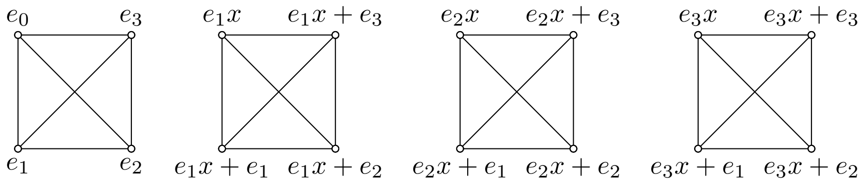


Figure 2. $G_{\rm Id}((\mathbb{Z}_2\times\mathbb{Z}_2)[x]/\langle x^2\rangle)$.

The following example shows that the condition 'abelian' in Theorem 2.7 is sufficient but not necessary.

Example 2.9. Let

$$\begin{aligned} R = M_2(\mathbb{Z}_2) = \Big\{ &\mathbf{1} = \begin{bmatrix} 0 & 0\\ 0 & 0\end{bmatrix}, \mathbf{2} = \begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}, \mathbf{3} = \begin{bmatrix} 0 & 1\\ 0 & 0\end{bmatrix}, \mathbf{4} = \begin{bmatrix} 0 & 0\\ 1 & 0\end{bmatrix},\\ &\mathbf{5} = \begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}, \mathbf{6} = \begin{bmatrix} 1 & 1\\ 0 & 0\end{bmatrix}, \mathbf{7} = \begin{bmatrix} 1 & 0\\ 1 & 0\end{bmatrix}, \mathbf{8} = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix},\\ &\mathbf{9} = \begin{bmatrix} 0 & 1\\ 1 & 0\end{bmatrix}, \mathbf{10} = \begin{bmatrix} 0 & 1\\ 0 & 1\end{bmatrix}, \mathbf{11} = \begin{bmatrix} 0 & 0\\ 1 & 1\end{bmatrix}, \mathbf{12} = \begin{bmatrix} 1 & 1\\ 1 & 0\end{bmatrix},\\ &\mathbf{13} = \begin{bmatrix} 1 & 1\\ 0 & 1\end{bmatrix}, \mathbf{14} = \begin{bmatrix} 1 & 0\\ 1 & 1\end{bmatrix}, \mathbf{15} = \begin{bmatrix} 0 & 1\\ 1 & 1\end{bmatrix}, \mathbf{16} = \begin{bmatrix} 1 & 1\\ 1 & 1\end{bmatrix} \Big\}\end{aligned}$$

(here bold numbers are used to denote matrices), with usual addition of matrices and matrix multiplication.

Here $\mathrm{Id}(R) = \{\mathbf{1},\mathbf{2},\mathbf{5},\mathbf{6},\mathbf{7},\mathbf{8},\mathbf{10},\mathbf{11}\}$ and $G_{\rm Id}(R)$ is as depicted (using Sage) in Figure 3. Observe that $\omega(G_{\rm Id}(R)) = \chi(G_{\rm Id}(R)) = 4$. Here R is not abelian but still $G_{\rm Id}(R)$ is weakly perfect.

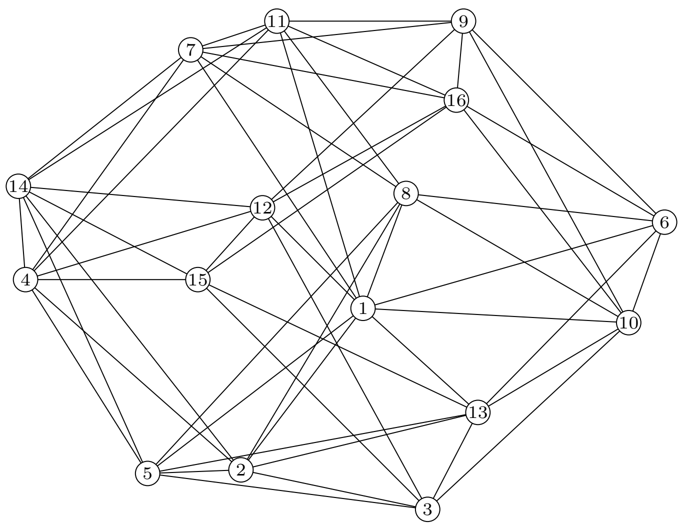


Figure 3. $G_{\rm Id}(M_2(\mathbb{Z}_2))$.

Recall that a graph G is a *perfect graph* if the clique number $\omega(H)$ is the same as the chromatic number $\chi(H)$ for every induced subgraph H of G. Chudnovsky et al. [7] characterized perfect graphs as follows.

Theorem 2.10 (Strong perfect graph theorem [7]). *A graph G is perfect if and only if it has no induced subgraph isomorphic either to a cycle of odd length at least 5, or to the complement of such a cycle.*

Next we provide a condition for $G_{\mathrm{Id}}(R)$ to be perfect.

Theorem 2.11. *Let R be a ring with unity. If $G_{\mathrm{Id}}(R)$ is perfect, then R has at most four primitive idempotents.*

Proof. Suppose that $G_{\mathrm{Id}}(R)$ is perfect. On the contrary, assume that R contains more than four primitive idempotents. Let e_1, e_2, e_3, e_4 and e_5 be distinct primitive idempotents in R. Then $(e_1+e_2) \sim (e_3+e_4) \sim (e_1+e_5) \sim (e_2+e_3) \sim (e_4+e_5)$ is an induced 5-cycle in $G_{\mathrm{Id}}(R)$, a contradiction to the fact that $G_{\mathrm{Id}}(R)$ is perfect (by Theorem 2.10). This completes the proof. □

For the zero-divisor graph $\Gamma(R)$ of a finite reduced commutative semiring R, Patil et al. [11], Corollary 2.5 proved that if $\omega(\Gamma(R)) \leqslant 4$, then $\Gamma(R)$ is perfect. The analogous result is not true for $G_{\mathrm{Id}}(R)$ (see example below). We provide rings with exactly two and three primitive idempotents for which $G_{\mathrm{Id}}(R)$ is not perfect.

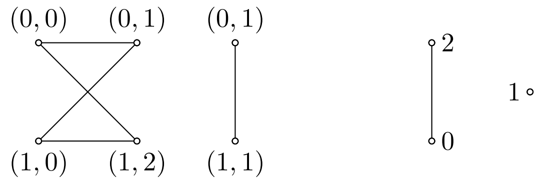
Example 2.12. Let $R_1 = \mathbb{Z}_3 \times \mathbb{Z}_5$. Then $\mathrm{Id}(R_1) = \{e_1 = (0,0), e_2 = (1,0), e_3 = (0,1), e_4 = (1,1)\}$, and $S = \{e_1, e_2, e_3\}$ is a maximal set of pairwise orthogonal idempotents in R_1. Hence, by Theorem 2.7, $\omega(G_{\mathrm{Id}}(R_1)) = 3$. Here e_2, e_3 are the only primitive idempotents in R_1. Let $a_1 = (2,0), a_2 = (0,4) \in R_1$. Then $e_2 \sim e_3 \sim a_2 \sim e_4 \sim a_1 \sim e_2$ is an induced 5-cycle in $G_{\mathrm{Id}}(R_1)$ which has no chord, hence, by Theorem 2.10, $G_{\mathrm{Id}}(R_1)$ is not perfect.

Let $R_2 = \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$. Then $\mathrm{Id}(R_2) = \{f_1 = (0,0,0), f_2 = (1,0,0), f_3 = (0,1,0), f_4 = (0,0,1), f_5 = (1,1,0), f_6 = (1,0,1), f_7 = (0,1,1), f_8 = (1,1,1)\}$ and $S = \{f_1, f_2, f_3, f_4\}$ is a maximal set of pairwise orthogonal idempotents in R_2. Hence, by Theorem 2.7, $\omega(G_{\mathrm{Id}}(R_2)) = 4$. Here f_2, f_3, f_4 are the only primitive idempotents in R_2. Let $b_1 = (2,2,2), b_2 = (1,2,2) \in R_2$. Then $f_1 \sim f_8 \sim b_1 \sim b_2 \sim f_7 \sim f_1$ is an induced 5-cycle in $G_{\mathrm{Id}}(R_2)$ which has no chord, hence by Theorem 2.10, $G_{\mathrm{Id}}(R_2)$ is not perfect.

We conclude by characterizing finite abelian rings with unity for which $G_{\mathrm{Id}}(R)^c$ is connected.

Theorem 2.13. *Let R be a finite abelian ring. Then the complement of $G_{\mathrm{Id}}(R)$ is connected if and only if R is not isomorphic to a Boolean ring or to $\mathbb{Z}_3$ or to $\mathbb{Z}_2 \times \mathbb{Z}_3$.*

Proof. If R is a Boolean ring, then $G_{\mathrm{Id}}(R)$ is a complete graph, hence, its complement is a null graph. The graphs $G_{\mathrm{Id}}(\mathbb{Z}_2 \times \mathbb{Z}_3)^c$ and $G_{\mathrm{Id}}(\mathbb{Z}_3)^c$ are as depicted in Figure 4, which are disconnected graphs.



$G_{\mathrm{Id}}(\mathbb{Z}_2 \times \mathbb{Z}_3)^c$ $\quad$ $G_{\mathrm{Id}}(\mathbb{Z}_3)^c$

Figure 4. $G_{\mathrm{Id}}(R)^c$ disconnected.

Suppose that R is not a Boolean ring. If R has exactly two idempotents, then by Razaghi et al. [12], $G_{\mathrm{Id}}(R)$ is either disconnected or is a path. Hence, its complement is disconnected only if $|R| = 2$ or 3, i.e., $R = \mathbb{Z}_2$ or $R = \mathbb{Z}_3$. Suppose that R has exactly four idempotents, say, $\mathrm{Id}(R) = \{0, 1, e, 1-e\}$. Suppose that $ch(R) = 2$. If the additive group R is not generated by $\mathrm{Id}(R)$, then by Razaghi et al. [12], $G_{\mathrm{Id}}(R)$ is disconnected, hence, its complement is connected. If the additive group R is generated by $\mathrm{Id}(R)$, then R is a Boolean ring (as we are considering $ch(R) = 2$), which is a contradiction, since we are considering non-Boolean rings. Suppose that $ch(R) \neq 2$. Then $R = eR \times (1-e)R$, where eR and $(1-e)R$ both contain exactly two idempotents. Then $\mathrm{Id}(R) = \{(0,0), (e,0), (0,1-e), (e,1-e)\}$. If $|eR| = 2$ and $|(1-e)R| = 3$ (since R is not Boolean), then $R = \mathbb{Z}_2 \times \mathbb{Z}_3$, which yields that $G_{\mathrm{Id}}(R)^c$ is disconnected. Suppose $|eR| = 2$ and $|(1-e)R| > 3$. Then $2e = 0$ and $2(1-e) \neq 0$, hence there exists $x \in (1-e)R$ such that $x \neq -x$ and $x \neq 1-e$. Then $(e, 1-e)$ and $(0, 1-e)$ are adjacent in $G_{\mathrm{Id}}(R)^c$, which gives a path $(0,0) \sim (0,x) \sim (0,1-e)$ in $G_{\mathrm{Id}}(R)^c$. Similarly, there are paths $(0,0) \sim (e,x) \sim (e,1-e)$ and $(0,0) \sim (e,x) \sim (e,0)$ in $G_{\mathrm{Id}}(R)^c$. Thus, $(0,0)$ is connected to every other vertex in $G_{\mathrm{Id}}(R)^c$ through a path. Consequently, $G_{\mathrm{Id}}(R)^c$ is connected. Similarly, $G_{\mathrm{Id}}(R)^c$ is connected if $|eR| \geqslant 3$ and/or $|(1-e)R| \geqslant 3$.

Next, suppose that $|\mathrm{Id}(R)| \geqslant 6$. By Patil et al. [10], Theorem 4, there is a path in $G_{\mathrm{Id}}(R)^c$ connecting any two nontrivial idempotents of R. Since R is a non-Boolean ring, there exists $x \in R$ which is not an idempotent. Then x is adjacent to 0 in $G_{\mathrm{Id}}(R)^c$. We claim that there is a nonzero idempotent in R that is adjacent to x in $G_{\mathrm{Id}}(R)^c$. Let S be a maximal set of pairwise idempotents in R. If there is an element in S which is adjacent to x in $G_{\mathrm{Id}}(R)^c$, then we are done. Suppose that x is nonadjacent to every element of S in $G_{\mathrm{Id}}(R)^c$, i.e., $x+e$ is an idempotent in R for each $e \in S$. Observe that $x+e_1 \neq x+e_2$ for distinct $e_1, e_2 \in S$. Let f be a nonzero idempotent in S. Then by assumption x is nonadjacent to f in $G_{\mathrm{Id}}(R)^c$, i.e., $x+f$ is an idempotent in R. Then as shown in the proof of Theorem 2.7, there exists an element $g \in S$ such that $x+f+g$ is not an idempotent in R (clearly $g \neq 0$, as $x+f$ is idempotent). Then $f+g$ is a nonzero idempotent in R that is adjacent to x in $G_{\mathrm{Id}}(R)^c$. Thus, $0 \sim x \sim f+g$ is a path in $G_{\mathrm{Id}}(R)^c$. Hence, in $G_{\mathrm{Id}}(R)^c$, 0 is connected to every idempotent and to every nonidempotent element of R through a path. Thus, $G_{\mathrm{Id}}(R)^c$ is connected in this case. $\square$

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