Rozhledy matematicko-fyzikální

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Hrátky s dělitelností. Řešení – Solution

Rozhledy matematicko-fyzikální, Vol. 100 (2025), No. 4, 39–45

Persistent URL: `http://dml.cz/dmlcz/153275`

Hrátky s dělitelností. Řešení – Solution

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V tomto článku vyřešíme úlohy z minulých Matematických oříšků.

Úloha 1

Ukažte, že mezi libovolnými 39 po sobě jdoucími přirozenými čísly je alespoň jedno, jehož součet číslic je dělitelný 11.

A v obecnější podobě.

Úloha 2

Jaké je minimální číslo M_z, pro které mezi každými M_z po sobě jdoucími přirozenými čísly existuje alespoň jedno číslo, jehož součet číslic v zápisu o základu z je dělitelný $z+1$?

V první části článku rozřešíme úlohu 1 pro dekadický zápis čísel, tj. pro $z = 10$. Ta je napsána česky. Druhá část, řešící úlohu obecně pro zápis čísel v soustavě o libovolném základu z, je prezentována v angličtině.

Použité značení

Každé přirozené číslo a lze jednoznačně zapsat v *číselné soustavě o základu* $z \in \mathbb{N}$, $z \geq 2$,

$$a = a_s z^s + a_{s-1} z^{s-1} + \cdots + a_2 z^2 + a_1 z + a_0, \tag{1}$$

kde $a_s \neq 0, a_0, a_1, \ldots, a_s \in \{0, 1, \ldots, z-1\}$. Tento zápis budeme značit stručněji

$$a = (a_s, a_{s-1} \ldots a_2 a_1 a_0)_z. \tag{2}$$

Dále definujme součet číslic

$$\sigma_z(a) = \sum_{t=0}^{s} a_t. \tag{3}$$

Pomocí funkce σ_z je možno formulovat úlohu 2 následovně:

Úloha 2

Pro daný základ z najděte nejmenší číslo M_z splňující podmínku, že každá posloupnost M_z po sobě jdoucích přirozených čísel obsahuje číslo x takové, že $\sigma_z(x)$ je dělitelné číslem $z+1$.

Číslo M_z budeme dále v textu značit $\mathrm{Min}(z)$.

Řešení úlohy 1

V této první části budeme čísly rozumět přirozená čísla spolu s nulou, zapsaná v desítkové číselné soustavě. Každému přirozenému číslu

$$a = (a_s a_{s-1} \dots a_2 a_1 a_0)_{10}$$

je tedy přiřazen součet číslic $\sigma(a) = \sum_{t=0}^{s} a_t$.

Konečnou posloupnost po sobě jdoucích (celých) čísel nazveme *segment*. Segment délky d začínající číslem a budeme značit symbolem ${}_a\mathcal{S}(d)$:

$${}_a\mathcal{S}(d) = (a, a+1, a+2, \dots, a+d-1).$$

Tedy

${}_0\mathcal{S}(10) = (0, 1, 2, 3, 4, 5, 6, 7, 8, 9)$,
pro $1 \le k \le 9$ je ${}_0\mathcal{S}(k) = (0, 1, 2, \dots, k-1)$ a
${}_{10-k}\mathcal{S}(k) = (10-k, 10-k+1, 10-k+2, \dots, 9)$.

Každá posloupnost *posledních číslic* čísel segmentu ${}_a\mathcal{S}(39)$ zapsaných v desítkové soustavě má tedy jeden z těchto tvarů:

$${}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(9),$$

$${}_1\mathcal{S}(9)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10),$$

$$\dots\dots\dots$$

$${}_t\mathcal{S}(10-t)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(t-1) \text{ pro } 2 \le t \le 8,$$

$$\dots\dots\dots$$

$${}_9\mathcal{S}(1)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(8).$$

Proto každá z těchto posloupností obsahuje posloupnost

$${}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10)\ {}_0\mathcal{S}(10) = (0, 1, 2, \dots, 8, 9, 0, 1, 2, \dots, 8, 9, 0, 1, 2, \dots, 8, 9).$$

To jsou poslední číslice segmentu, který označíme

$${}_m\mathcal{S}(30) = (m, m+1, m+2, \dots, m+29).$$

Nyní uvažujme příslušnou posloupnost *číslic*

$(k_1, k_2, k_3, \dots, k_9, k_{10}, k_{11}, k_{12}, k_{13}, \dots, k_{19}, k_{20}, k_{21}, k_{22}, k_{23}, \dots, k_{29}, k_{30})$, které jsou v tomto zápisu čísel příslušného segmentu ${}_m\mathcal{S}(30)$ na předposledním místě. Je-li $k_1 = k \neq 9$, je příslušná posloupnost párů posledních číslic

$(k\,0, k\,1, k\,2, k\,3, \ldots, k\,9, k+1\,0, k+1\,1, k+1\,2, \ldots, k+1\,9, \ldots, k_{29}\,8, k_{30}\,9).$

Mimochodem, zde $k_{29} = k_{30} = 0$ nebo $k+2$ v závislosti na tom, zda $k = 8$, nebo $k \neq 8$. Omezme se na prvních 20 čísel tohoto segmentu a označme $\sigma(m) = N$. Potom posloupnost

$$(\sigma(m), \sigma(m+1), \ldots, \sigma(m+9), \sigma(m+10), \sigma(m+11), \ldots, \sigma(m+19)) = \\ = (N, N+1, \ldots, N+9, N+1, N+2, \ldots, N+10),$$

a proto mezi těmito čísly musí být číslo dělitelné 11.

Jestliže $k_1 = k = 9$, pak $k_{11} \neq 9$ a předešlý postup aplikujeme na posloupnost $(k_{11}, k_{12}, \ldots, k_{30})$. Tím je dokázáno, že v dekadickém případě každý segment délky 39 obsahuje číslo, jehož součet číslic, tj. $\sigma-$hodnota je násobek 11.

Dodejme ještě, že kromě segmentu

$${}_{999981}\mathcal{S}(38) = (999981, \ldots\ldots, 1000018)$$

zmíněného v zadání úlohy, kdy $\sigma-$hodnota žádného z jeho prvků není násobkem 11, je takových segmentů délky 38 nekonečně mnoho. Čtenář se může přesvědčit, že pro každé $t = 0, 1, 2, \ldots,$

$${}_{99\cdots 9981}\mathcal{S}(38) = (99\cdots 9981, \ldots\ldots, 100\cdots 0018),$$

kde počet devítek (a příslušných nul) je $11t+4$, má tuto vlastnost. Tedy $\mathrm{Min}(10) = 39$.

Solution of Problem 2

The solution presented in the first part is a special case of the general situation where the numbers are expressed in base-z numeral system. Let us express in English the general problem that we want to solve.

Problem 2

For a given base z find the minimum number M_z satisfying the condition that every sequence of M_z consecutive non-negative integers contains a number x such that $\sigma_z(x)$ of the digits of x in the base-z system is divisible by the number $z+1$.

In the following, we denote M_z by $\mathrm{Min}(z)$.

Following the first part of the article, denote by ${}_a\mathcal{S}(d)$ the sequence of d consecutive numbers starting with the number a and call it a segment. Let us call the segment ${}_a\mathcal{S}(z)$ a *basic z-sequence* if the starting number a is divisible by z. Thus in the base-z form (1) of the numbers of this segment only last digits change, running from 0 to $z-1$. The σ_z-values (3) of these numbers are also consecutive numbers.

We begin the solution of the problem with a few preliminary observations.

Observation 1. *A basic z-sequence contains a number whose σ_z-value is divisible by $z+1$, unless it starts with a number whose σ_z-value has a remainder 1 when divided by $z+1$.*

Let ${}_aS(z)$ be a basic z-sequence and let $\sigma_z(a) = k$. Then the σ_z-values of the numbers in ${}_aS(z)$ form a sequence ${}_kS(z) = (k, k+1, \ldots, k+z-1)$. Among the $z+1$ consecutive numbers $k-1, k, \ldots, k+z-1$, exactly one is divisible by $z+1$; this number lies in ${}_kS(z)$ if and only if $k-1$ is not a multiple of $z+1$.

Observation 2. *Let ${}_aS(2z)$ be a sequence consisting of two consecutive basic z-sequences, where $a = (a_s a_{s-1} \ldots a_2 a_1 0)_z$. If $a_1 < z-1$, then ${}_aS(2z)$ contains a number whose σ_z-value is divisible by $z+1$.*

In this case ${}_aS(2z) = {}_aS(z)_{a+z}S(z)$, where $a + z = (a_s a_{s-1} \ldots a_2 a_1 + 10)_z$, so $\sigma_z(a+z) = \sigma_z(a)+1$. Therefore, by Observation 1, either ${}_aS(z)$ or ${}_{a+z}S(z)$ contains a number with σ_z-value divisible by $z+1$.

Observation 3. *If z is odd, then in any sequence ${}_aS(2z)$ consisting of two consecutive basic z-sequences, there is a number with σ_z-value divisible by $z+1$.*

Let $a = (a_s a_{s-1} \ldots a_2 a_1 0)_z$ be the starting number of the first sequence. By Observation 2, the statement is true if $a_1 < z-1$. Thus, it suffices to consider the case where

$$a = (a_s \ldots a_{k+1}(z-1)\ldots(z-1)0)_z$$

for some $k > 0$, where $a_{k+1} < z-1$. Then the starting number of the second basic z-sequence is

$$a + z = (a_s \ldots a_{k+2}(a_{k+1}+1)0\ldots 0)_z.$$

Thus, $\sigma_z(a+z)-\sigma_z(a)=-k(z-1)+1$. But this is an odd number, so it cannot be divisible by $z+1$. Hence $\sigma_z(a)$ and $\sigma_z(a+z)$ yield different remainders when divided by $z+1$. It now follows from Observation 1 that at least one of ${}_aS(z)$ and ${}_{a+z}S(z)$ contains a number with σ_z-value divisible by $z+1$.

Based on these observations, we can deduce the following statement.

Corollary. *Any sequence of $4z-1$ consecutive numbers contains a number with σ_z-value divisible by $z+1$. In fact, if z is odd, then any sequence of $3z-1$ consecutive numbers contains a number with σ_z-value divisible by $z+1$.*

Indeed, a sequence of $4z-1$ consecutive numbers must contain three consecutive basic z-sequences (otherwise the length of the sequence could not be greater than $(z-1)+2z+(z-1)=4z-2$), and at least one of the first and second of these sequences would start with a number satisfying the condition for Observation 1. When z is odd then we can do even better: a sequence of $3z-1$ numbers contains at least two consecutive basic z-sequences and thus following Observation 3, there is a number in the sequence whose σ_z-value is divisible by $z+1$.

Now we want to show that these are actually the minimum values we are looking for.

Theorem.

$$\operatorname{Min}(z)=\begin{cases}4z-1 & \text{if } z \text{ is even,}\\ 3z-1 & \text{if } z \text{ is odd.}\end{cases}$$

Let us first consider the case where z is odd. As a counterexample we need a sequence ${}_{a-z+1}S(3z-2)={}_{a-z+1}S(z-1)_aS(z)_{a+z}S(z-1)$, where $a=(a_s\dots a_1 0)_z$ and the remainder of $\sigma_z(a)$ by $z+1$ is 1 (see Observations 3 and 1). Actually, ${}_1S(3z-2)=(1,2,\dots,3z-2)$ is such a sequence and the table below shows that none of the σ_z-values are divisible by $z+1$.

a	1	$\cdots$	$z-1$	z	$\cdots$	$(2z-1)$	$2z$	$\cdots$	$3z-2$
in base z	$\overline{1}$	$\cdots$	$\overline{(z-1)}$	$\overline{10}$	$\cdots$	$\overline{1(z-1)}$	$\overline{20}$	$\cdots$	$\overline{2(z-2)}$
$\sigma_z(a)$	1	$\cdots$	$z-1$	1	$\cdots$	z	2	$\cdots$	z

In fact, if we choose $0<a_1<z-1$, while assuring that $\sigma_z(a)$ has a remainder 1 by $z+1$, then the σ_z-values of the numbers in the sequence

will still have the same remainders as above and thus those sequences will also be counterexamples.

Suppose now that z is even and let us try to construct a sequence of $4z-2$ consecutive numbers such that the σ_z-value of none of these numbers is divisible by $z+1$. By Observation 2, this sequence of numbers must not contain more than two consecutive basic z-sequences. Hence it contains exactly two such sequences, which are preceded and followed by $z-1$ numbers. Furthermore, the two basic z-sequences must start with a and $a+z$ such that $\sigma_z(a)$ and $\sigma_z(a+z)$ both give remainder 1 when divided by $z+1$, otherwise Observation 1 would ensure the existence of a number with σ_z-value divisible by $z+1$. So a must be of the form described in the proof of Observation 3. With the notation used there, we have $\sigma_z(a+z)-\sigma_z(a) = -k(z-1)+1$. The divisibility conditions imply that both this and $\sigma_z(a+z)-1 = a_s+\ldots+a_{k+1}$ should be divisible by $z+1$. The first is equivalent to saying that

$$(z+1) \mid -k(z-1)+1+k(z+1) = 2k+1,$$

and clearly, $k=\frac{z}{2}$ satisfies this. The easiest way to satisfy the second divisibility condition is to set $a_s = \ldots = a_{k+1} = 0$. So we choose the sequence

$${}_{a-z+1}S(4z-2) = {}_{a-z+1}S(z-1)_a S(z)_{a+z} S(z)_{a+2z} S(z-1)$$

with $a = ((z-1)\ldots(z-1)(z-1)0)_z$, where the number of the digits equal to $(z-1)$ is $k=\frac{z}{2}$. Now we need to check that also the σ_z-values of the numbers in the "incomplete" subsequences ${}_{a-z+1}S(z-1)$ and ${}_{a+2z}S(z-1)$ are not divisible by $z+1$. The first number of the sequence ${}_{a-z+1}S(z-1)$ is $a-z+1 = ((z-1)\ldots(z-1)(z-2)1)_z$, and thus

$$\sigma_z(a-z+1) = \left(\frac{z}{2}-1\right)(z-1)+(z-2)+1 = \frac{z}{2}(z-1) = \sigma_z(a).$$

Since $\frac{z}{2}(z-1) = (z+1)\left(\frac{z}{2}-1\right)+1$, the remainder of both $\sigma_z(a)$ and of $\sigma_z(a-z+1)$ is 1 when divided by $z+1$. Furthermore, $a+z = z^{k+1} = (1\ 0\ \ldots\ 0\ 0\ 0)_z$, and $a+2z = z^{k+1}+z = (1\ 0\ \ldots\ 0\ 1\ 0)_z$, with σ_z-values 1 and 2. Thus the remainder terms of the σ_z-values of the $4z-2$ numbers are

$$(1,\ldots,z-1)(1,\ldots,z)(1,\ldots,z)(2,\ldots,z).$$

This shows that neither of the σ_z-values of the $4z-2$ consecutive numbers in ${}_{a-z+1}S(4z-2)$ is divisible by $z+1$.

As a concrete illustration, consider the case where $z = 60$. To find a counterexample in this case, we need numbers that have at least 32 digits (in base 60) and at least 56 digits in the decimal system. To be more specific, using the following standard way of denoting the digits in the sexagesimal system

1	2	3	4	5	6
7	8	9	$A = 10$	$B = 11$	$C = 12$
...	...	...	...	...	...
$V = 31$	$W = 32$	$X = 33$	$Y = 34$	$Z = 35$	$a = 36$
$b = 37$	$c = 38$	$d = 39$	$e = 40$	$f = 41$	$g = 42$
...	...	...	...	...	...
$t = 55$	$u = 56$	$v = 57$	$w = 58$	$x = 59$	0

we may display the situation in the following table:

segment	value of σ
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxw1	1770
13264435183244001473986559999999999999999999999999999881	decimal
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxw2	1771
13264435183244001473986559999999999999999999999999999882	decimal
....	...
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxww	1827
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxwx	1828
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx0	1770
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx1	1771
....	...
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	1829
10000000000000000000000000000000	1
10000000000000000000000000000001	2
....	...
1000000000000000000000000000000x	60
10000000000000000000000000000010	2
10000000000000000000000000000011	3
....	...
100000000000000000000000000001w	60
132644351832440014739865560000000000000000000000000000118	decimal

Table 1 The segment of the length 238 in the sexagesimal system and the translation into the decimal system