Kybernetika

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Kybernetika, Vol. 62 (2026), No. 2, 190–205

Persistent URL: `http://dml.cz/dmlcz/153631`

KYBERNETIKA — VOLUME *62* (2026), NUMBER 2, PAGES 190–205

PRESCRIBED-TIME OUTER SYNCHRONIZATION OF STOCHASTIC COMPLEX SYSTEMS WITH AND WITHOUT PINNING CONTROL

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This paper investigates prescribed-time outer synchronization in drive-response complex networks with the perturbation of noise. We propose two control frameworks: a general prescribed-time controller and its pinning control variant. Through the stability theory of stochastic differential equations, we establish sufficient conditions that ensure outer synchronization in drive-response systems within a prescribed finite convergence time, independent of both initial conditions and system parameters. The proposed controllers are shown to be continuous and bounded. Numerical simulations demonstrate the effectiveness and feasibility of the proposed control schemes.

Keywords: prescribed-time, outer synchronization, drive-response systems, pinning control, noise

Classification: 34F05, 34H10

1. INTRODUCTION

Complex networks are ubiquitous in natural and engineered systems, including social networks, biological networks, ecological systems, and transportation systems [1, 27, 32]. Among the diverse collective behaviors in complex networks, synchronization has emerged as a central focus of research. From the rhythmic flashes of fireflies and the synchronized applause of audience to the firing of neurons and the coordinated pulses of cardiac pacemakers, synchronization is a universal principle observable in both natural and engineered systems [6, 10, 11, 25, 30]. Synchronization in complex networks primarily encompasses two categories: inner synchronization, which refers to coordinated dynamics among nodes within a single network, and outer synchronization, which describes the coordination between two or more networks. The study of synchronization in complex networks is of significant importance due to its wide-ranging applications across disciplines [2, 34]. This phenomenon is exemplified both in the synchronization of infectious disease spread across interconnected communities [5] and in the synchronization of neural networks, such as the clock neurons in fruit flies, which plays a crucial role in regulating circadian rhythms, cognition, and behavior [26].

DOI: 10.14736/kyb-2026-2-0190

For practical applications that require synchronization to be achieved within a finite time, finite-time controllers are designed specifically to achieve such outer synchronization [7, 9, 18, 29]. However, the convergence time in finite-time synchronization depends on the initial state, rendering it difficult to determine accurately in large-scale networks where the initial state cannot be measured. To overcome this limitation, fixed-time control strategies have been proposed and implemented in two-layer complex networks [20, 21]. However, the convergence time of the such methods has been bounded by a fixed upper limit determined by the control parameters. In practical applications, a desirable objective is to achieve synchronization within a convergence time that is independent of both initial conditions and control parameters, and can be prescribed arbitrarily according to task requirements. Thus, prescribed-time control strategies have been developed [15], where the convergence time can be arbitrarily pre-specified independently of both the system's initial state and control parameters. Prescribed-time synchronization has emerged as an active research area and has achieved a series of important results [4, 13, 33]. However, achieving prescribed-time outer synchronization in drive-response networks remains unclear and needs further investigation.

Although prescribed-time controller can ensure synchronization within a prescribed time, their practical implementation is fundamentally constrained by the requirement for full state observability. This limitation has motivated the development of pinning control strategies [22], which apply control inputs to only a subset of network nodes, thereby significantly reducing the number of required entities. The effectiveness of pinning control technique has been demonstrated in the study of network synchronization [8, 17, 28]. Building upon these developments, the present work extends pinning control to investigate outer synchronization phenomena in complex networks. Moreover, in practical implementations, stochastic disturbances are inevitable due to signal transmission interference, measurement inaccuracies, and environmental fluctuations. It is therefore imperative to account for stochastic effects in the development of networked control and coordination schemes [14, 16, 23].

Based on the above discussion, we investigate the prescribed-time outer synchronization in drive-response networked systems under stochastic perturbations, both with and without pinning control. A prescribed-time controller is designed based on a time-varying function, ensuring that the system's convergence time is completely independent of initial conditions and control parameters, and can be set arbitrarily by the designer. Based on the principle of prescribed-time stability theory, sufficient conditions are established for achieving prescribed-time outer synchronization. Further, we propose a prescribed-time pinning control strategy, which effectively overcomes the limitation of requiring full-state observability and tackles the practical challenge of controlling an excessive number of nodes in large-scale networks.

The rest of the paper is organized as follows. Section 2 introduces several preliminaries and the problem statement. Section 3 develops the theoretical framework for prescribed-time outer synchronization in drive-response systems, and then extends this analysis to pinning control strategies in drive-response systems. Numerical simulations validating the theoretical results are presented in Section 4. Section 5 concludes the paper.

Notations: Throughout this paper, we adopt the following mathematical notations. $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^N$ is real-valued N-dimensional vector space and $\mathbb{R}^{N\times N}$

denotes the space of real $N \times N$ matrices. For a matrix/vector A, A^T denotes its transpose. $\lambda_{\max}(\cdot)$ is the largest eigenvalue of a matrix. The symmetric matrix A is negative definite if all eigenvalues of A are negative. I_n is the n-dimension identity matrix. The Kronecker product is denoted as $\otimes$ and $\mathbb{E}(\cdot)$ represents the expectation operator.

2. PROBLEM FORMULATION AND PRELIMINARIES

In this section, we consider a complex network system comprising N nodes, with their dynamics described by the following equation:

$$dx_i(t) = [f(x_i(t)) + \sum_{j=1}^{N} c_{ij}\Gamma x_j(t)]dt, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i(t) = (x_{i1}(t), x_{i2}(t), \ldots, x_{in}(t))^T \in \mathbb{R}^n$ represents the state vector of the i-th node, $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuously differentiable nonlinear vector function describe the node dynamics, and $\Gamma \in \mathbb{R}^{n\times n}$ represents the inner coupling matrix between connected nodes. The network topology is represented by the coupling matrix $C = (c_{ij})_{N\times N}$, where entries c_{ij} are defined as follows: if there exists a link from node j to node i, then $c_{ij} > 0(i \neq j)$, otherwise, $c_{ij} = 0(i \neq j)$. The diagonal elements of matrix C are defined as $c_{ii} = -\sum_{j=1, j\neq i}^{N} c_{ij}$.

To study the outer synchronization between two coupled complex network systems, we consider the system described in Eq. (1) as the drive network and construct the corresponding response network as follows:

$$dy_i(t) = [f(y_i(t)) + \sum_{j=1}^{N} d_{ij}\Gamma y_j(t) + u_i(t)]dt + \sigma_i(e_i(t))\,dW(t), \quad i = 1, 2, \ldots, N, \tag{2}$$

where $y_i(t) = (y_{i1}(t), y_{i2}(t), \ldots, y_{in}(t))^T \in \mathbb{R}^n$ represents the state vector of the i-th node, Γ and the coupling matrix $D = (d_{ij})_{N\times N}$ are defined analogously to those in the drive system (1). The control input $u_i(t)$ is a prescribed-time synchronization controller, and the synchronization error between the response system (2) and the drive system (1) is defined as $e_i(t) = y_i(t) - x_i(t)\,(i = 1, 2, \ldots, N)$. The stochastic dynamics are characterized by the noise intensity function $\sigma_i : \mathbb{R}^n \to \mathbb{R}^{n\times m}$, an m-dimensional Brownian motion $W(t) = (w_1(t), w_2(t), \ldots, w_m(t))^T$, whose derivative $\dot{W}$ represents m-dimensional white noise. The initial states of the drive and response network systems are given by $x(0) = \left(x_1(0)^T, x_2(0)^T, \ldots, x_N(0)^T\right)^T$ and $y(0) = \left(y_1(0)^T, y_2(0)^T, \ldots, y_N(0)^T\right)^T$.

The definition of prescribed-time stochastic outer synchronization is given as follows:

Definition 2.1. The systems (1) and (2) are said to achieve prescribed-time stochastic outer synchronization, if for any initial states $x_i(t_0)$ and $y_i(t_0)$, there exists a user-prescribed positive constant T_0 (settling-time) such that

$$\mathbb{P}\{\|y_i(t, y_i(t_0)) - x_i(t, x_i(t_0))\| = 0, \forall t \geq t_1\} = 1,$$

where $t_1 = t_0 + T_0$, and t_0 is the initial time.

The following assumptions and lemmas are introduced to assist in the proof of the main results.

Assumption 2.1. Assume that the function $f:\mathbb{R}^n\to\mathbb{R}^n$ satisfies the following condition, i. e., there exists a constant $p>0$ such that $\forall\, x,\ y\in\mathbb{R}^n$

$$(y-x)^T[f(y)-f(x)]\le p(y-x)^T\Gamma(y-x),$$

where Γ is the inner-coupling matrix of complex networks.

Assumption 2.2. The noise intensity function $\sigma_i(x_i(t))$ satisfies the following condition, i. e., there exists a constant $q>0$ such that

$$\mathrm{trace}(\sigma_i^T(x_i(t))\sigma_i(x_i(t)))\le 2qx_i^T(t)\Gamma x_i(t).$$

Assumption 2.3. Assume that the inner connection matrix $\Gamma\in\mathbb{R}^{n\times n}$ is a symmetric positive definite matrix.

Lemma 2.1. (Wang et al. [24]) A time-varying scale function is presented as follows:

$$\mu(t)=\begin{cases}\left(\frac{T_0}{T_0+t_0-t}\right)^h, & t\in[t_0,t_1),\\ 1, & t\in[t_1,\infty),\end{cases}\tag{3}$$

where $h>2$ is a user-defined real constant. $T_0\ge T_s\ge 0$, where T_s is the time required for signal processing and information transmission.

Clearly, for any positive real number ($p>0$), the function $\mu(t)^{-p}$ is monotonically decreasing on $[t_0,t_1)$, and $\mu(t_0)^{-p}=1$, $\lim_{t\to t_1^-}\mu(t)^{-p}=0$. Therefore, for all $t\ge 0$, we have $0<\mu(t)^{-p}\le 1$, and

$$\dot{\mu}(t)=\begin{cases}\frac{h}{T_0}\mu(t)^{1+\frac{1}{h}}, & t\in[t_0,t_1),\\ 0, & t\in[t_1,\infty).\end{cases}$$

here we use the right-hand derivative of $\mu(t)$ at $t=t_1$, denoted as $\dot{\mu}(t_1)$.

Consider the system

$$\dot{x}(t)=f(t,x(t)),\quad t\in\mathbb{R}^+,\quad x(t_0)=x_0,\tag{4}$$

where $x(t)\in\mathbb{R}^n$, $f:\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}^n$ is a locally uniformly bounded vector function in time. Let $V(x(t),t):\mathbb{R}^+\times\mathbb{R}^n\to\mathbb{R}$ be a continuously differentiable function satisfying $V(0,t)=0$, and $V(x(t),t)>0$ when $x(t)\neq 0$. If there exists a real number $b>0$ such that

$$\dot{V}(x(t),t)=-bV(x(t),t)-2\frac{\dot{\mu}}{\mu}V(x(t),t),\quad t\in[t_0,\infty),\tag{5}$$

where $\dot{V}(x(t),t)=\frac{\partial V}{\partial x}(x)f(t,x(t))$, then system (4) is globally prescribed-time stable with prescribed time T_0. Moreover, the following equations hold:

$$V(t)\le\mu^{-2}\exp^{-b(t-t_0)}V(t_0),t\in[t_0,t_1),$$

$$V(t)\equiv 0,t\in[t_1,\infty).$$

Lemma 2.2. (Yu et al. [31]) Let M be a symmetric matrix of the form

$$M = \begin{pmatrix} P & Q \\ Q^T & S \end{pmatrix},$$

where $P \in \mathbb{R}^{m\times m}$, $Q \in \mathbb{R}^{m\times(n-m)}$, $S \in \mathbb{R}^{(n-m)\times(n-m)}$. Then M is negative definite ($M \prec 0$) if and only if one of the following conditions holds:
(1) $P \prec 0, \quad S - Q^T P^{-1} Q \prec 0$;
(2) $S \prec 0, \quad P - Q S^{-1} Q^T \prec 0$.

Lemma 2.3. (Chen and Jiao [3]) Consider the stochastic differential equation

$$\mathrm{d}x = f(x)\,\mathrm{d}t + g(x)\,\mathrm{d}W(t), \tag{6}$$

where $f(x) : \mathbb{R}^n \to \mathbb{R}^n$ and $g(x) : \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are continuous functions with initial conditions satisfying $f(0) = 0$, $g(0) = 0$. The vector $x(t) \in \mathbb{R}^n$ is the state vector of the system, and $x(0)$ represents the initial state of the system. $W = (W_1(t), W_2(t), \ldots, W_m(t))^T$ is an m-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Assuming equation (6) has a unique global solution $x(t, x(0))$, $0 \le t < \infty$, then, Itô formula can be derived. For $V : \mathbb{R}^n \times \mathbb{R}^+ \to \mathbb{R}^+$ being an Itô process, then

$$\mathrm{d}V = \mathcal{L}V \cdot \mathrm{d}t + \frac{\partial V}{\partial x} \cdot g \cdot \mathrm{d}W(t),$$

and the definition of operator $\mathcal{L}V$ is

$$\mathcal{L}V = \frac{\partial V}{\partial x} \cdot f + \frac{1}{2}\mathrm{trace}\left[g^T \cdot \frac{\partial^2 V}{\partial x^2} \cdot g\right],$$

where $\frac{\partial V}{\partial x} = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right)$, $\frac{\partial^2 V}{\partial x^2} = \left[\frac{\partial^2 V}{\partial x_i \partial x_j}\right]_{n\times n}$.

3. MAIN RESULTS

In this section, we present theoretical conditions for achieving stochastic outer synchronization between the drive system ((1)) and the response system (2). The analysis systemically addresses two distinct cases: the case without pinning control and the case with pinning control.

3.1. Prescribed-time outer synchronization of stochastic complex network systems without pinning control

To achieve prescribed-time outer synchronization in drive-response systems, we design the following controller:

$$u_i(t) = \sum_{j=1}^{N} (c_{ij} - d_{ij})\,\Gamma x_j(t) - \left(k + c\frac{\dot{\mu}(t)}{\mu(t)}\right)\Gamma e_i(t), \tag{7}$$

where $k > 0$ and $c > 0$ are design parameters, and $\mu(t)$ is the scaling function defined in Eq. (3).

Theorem 3.1. Suppose the function $f(\cdot)$ satisfies Assumption 2.1, the noise intensity function $\sigma_i(\cdot)$ satisfies Assumption 2.2, and the inner coupling matrix $\Gamma \in \mathbb{R}^{n\times n}$ satisfies Assumption 2.3. If the control gains k and c satisfy

$$k > p + q + \lambda_{\max}\left(\tilde{D}\right), c \geq \frac{1}{\lambda_{\min}(\Gamma)},$$

where $\tilde{D} = \frac{D+D^T}{2} = \begin{pmatrix} (D_1)_{l\times l} & (D_2)_{l\times(N-l)} \\ (D_2^T)_{(N-l)\times l} & (D_4)_{(N-l)\times(N-l)} \end{pmatrix}$, then the systems (1) and (2) can achieve prescribed-time outer synchronization with a settling time T_0 under the designed controller (7).

Proof. Firstly, based on networks (1) and (2), as well as controller (7), we can obtain the following error system:

$$de_i(t) = \left[f(y_i) - f(x_i) + \sum_{j=1}^{N} d_{ij}\Gamma e_j(t) - \left(k + c\frac{\dot{\mu}(t)}{\mu(t)}\right)\Gamma e_i(t)\right] dt + \sigma_i(e_i(t))\, dW(t). \tag{8}$$

Let $e(t) = (e_1(t), e_2(t), \ldots, e_N(t))^T$, and consider the following Lyapunov function:

$$V(t) = \frac{1}{2}e^T(t)\, e(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)\, e_i(t).$$

From the Itô formula and the definition of operator $\mathcal{L}V$, one has

$$\begin{aligned} \mathcal{L}V(t) = &\sum_{i=1}^{N} e_i^T(t)\left[f(y_i) - f(x_i) + \sum_{j=1}^{N} d_{ij}\Gamma e_j(t) - \left(k + c\frac{\dot{\mu}(t)}{\mu(t)}\right)\Gamma e_i(t)\right] \\ &+ \frac{1}{2}\sum_{i=1}^{N} \operatorname{trace}\left(\sigma_i^T(e_i(t))\, \sigma_i(e_i(t))\right). \end{aligned}$$

According to Assumptions 2.1 and 2.2, we can obtain

$$\begin{aligned} \mathcal{L}V(t) \leq\, & p\sum_{i=1}^{N} e_i^T(t)\, \Gamma e_i(t) + \sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} d_{ij}\Gamma e_j(t) \\ &- \left(k + c\frac{\dot{\mu}(t)}{\mu(t)}\right)\sum_{i=1}^{N} e_i^T \Gamma e_i(t) + q\sum_{i=1}^{N} e_i^T(t)\, \Gamma e_i(t), \end{aligned} \tag{9}$$

and

$$\sum_{i=1}^{N} e_i^T(t)\, \Gamma e_i(t) = e^T(t)\, (I_n \otimes \Gamma)\, e(t), \tag{10}$$

$$\sum_{i=1}^{N} e_i^T(t)\sum_{j=1}^{N} d_{ij}\Gamma e_j(t) = e^T(t)\, (D \otimes \Gamma)\, e(t) = e^T(t)\left(\hat{D} \otimes \Gamma\right) e(t), \tag{11}$$

wherein, $\hat{D} = \frac{D+D^T}{2}$.

Thus, substituting Eqs. (10) − (11) into Eq.(9), we can obtain

$$\mathcal{L}V(t) \leq e^T(t)\left(\left((p+q-k)I_n + \hat{D}\right)\otimes\Gamma\right)e(t) - c\frac{\dot{\mu}(t)}{\mu(t)}e^T(t)(I_n\otimes\Gamma)e(t).$$

As we assume the coupling matrix Γ is positive definite, then according to Lemma 2.1, if we can prove $(p+q-k)I_n + \hat{D}$ is negative definite, then the drive-response systems achieve prescribed-time outer synchronization. If the control gain k satisfies

$$k > p+q+\lambda_{\max}\left(\hat{D}\right),$$

then $(p+q-k)I_n + \hat{D}$ is negative definite. Further, as Γ is positive definite, according to the property of kKronecker product, the matrix $\left(\left((p+q-k)I_n + \hat{D}\right)\otimes\Gamma\right)$ is negative definite, and then $\lambda_{\max}\left(\left((p+q-k)I_n + \hat{D}\right)\otimes\Gamma\right) < 0$ holds. Let $\bar{b} = -\lambda_{\max}\left(\left((p+q-k)I_n + \hat{D}\right)\otimes\Gamma\right)/2$, where $\bar{b} > 0$. Furthermore, according to the condition $c \geq \frac{1}{\lambda_{\min}(\Gamma)}$, we can obtain

$$\mathcal{L}V(t) \leq -\bar{b}V(t) - 2\frac{\dot{\mu}(t)}{\mu(t)}V(t).$$

Using the properties of the Wiener process yields

$$\frac{\mathrm{d}\mathbb{E}[V(t)]}{\mathrm{d}t} = \mathbb{E}[\mathcal{L}V(t)] \leq -\bar{b}\mathbb{E}[V(t)] - 2\frac{\dot{\mu}(t)}{\mu(t)}\mathbb{E}[V(t)].$$

According to Lemma 2.1, the expectation of the Lyapunov function satisfies

$$\mathbb{E}[V(t)] \leq \mu(t)^{-2}e^{-\bar{b}(t-t_0)}\mathbb{E}[V(t_0)].$$

As $t\to t_1$, $\mu(t)^{-2}\to 0$, and $e^{-\bar{b}(t-t_0)}$, $\mathbb{E}[V(t_0)]$ are bounded, therefore $\mathbb{E}[V(t)]\to 0$. As $V(t)$ measures synchronization error, then the system (1) and (2) is globally prescribed-time stable, i. e., the drive-response systems achieve synchronization with setting time t_1.

Consider $t\in[t_1,\infty)$,

$$\frac{\mathrm{d}\mathbb{E}[V(t)]}{\mathrm{d}t} \leq 0,$$

thus $\mathbb{E}[V(t)]$ is a monotonically decreasing function, so

$$0 \leq \mathbb{E}[V(t)] \leq \mathbb{E}[V(t_1^-)] = 0,$$

we can obtain that, for $t\in[t_1,\infty)$, $\mathbb{E}[V(t)]\equiv 0$, implying that the drive-response system can maintain synchronization. The drive-response system can achieve synchronization within the prescribed time T_0, and consistently maintain it for $t\in[t_1,\infty)$. □

Corollary 3.1. Consider a drive-response system, where the response system shares the same coupling matrices as the drive system (1). The dynamics are governed by

$$dy_i(t) = [f(y_i(t)) + \sum_{j=1}^{N} c_{ij}\Gamma y_j(t) + u_i(t)]dt + \sigma_i(e_i(t))\,dW(t), \quad i = 1, 2, \ldots, N, \tag{12}$$

here, the coupling matrix $C = (c_{ij})_{N\times N}$ is the same as that of the drive system (1). The controller becomes

$$u_i(t) = -\left(k + c\frac{\dot{\mu}(t)}{\mu(t)}\right)\Gamma e_i(t).$$

Under the conditions of Theorem 3.1, the drive system (1) and response system (12) achieve prescribed-time outer synchronization under the above controller.

Remark 3.1. According to Definition 2.1, outer synchronization is achieved when corresponding nodes between the drive and response systems synchronize, regardless of whether the individual drive or response systems are internally synchronized. This definition imposes no restrictions on the coupling matrices C (drive network) and D (response network).

Remark 3.2. From the proof of Theorem 3.1, it is evident that the time-varying function $\mu(t)$ plays an important role in achieving prescribed-time synchronization. Although $\mu(t) \to \infty$ when $t \to t_1$, its reciprocal $\mu(t)^{-1} \to 0$. Therefore, our approach not only guarantees that drive-response systems achieve outer synchronization within the prescribed-time T_0, but also ensures the boundedness of the controller. Numerical simulations further confirm that the designed controller is both continuous and uniformly bounded, validating our theoretical results.

3.2. Prescribed-time outer synchronization of stochastic complex network systems with pinning control

In this section, the outer synchronization between the drive and response systems under pinning control is investigated. Assume that the first l nodes in the system are controlled. Incorporating the pinning control strategy, we design the controller in the following form:

$$u_i(t) = \begin{cases} \sum_{j=1}^{N}(c_{ij} - d_{ij})\Gamma x_j(t) - \frac{\dot{\mu}}{\mu}e_i(t) - k\Gamma e_i(t), & i \in \{1, 2, \ldots l\}, \\ \sum_{j=1}^{N}(c_{ij} - d_{ij})\Gamma x_j(t) - \frac{\dot{\mu}}{\mu}e_i(t), & \text{otherwise}. \end{cases} \tag{13}$$

Theorem 3.2. Under Assumptions 2.1 - 2.3, if the coupling matrix D of the response system (2) and the parameter k satisfy the following conditions:

$$D_4 + (p+q)I_{N-l} \prec 0,$$
$$k > p + q + \lambda_{\max}\left(D_1 - D_2(D_4 + (p+q)I_{N-l})^{-1}D_2^T\right),$$

where $\tilde{D} = \frac{D+D^T}{2} = \begin{pmatrix} (D_1)_{l\times l} & (D_2)_{l\times(N-l)} \\ (D_2^T)_{(N-l)\times l} & (D_4)_{(N-l)\times(N-l)} \end{pmatrix}$. Then the response system (2) achieves prescribed-time outer synchronization with the drive system (1) under the designed controller with pinning control (13).

Proof. Under the controller specified in (13), the evolution of synchronization error between drive system (1) and response system (2) is governed by:

$$de_i(t) = [f(y_i) - f(x_i) + \sum_{j=1}^{N} d_{ij}\Gamma e_j(t) - \frac{\dot{\mu}(t)}{\mu(t)} e_i(t) - \alpha_i k \Gamma e_i(t)]dt + \sigma_i(e_i(t))\, dW(t),$$

where $\alpha_i = 1$ when $i \in \{1, 2, \ldots, l\}$, otherwise $\alpha_i = 0$.

Define Lyapunov function as

$$V(t) = \frac{1}{2}\sum_{i=1}^{N} e_i^T(t)\, e_i(t).$$

Applying Itô formula, and using Assumptions 2.1-2.2, we can obtain:

$$\begin{aligned}
\mathcal{L}V(t) &= \sum_{i=1}^{N} e_i^T(t)[f(y_i) - f(x_i)] + \sum_{i=1}^{N} e_i^T(t) \sum_{j=1}^{N} d_{ij}\Gamma e_i(t) \\
&\quad - \sum_{i=1}^{N} e_i^T(t) \frac{\dot{\mu}(t)}{\mu(t)} e_i(t) - \sum_{i=1}^{N} e_i^T \alpha_i k \Gamma e_i(t) + \frac{1}{2}\sum_{i=1}^{N} \operatorname{trace}\left(\sigma_i^T(e_i(t))\sigma_i(e_i(t))\right) \\
&\le (p+q)\, e^T(t)(I_n \otimes \Gamma)\, e(t) + e^T(t)\left(\frac{D + D^T}{2} \otimes \Gamma\right) e(t) - \frac{\dot{\mu}(t)}{\mu(t)} e^T(t)\, e(t) \\
&\quad - k e^T(t)(D_\alpha \otimes \Gamma)\, e(t) \\
&= e^T(t)\left(\left((p+q) I_n + \tilde{D} - kD_\alpha\right) \otimes \Gamma\right) e(t) - \frac{\dot{\mu}(t)}{\mu(t)} e^T(t)\, e(t),
\end{aligned}$$

where $D_\alpha = \operatorname{diag}(\underbrace{1, \ldots, 1}_{l}, \underbrace{0, \ldots, 0}_{N-l})$.

Next, we prove that the matrix $((p+q)I_n + \tilde{D} - kD_\alpha)$ is negative definite. This matrix can be written as

$$(p+q) I_n + \tilde{D} - kD_\alpha = \begin{pmatrix} D_1 + (p+q-k) I_l & D_2 \\ D_2^T & D_4 + (p+q) I_{N-l} \end{pmatrix}.$$

According to Lemma 2.2, a sufficient condition for $(p+q)I_n + \tilde{D} - kD_\alpha$ to be negative definite is that the following conditions hold:

$$\begin{cases} D_4 + (p+q) I_{N-l} \prec 0, \\ D_1 + (p+q-k) I_l - D_2 \left(D_4 + (p+q) I_{N-l}\right)^{-1} D_2^T \prec 0. \end{cases}$$

The matrix $D_4 + (p+q)I_{N-l}$ should be negative definite, i.e., the first condition in the Theorem 3.2 needs to satisfy. And we prove the matrix $D_1 + (p+q-k) I_l - D_2 \left(D_4 + (p+q) I_{N-l}\right)^{-1} D_2^T$ is negative definite.

By choosing an appropriate k to ensure that

$$k > p + q + \lambda_{\max}\left(D_1 - D_2 \left(D_4 + (p+q) I_{N-l}\right)^{-1} D_2^T\right).$$

Then the largest eigenvalue of the matrix $D_1+(p+q-k)\,I_l-D_2\left(D_4+(p+q)\,I_{N-l}\right)^{-1}D_2^T$ is less than 0, i. e., this matrix is negative definite. Combining this with the Assumptions 2.3 that Γ is positive definite, according to the property of Kronecker product, we can obtain that $((p+q)I_n+\tilde{D}-kD_\alpha)\otimes\Gamma$ is negative definite.

Let $\tilde{b}=-\lambda_{\max}\left(\left((p+q)\,I_n+\tilde{D}-kD_\alpha\right)\otimes\Gamma\right)/2>0$, then $\mathcal{L}V\,(t)$ can be simplified as:

$$\mathcal{L}V\,(t)\leq-\tilde{b}V\,(t)-2\frac{\dot{\mu}\,(t)}{\mu\,(t)}V\,(t)\,.$$

Using the properties of the Wiener process, we can obtain

$$\frac{\mathrm{d}\mathbb{E}\,[V\,(t)]}{\mathrm{d}t}=\mathbb{E}\,[\mathcal{L}V\,(t)]\leq-\bar{b}\mathbb{E}\,[V\,(t)]-2\frac{\dot{\mu}\,(t)}{\mu\,(t)}\mathbb{E}\,[V\,(t)]\,.$$

According to Lemma 2.1, the drive-response system can achieve prescribed-time outer synchronization under the designed controller with pinning control. □

4. NUMERICAL SIMULATIONS

In this section, numerical simulations are conducted to demonstrate the validity and effectiveness of the proposed theoretical analysis. The Euler–Maruyama method is employed to solve the stochastic differential equations [12]. For simplicity, the drive and response networks are assumed to share the same coupling matrix. Suppose

$$C=D=\begin{bmatrix}-1&1&0&0&0&0&0&0&0&0\\0&-2&0&1&0&0&0&0&1&0\\0&0&-1&0&0&1&0&0&0&0\\0&0&1&-2&0&0&1&0&0&0\\1&1&0&0&-4&0&0&0&1&0\\1&0&0&0&1&-4&0&1&0&0\\0&1&0&1&0&0&-4&0&0&1\\0&0&1&0&1&0&0&-3&0&0\\1&0&0&1&0&0&0&1&-4&0\\1&0&0&0&0&1&0&0&0&-3\end{bmatrix}.$$

The synchronization error between the drive and response systems is given by $e_i\,(t)=y_i\,(t)-x_i\,(t)$ $(i=1,2,\ldots,N)$, and $E\,(t)=\|e\,(t)\|$ is used to quantify the total magnitude of the error. If $E(t)<10^{-3}$ when $t\geq T_0$, where T_0 be the prescribed-time, then the drive-response systems achieve prescribed-time outer synchronization.

Suppose the inner coupling matrix $\Gamma=I_{10}$, and the node dynamics follow a Rössler − like chaotic system,

$$\dot{x}=f\,(x)=\alpha_1\begin{pmatrix}-\gamma&-\alpha_2&-\lambda\\1&\alpha_3&0\\0&0&-\eta\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}+\begin{pmatrix}0\\0\\\alpha_1\eta\phi\,(x_1)\end{pmatrix}\triangleq Bx+\Psi\,(x)\,,$$

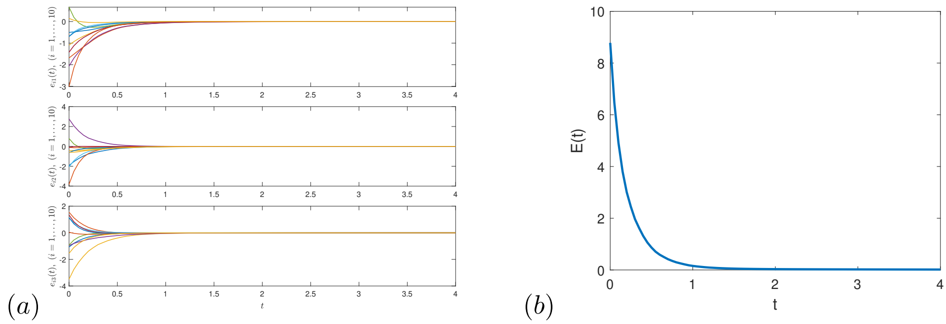

Fig. 1. (a) The trajectories of synchronization error of each node pair. (b) The total magnitude of error between the drive system (1) and the response system (2) under the prescribed-time controller without pinning control (7).

where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is the state vector, and

$$\phi(\tau) = \begin{cases} 0, & \tau < 2.56, \\ \xi\,(\tau - 2.56)\,, & \tau \geq 2.56. \end{cases}$$

Therein, the system parameters are set as $\alpha_1 = 0.03$, $\alpha_2 = 1.5$, $\alpha_3 = 0.2$, $\eta = 1.5$, $\lambda = 0.75$, $\xi = 21.43$, $\gamma = 0.075$. Based on these parameters, the Lipschitz constant p for the Rössler $-$ like chaotic system is computed as $p = 0.4926$. Further, the noise factor is defined as $\sigma_i(e_i(t)) = \sqrt{2\sigma_0}e_i(t)$, $(i = 1, 2, \dots, N)$, where σ_0 is a positive constant. In the following simulations, we set $\sigma_0 = 1$ to examine the collective behavior. Under this definition, $\sigma_i(e_i(t))$ satisfies the Lipschitz condition, i. e., $\mathrm{trace}(\sigma_i(e_i(t))^T\sigma_i(e_i(t))) \leq 2\sigma_0 e_i^T(t)e_i(t) = 2e_i^T(t)e_i(t)$, implying $q = 1$.

We examine the synchronization dynamics between the drive system (1) and response system (2) under the prescribed-time controller (7) without pinning control. Based on preliminary calculations, we obtain $\lambda_{\max}(\hat{D}) = -0.0583$. Therefore, we choose $k = 3 > p+q+\lambda_{\max}(\hat{D}) = 1.4343$, $c = 2 > 1/\lambda_{\min}(\Gamma) = 1$ to satisfy the conditions in Theorem 3.1. Figure (1) plots the trajectories of the errors $e_i(t), (i = 1, 2, \dots, N)$ for each node pair and the total magnitude of synchronization error $E(t)$ over time, with a prescribed-time $T_0 = 4$. Therein, the initial states $x_i\,(0)$ and $y_i\,(0)$ of the drive and response systems are randomly selected from $[-2, 2]$. It can be observed that the corresponding nodes of systems (1) and (2) converge to the same value, with the synchronization error approaching zero within the prescribed time $T_0 = 4$. These results confirm that the drive-response systems achieve prescribed-time outer synchronization within the prescribed-time T_0, thereby verifying the effectiveness of the proposed controller (7).

The synchronization behavior between systems (1) and (2) under the prescribed-time controller (13) with pinning control is investigated, where we select four nodes $(l = 4)$ to control. We calculate $\lambda_{\max}(D_4 + (p+q)I_{N-l}) = -0.7266$, indicating that the matrix is $D_4 + (p+q)I_{N-l}$ negative definite, i. e., the first condition in Theorem 3.2 is satisfied.

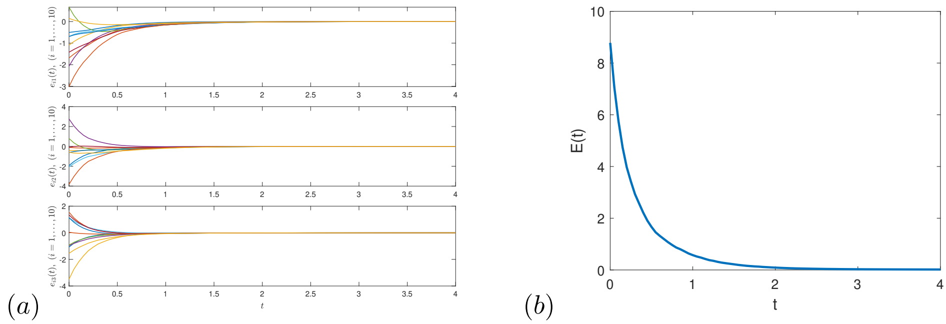

Fig. 2. (a) The trajectories of synchronization error of each node pair. (b) The total magnitude of the error between the drive system (1) and the response system (2) under the prescribed-time controller with pinning control (13).

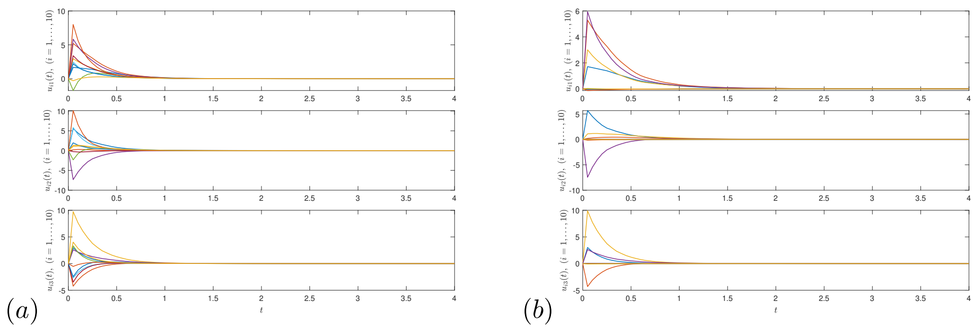

Fig. 3. (a) The evolution of the prescribed-time controller without pinning control (7). (b) The evolution of the prescribed-time controller with pinning control (13).

And $p + q + \lambda_{\max}\left(D_1 - D_2\left(D_4 + (p+q)\, I_{N-l}\right)^{-1} D_2^T\right) = 2.4469$. Thus, we choose $k = 3 > 2.4469$, $c = 2$ to satisfy the second condition. To compare the control schemes, the initial conditions for both the drive and response systems are kept consistent with those in Fig. (1). Figure (2) shows the trajectories of the synchronization errors $e_i(t)$ for each node pair and the total magnitude of synchronization error $E(t)$ over time, with a prescribed-time $T_0 = 4$. It can be seen that the synchronization error approaches zero within the prescribed-time $T_0 = 4$. The simulation results show that the drive-response systems can achieve prescribed-time outer synchronization by controlling only a few nodes, proving that the effectness of designed controller (13).

To intuitively illustrate the controller proposed in this study, we present the time evolution of the controller without pinning control (7) and the controller with pinning control (13) in Fig. (3). The results confirm that both control strategies are continuous

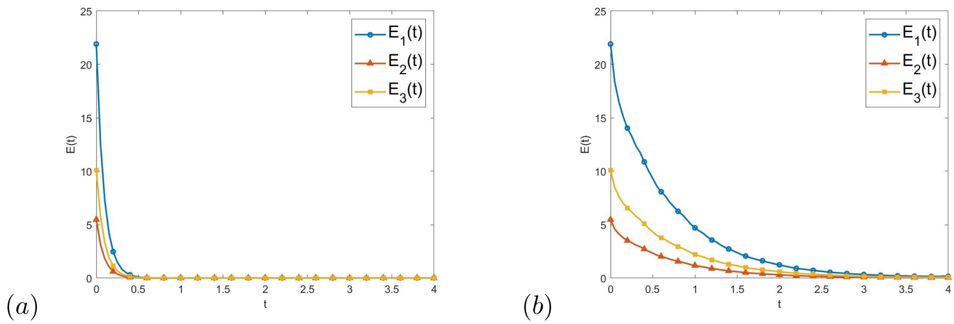

Fig. 4. Effects of different initial conditions $x_i(0)$, $y_i(0)$ on the synchronization dynamics, measured by the total synchronization error. (a) Prescribed-time controller without pinning control (Eq. (7)). (b) Prescribed-time controller with pinning control applied to four nodes (Eq. (13)). The other parameters are set as $T_0 = 4, k = 8, c = 3$.

and bounded, demonstrating their practical feasibility. Moreover, once synchronization between the drive system (1) and the response system (2) is achieved, the control inputs stabilize at zero and exhibit no further fluctuations. This observation conclusively demonstrates the successful attainment of synchronization control.

To investigate the influence of the systems' initial conditions on synchronization dynamics, we plot the evolution of the overall magnitude of synchronization errors for three distinct initial conditions under different controllers (7) and (13) in Fig. (4). Three set of initial values are considered: (1) $x_i(0) = (2, 2, 2)$, $y_i(0) = (-2, -2, -2)$; (2) $x_i(0) = (0.5, 0.5, 0.5)$, $y_i(0) = (-0.5, -0.5, -0.5)$; (3) $x_i(0) = (1+0.2\sin(i), -1+0.2\cos(i), 0.1i)$, $y_i(0) = (-1-0.2\sin(i), 1-0.2\cos(i), -0.1i)$. The results demonstrate that outer synchronization between the drive and response systems is consistently achieved regardless of the initial conditions. This finding suggests that outer synchronization is independent of the systems' initial state.

5. CONCLUSIONS

In this paper, we investigated the prescribed-time outer synchronization in stochastic drive-response networked systems. Based on prescribed-time stability principles, we proposed continuous and bounded controllers, designed both with and without pinning control. Using the stability theory of stochastic differential equations, we have established sufficient criteria to ensure the effectiveness of the proposed controller, guaranteeing prescribed-time outer synchronization between the drive and response systems. Notably, the proposed controllers are independent of initial conditions and predefined parameters, enhancing their practical applicability. Finally, numerical simulations validate the efficiency of the proposed control protocols.

It should be noted that this study assumes instantaneous information transmission between nodes, in contrast to real-world networked systems where time delays are prevalent. Consequently, investigating prescribed-time outer synchronization in drive-response systems with time-delayed coupling constitutes a significant direction for future research. Additionally, our current analysis considered only fixed network topologies, whereas many practical applications e. g., social networks) involve time-varying connections. Extending this work to prescribed-time synchronization under switching topologies would greatly enhance its practical applicability.

ACKNOWLEDGEMENT

This work was supported by the National Natural Science Foundation of China (Grants No. 12302040), Natural Science Foundation of Jiangsu Province (Grants No. BK20221107), and the Fundamental Research Funds for the Central Universities (Grant No. 2024KYJD2003).

(Received May 8, 2025)

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