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EXPLORING FIXED POINT RESULTS IN FUZZY $\mathscr{F}$-METRIC SPACES WITH AN APPLICATION TO SATELLITE WEB COUPLING PROBLEM

Dipti Barman, Abhishikta Das, and Tarapada Bag

In this article, we study some basic properties of $\mathscr{F}$-compactness and $\mathscr{F}$-totally boundedness in fuzzy $\mathscr{F}$-metric spaces. We establish a fixed-point theorem in this setting and apply it to the satellite web coupling problem. To justify the fixed-point result, a counterexample and a graphical illustration of the contraction condition are presented. Furthermore, a numerical illustration is provided to justify the applicability of the result, where the successive iterates and the decay of the sup-norm error demonstrate the effectiveness of the proposed approach.

Keywords: t-norm, fuzzy $\mathscr{F}$-metric space, fixed point, ODE, satellite web coupling problem

Classification: 46S40, 54H27, 55M20

1. INTRODUCTION

In 1965, Zadeh [21] made a breakthrough contribution by introducing the concept of fuzzy sets. Subsequently, this concept stimulated the study of logic, topology, functional analysis, and algebra within the framework of fuzzy settings. Fuzzy metrics play an essential role in the development of fuzzy functional analysis, which was introduced in 1975 by Kramosil and Michalek [12]. Later, in 1994, George & Veeramani [6] redefined this definition by incorporating continuous t-norm [11] to induce Hausdorff topology. In the last few years, several generalizations of fuzzy metrics have been introduced, viz. 2-fuzzy metric [17], M-fuzzy metric [16], D^*-fuzzy metric [1], fuzzy b-metric [14], fuzzy cone metric [15], G-fuzzy metric [19], among others.

On the other hand, metric fixed point results have been extended to these generalized fuzzy metric spaces (for references, see [2, 7, 10, 13, 20]) and have been applied in diverse areas, such as uncertainty models, image processing, decision-making, dynamical systems, etc. In fact, recent studies [4, 5, 9] have shown growing interest in exploring ordered-theoretic and relation-theoretic approaches to fixed-point theory, which yield deeper insights and new applications. Guo et al. [8] proposed numerical algorithms for coupled fixed points in normed spaces, highlighting applications to fractional differential equations and economics. These advancements demonstrate that modern fixed point

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theory is not only rich in abstract generalizations but also powerful in addressing real-world problems.

This ongoing research encourages further developments in fuzzy metric space theory. Inspired by these works, in our previous study [3], we introduced the concept of a fuzzy $\mathscr{F}$-metric to explore new theoretical properties and practical applications. This extension involves a function $f : [0,1] \to [0,1]$ belonging to a specific class of functions that relaxes the axioms of traditional fuzzy metric spaces. The resulting generalized structure enriches the notion of distance in a fuzzy setting and also provides new avenues for exploring fixed-point theorems and their applications.

Continuing this line of research, the present article investigates the fundamental properties of fuzzy $\mathscr{F}$-metric spaces and develops fixed-point results. We study results on $\mathscr{F}$-compactness, $\mathscr{F}$-totally boundedness, and other related concepts. Additionally, we establish a fixed-point theorem for generalized contraction conditions and apply it to problem-solving, such as the existence and uniqueness of solutions to nonlinear ordinary differential equations. A geometric illustration is included to support the theoretical analysis, and the necessity and sufficiency of the contraction mappings are demonstrated through suitable examples. In particular, a numerical example based on the involved boundary value problem is implemented in MATLAB to justify the applicability of the result, where the successive iterates and sup-norm error decay confirm the predicted convergence behavior.

The structure of this paper is as follows. Section 2 presents the necessary preliminary concepts and results. In Section 3, we establish several fundamental properties of fuzzy $\mathscr{F}$-metric spaces. Section 4 is devoted to developing a new type of fixed-point theorem under generalized contraction conditions involving ψ-functions. Furthermore, we demonstrate the applicability of the obtained result by addressing the satellite web coupling problem. Finally, numerical results on the associated boundary value problem are provided to validate the theoretical findings and illustrate the convergence behavior of the proposed iterative scheme.

2. PRELIMINARIES

To carry out this study, we provide the following definitions and results.

Definition 2.1. (Klir and Yuan [11]) A binary operation $\star : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(i) $\star$ is associative and commutative;

(ii) $\alpha \star 1 = \alpha, \quad \forall \alpha \in [0,1]$;

(iii) $\alpha \star \gamma \leq \beta \star \delta$ whenever $\alpha \leq \beta$ and $\gamma \leq \delta, \quad \forall \alpha, \beta, \gamma, \delta \in [0,1]$.

If $\star$ is continuous, then it is called a continuous t-norm.

Example 2.2. (Klir and Yuan [11]) The following are examples of t-norms given.

(i) Standard intersection: $\alpha \star \beta = \min\{\alpha, \beta\}$.

(ii) Algebraic product: $\alpha \star \beta = \alpha\beta$.

(iii) Bounded difference: $\alpha \star \beta = \max\{0, \alpha + \beta - 1\}$.

The following is the definition of fuzzy metric given by George & Veeramani [6].

Definition 2.3. (George and Veeramani [6]) The 3-tuple $(X, M, \star)$ is said to be a fuzzy metric space if the fuzzy set M on $X^2 \times (0, \infty)$ satisfies the following conditions:
$(M1)$ $M(x, y, t) > 0$, $\forall x, y \in X \,\&\, t > 0$;
$(M2)$ $M(x, y, t) = 1$ if and only if $x = y$;
$(M3)$ $M(x, y, t) = M(y, x, t)$, $\forall x, y \in X \,\&\, t > 0$;
$(M4)$ $M(x, y, t) \star M(y, z, s) \leq M(x, z, s + t)$, $\forall x, y, z \in X \,\&\, t, s > 0$;
$(M5)$ $M(x, y, .) : (0, \infty) \to (0, 1]$ is continuous, for all $x, y \in X$.

Lemma 2.4. (George and Veeramani [6]) A fuzzy metric $M(x, y, \cdot)$ over a nonempty set X is non-decreasing on $(0, \infty)$, for all $x, y \in X$.

Remark 2.5. (George and Veeramani [6]) In a fuzzy metric space $(X, M, \star)$, the following holds:
(i) If for $0 < r < 1$, $M(x, y, t) > 1 - r$, for all $x, y \in X \quad t > 0$, we can find a $t_0, 0 < t_0 < t$ such that $M(x, y, t_0) > 1 - r$.
(ii) For any $r_1 > r_2$ in $(0, 1)$, we can find $r_3 \in (0, 1)$ such that $r_1 \star r_3 \geq r_2$ and for any $r_4 \in (0, 1)$, we can find $r_5 \in (0, 1)$ such that $r_5 \star r_5 \geq r_4$.

Next, we recall the definition of fuzzy $\mathscr{F}$-metric space with an example and related definitions and results, introduced in our earlier study [3]. Throughout the article, $\mathscr{F}$ denotes the set of all functions $f\colon [0, 1] \to [0, 1]$ which satisfy the following conditions:
$(\mathscr{F}1)$ f is strictly increasing in $[0, 1)$;
$(\mathscr{F}2)$ For every sequence $\{t_n\}$ in $[0, 1]$ we have, $\lim\limits_{n\to\infty} t_n = 1 \iff \lim\limits_{n\to\infty} f(t_n) = 1$.

Example 2.6. (Das et al. [3]) The following are some examples of members of $\mathscr{F}$.
(i) $f(x) = x^n, \quad \forall x \in [0, 1], \quad n \in \mathbb{N}$.
(ii) $f(x) = \sqrt{x}, \quad \forall x \in [0, 1]$.

Definition 2.7. (Das et al. [3]) Let X be a non-empty set, $M : X \times X \times (0, \infty) \to [0, 1]$ be a mapping, and $\star$ be a continuous t-norm. If there exists $(f, \alpha) \in \mathscr{F} \times (0, 1]$ such that M satisfies the following conditions:
$(\mathscr{F}M1)$ $M(x, y, t) > 0$, $\forall x, y \in X$ & $t > 0$;
$(\mathscr{F}M2)$ $(M(x, y, t) = 1, \forall t > 0)$ iff $x = y$;
$(\mathscr{F}M3)$ $M(x, y, t) = M(y, x, t)$, $\forall x, y \in X$ & $t > 0$;
$(\mathscr{F}M4)$ for every $(x, y) \in X \times X$, for every $N \in \mathbb{N}$, $N \geq 2$ and for every $\{u_i\}_i^N \subseteq X$ with $u_1 = x$ and $u_N = y$, $M(x, y, t) < 1$ implies

$$(f(M(x, y, t)))^\alpha \geq f(M(u_1, u_2, t_1) \star M(u_2, u_3, t_2) \star \cdots \star M(u_{N-1}, u_N, t_{N-1}))$$

where $t = t_1 + t_2 + \ldots + t_{N-1}$; $t_i > 0$ for $i = 1, 2, \cdots, (N - 1)$.

Then M is called a fuzzy $\mathscr{F}$-metric on X and the 5-tuple $(X, M, f, \alpha, \star)$ is called a fuzzy $\mathscr{F}$-metric space.

Example 2.8. (Das et al. [3]) Let $X = \mathbb{R}$ and define a function $M : X^2 \times (0, \infty) \to [0, 1]$ by $M(x, y, t) = \left(\dfrac{t}{t+1}\right)^{|x-y|^2}$, $\forall (x, y) \in X \times X$ & $t \in (0, \infty)$. Then M is a fuzzy $\mathscr{F}$-metric on X with respect to $f(x) = x^2$, $\forall x \in [0, 1]$, $\alpha = \frac{1}{2}$, and *product* t-norm.

Definition 2.9. (Das et al. [3]) A sequence $\{x_n\}$ in a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is called

(i) an $\mathscr{F}$-convergent sequence if there exists $x \in X$ such that for any $0 < r < 1$, there exists a natural number N such that for all $t > 0$, $M(x_n, x, t) > 1 - r,\ \forall n \geq N$.

(ii) an $\mathscr{F}$-Cauchy sequence if for each $t > 0$ and $0 < r < 1$, there exists a natural number N such that $M(x_n, x_m, t) > 1 - r,\ \forall m, n \geq N$.

Proposition 2.10. (Das et al. [3]) In a fuzzy $\mathscr{F}$-metric space,

(i) a sequence $\{x_n\} \subseteq X$ is $\mathscr{F}$-convergent to $x \in X$ iff $\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall\, t > 0$.

(ii) a sequence $\{x_n\} \subseteq X$ is $\mathscr{F}$-Cauchy iff $\lim_{m,n\to\infty} M(x_n, x_m, t) = 1, \quad \forall\, t > 0$.

(iii) limit of an $\mathscr{F}$-convergent sequence is unique.

(iv) every $\mathscr{F}$-convergent sequence is an $\mathscr{F}$-Cauchy sequence.

Definition 2.11. (Das et al. [3]) A fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is said to be $\mathscr{F}$-complete if every $\mathscr{F}$-Cauchy sequence converges to some point in X.

Proposition 2.12. (Das et al. [3]) In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, $\tau_{\mathscr{F}} = \{A \subseteq X : \text{for each } x \in A, \exists r > 0, t > 0 \text{ such that } B_{\mathscr{F}}(x, r, t) \subseteq A\}$ forms a topology on X.

Definition 2.13. (Das et al. [3]) In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, $G \subset X$ is called $\mathscr{F}$-open if $G \in \tau_{\mathscr{F}}$ and called $\mathscr{F}$-closed if $X \backslash G \in \tau_{\mathscr{F}}$.

Definition 2.14. (Das et al. [3]) In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, $G \subseteq X$ is called fuzzy $\mathscr{F}$-bounded if and only if there exist $t_0 \in (0, \infty)$ and $r_0 \in (0, 1)$ such that $M(x, y, t_0) > 1 - r$, for all $x, y \in G$.

Proposition 2.15. In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, for any subset $A \subset X$, $x \in \bar{A}$, $0 < r < 1$ implies $B_{\mathscr{F}}(x, r, t) \cap A \neq \emptyset$ for any $t > 0$.

P r o o f. Since $x \in \bar{A}$, for $0 < r < 1$, there exists a sequence $\{x_n\} \subseteq A$ such that

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \forall t > 0.$$

This implies

$$\begin{aligned}
&\text{there exists } N(t, r) \in \mathbb{N} \text{ such that } M(x_n, x, t) > 1 - r, \forall n \geq N(t, r), \text{ for any } t > 0 \\
&\textit{or}\ \ x_n \in B_{\mathscr{F}}(x, r, t), \quad \forall n \geq N(t, r), \text{ for any } t > 0 \\
&\textit{or}\ \ B_{\mathscr{F}}(x, r, t) \cap A \neq \emptyset, \text{ for any } t > 0.
\end{aligned}$$

□

We develop a fixed-point theorem in Section 4 using the following class of mappings.

Definition 2.16. (Turkoglua and Sangurlu [20]) Let Ψ be the class of all mappings $\psi : [0, 1] \to [0, 1]$ satisfying

(i) ψ is continuous and non-decreasing on $[0, 1]$;

(ii) $\psi(t) > t, \quad \forall t \in (0, 1)$.

Example 2.17. (Turkoglua and Sangurlu [20]) A function $\psi(t) = \dfrac{t}{t + k(1-t)}$, $\forall t \in [0,1]$, $k \in (0,1)$ lies in Ψ.

The following example & properties of Ψ and the ψ-contraction theorem from [13, 20] are recalled here for use in the main result.

Lemma 2.18. (Turkoglua and Sangurlu [20]) If $\psi \in \Psi$ then $\psi(1) = 1$.

Lemma 2.19. (Turkoglua and Sangurlu [20]) If $\psi \in \Psi$ then $\lim\limits_{n\to\infty} \psi^n(t) = 1$, for all $t \in (0,1)$.

Theorem 2.20. (Mihet [13]) Let $(X, M, \star)$ be a complete fuzzy metric space and $S : X \to X$ be a mapping. S has a unique fixed point in X if it satisfies the ψ-contraction condition

$$M(x,y,t) > 0 \implies M(S(x), S(y), t) \geq \psi(M(x,y,t)), \quad \forall t > 0.$$

3. SOME BASIC PROPERTIES OF FUZZY $\mathscr{F}$-METRIC SPACE

In this section, we develop results related to $\mathscr{F}$-completeness, $\mathscr{F}$-compactness, and $\mathscr{F}$-totally boundedness, which are fundamental for studying the structural properties of the concerned fuzzy $\mathscr{F}$-metric space.

Lemma 3.1. If $f \in \mathscr{F}$, then $f(1) = 1$.

Proof. Since $f \in \mathscr{F}$, so f satisfies ($\mathscr{F}1$) & ($\mathscr{F}2$). Assume, for contradiction, that $f(1) \neq 1$. So, $f(1) < 1$.

Consider the constant sequence $\{t_n\}$ defined by $t_n = 1$, for all $n \in \mathbb{N}$. Then, by condition ($\mathscr{F}2$), $\lim\limits_{n\to\infty} f(t_n) = 1$.

Again, $t_n = 1, \forall n \in \mathbb{N} \implies f(t_n) = f(1), \forall n \in \mathbb{N}$ and hence $\lim\limits_{n\to\infty} f(t_n) = f(1) < 1$. This yields a contradiction with ($\mathscr{F}2$), and consequently $f(1) = 1$. □

Proposition 3.2. If $\{x_n\}$ be a $\mathscr{F}$-convergent sequence in $(X, M, f, \alpha, \star)$ converging to $x \in X$, then every subsequence of $\{x_n\}$ also converges to the same point x.

Proof. Let $\{x_{n_k}\}$ be a subsequence of $\{x_n\}$. Since $\{x_n\}$ converges to x, so

$$\lim_{n\to\infty} M(x_n, x, t) = 1, \quad \forall t > 0. \tag{1}$$

Again, because $\{x_n\}$ is an $\mathscr{F}$-Cauchy sequence (Proposition 2.10(iv)), it follows that

$$\lim_{k,n\to\infty} M(x_{n_k}, x_n, t) = 1, \quad \forall t > 0. \tag{2}$$

This yeilds

$$\lim_{k,n\to\infty} \{M(x_n, x, t) \star M(x_n, x_{n_k}, t)\} = 1, \quad \forall t > 0 \quad \text{(using (1) \& (2))}. \tag{3}$$

Now from $(\mathscr{F}M4)$, we have

$$\begin{aligned} \left(f\left(M\left(x_{n_k}, x, t\right)\right)\right)^{\alpha} &\geq f\left(M\left(x_{n_k}, x_n, \frac{t}{2}\right) \star M\left(x_n, x, \frac{t}{2}\right)\right), \quad \forall t>0 \\ \text{or} \lim_{k\to\infty}\left(f\left(M\left(x_{n_k}, x, t\right)\right)\right)^{\alpha} &\geq \lim_{k,n\to\infty} f\left(M\left(x_{n_k}, x_n, t\right) \star M\left(x_n, x, t\right)\right), \forall t>0 \\ &= f(1)=1, \forall t>0 \quad \text{(using (3) \& } (\mathscr{F})) \\ \text{or} \lim_{k\to\infty} M(x_{n_k}, x, t) &= 1, \quad \forall t>0. \end{aligned}$$

This shows that $\{x_{n_k}\}$ converges to x. Hence, the proof is completed. □

We define $\mathscr{F}$-compactness of a set in a fuzzy $\mathscr{F}$-metric space as follows.

Definition 3.3. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space and $A \subset X$. A is said to be $\mathscr{F}$-compact if every sequence in A has a convergent subsequence which converges to some point in A.

The following theorem establishes the relationship between $\mathscr{F}$-compactness and $\mathscr{F}$-boundedness in fuzzy $\mathscr{F}$-metric spaces.

Theorem 3.4. In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, every $\mathscr{F}$-compact subset is $\mathscr{F}$-closed and $\mathscr{F}$-bounded.

Proof. Let A be a $\mathscr{F}$-compact subset of X. Suppose that A is not $\mathscr{F}$-closed. So there exists a sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$ but $x \notin A$.

Since A is $\mathscr{F}$-compact, so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to some point in A.

But $x_n \to x$ as $n \to \infty$ implies $x_{n_k} \to x$ as $n \to \infty$ and hence $x \in A$. Which is a contradiction. Thus A is $\mathscr{F}$-closed.

We now show that A is $\mathscr{F}$-bounded. Suppose that A is unbounded and take a sequence $\{x_n\}$ in A. Since A is $\mathscr{F}$-compact, so it has a $\mathscr{F}$-convergent subsequence, say $\{x_{n_k}\}$, which converges to some $x \in A$.

Let $x_0 \in A$ be a fixed element and choose $0<\epsilon<1$. Then by $(\mathscr{F}2)$, there exists $\delta \in (0,1)$ such that

$$1-\delta<t\leq 1 \implies 1-\epsilon<f(t)\leq 1. \tag{4}$$

Again, there exists $t_0>0$ such that $M(x_0, x, t_0) > (1-\delta)$.

By Remark 2.5, we can find $r \in (0,1)$ such that

$$M(x_0, x, t_0) \star (1-r) > (1-\delta). \tag{5}$$

We now choose a sequence $\{\alpha_n\} \in (0,1)$ such that

$$\lim_{n\to\infty} \alpha_n = 1. \tag{6}$$

So for given $t'(>t_0)$, for each α_k, there exists $x_{n_k} \in A$ such that

$$M(x_{n_k}, x_0, t') \leq (1-\alpha_k). \tag{7}$$

Since $x_{n_k} \to x$ as $k \to \infty$, so for $r \in (0,1)$ and $(t' - t_0) = t_1$(say), there exists $m(t_1, r) \in \mathbb{N}$ such that

$$M(x, x_{n_k}, t_1) > (1 - r), \quad \forall k \geq m(t_1, r).$$

Hence, we have

$$M(x_0, x, t_0) \star M(x, x_{n_k}, t_1) > M(x_0, x, t_0) \star (1-r) > (1-\delta), \forall k \geq m(t_1, r) \quad (\text{using } (5))$$

which implies

$$\begin{aligned} & f(M(x_0, x, t_0) \star M(x, x_{n_k}, t_1)) > 1 - \epsilon, \forall k \geq m(t_1, r) \quad (\text{using } (4)) \\ \textit{or} \ & \left(f(M(x_0, x_{n_k}, t'))\right)^{\alpha} > 1 - \epsilon, \forall k \geq m(t', r) \quad (\text{using } (\mathscr{F}M4)) \\ \textit{or} \ & \left(f(1 - \alpha_k)\right)^{\alpha} > 1 - \epsilon, \forall k \geq m \quad (\text{using } (7)) \\ \textit{or} \ & \lim_{k \to \infty} \left(f(1 - \alpha_k)\right)^{\alpha} \geq 1 - \epsilon. \end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrary, we can write

$$\lim_{k \to \infty} \left(f(1 - \alpha_k)\right)^{\alpha} = 1 \ \textit{or} \ \lim_{k \to \infty} f(1 - \alpha_k) = 1 \ \textit{or} \ \lim_{k \to \infty} \alpha_k = 0 \quad (\text{using } (\mathscr{F}2)),$$

a contradiction to (6). Hence A is $\mathscr{F}$-bounded. □

We now proceed to prove that $\mathscr{F}$-compactness implies $\mathscr{F}$-completeness in fuzzy $\mathscr{F}$-metric spaces.

Theorem 3.5. In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, every $\mathscr{F}$-compact set is $\mathscr{F}$-complete.

Proof. Let $A \subseteq X$ be a nonempty $\mathscr{F}$-compact set in X and $0 < \epsilon < 1$. Then by $(\mathscr{F}2)$, there exists $0 < \delta < 1$ such that

$$1 - \delta < t \leq 1 \implies 1 - \epsilon < f(t) \leq 1. \tag{8}$$

For $\delta \in (0,1)$, there exists $\lambda \in (0,1)$ such that $(1 - \lambda) > (1 - \delta)$. Then using Remark 2.5, for $\lambda \in (0,1)$, we can find $r_1 \in (0,1)$ such that

$$(1 - r_1) \star (1 - r_1) \geq (1 - \lambda) > (1 - \delta).$$

Again using Remark 2.5, we can find $r_3 \in (0,1)$

$$(1 - r_1) \star (1 - r_1) \star (1 - r_3) \geq (1 - \delta). \tag{9}$$

Let $\{x_n\}$ be an $\mathscr{F}$-Cauchy sequence in A. Then for $0 < r_1 < 1$ and for each $t_1 > 0$, there exists $N_1(t_1, r_1) \in \mathbb{N}$ such that

$$M(x_n, x_m, t_1) > 1 - r_1, \quad \forall m, n \geq N_1(t_1, r_1).$$

Since A is $\mathscr{F}$-compact, so $\{x_n\}$ has a $\mathscr{F}$-convergent subsequence, say $\{x_{n_k}\}$ which converges to x in A. Then for each $t_2 > 0$, there exists $N_2(t_2, r_3) \in \mathbb{N}$ such that

$$M(x_{n_k}, x, t_2) > 1 - r_3, \quad \forall k \geq N_2(t_2, r_3).$$

Let $N(t, \delta) = \max\{N_1(t_1, r_1), N_2(t_2, r_3)\}$. Thus, for $m, n, k \geq N(t, \delta)$,

$$\begin{aligned} & M(x_n, x_m, t_1) \star M(x_m, x_{n_k}, t_1) \star M(x_{n_k}, x, t_2) > 1 - \delta \quad (\text{using } (9)) \\ \textit{or } & f\left(M(x_n, x_m, t_1) \star M(x_m, x_{n_k}, t_1) \star M(x_{n_k}, x, t_2)\right) > 1 - \epsilon \quad (\text{using } (8)) \\ \textit{or } & \left(f(M(x_n, x, T))\right)^\alpha > 1 - \epsilon \quad \text{ where } T = 2t_1 + t_2 (> 0) \quad (\text{using } (\mathscr{F}M4)). \end{aligned}$$

Since $0 < \epsilon < 1$ is arbitrary, so letting $n \to \infty$, we get

$$\lim_{n\to\infty} f\left(M(x_n, x, T)\right) = 1 \ \textit{ or } \lim_{n\to\infty} M(x_n, x, T) = 1 \quad (\text{using } (\mathscr{F}2)).$$

Since $t_1, t_2 > 0$ are arbitrary, thus $\lim\limits_{n\to\infty} M(x_n, x, s) = 1$, $\forall s > 0$. This proves $\{x_n\}$ is a $\mathscr{F}$-convergent sequence in A converging to $x \in A$. Thus A is $\mathscr{F}$-complete in X. □

The following proposition will be used to prove Theorem 3.9 in the setting of fuzzy $\mathscr{F}$-metric spaces.

Proposition 3.6. In a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$, every finite subset of X is $\mathscr{F}$-bounded.

Proof. Let A be a finite subset of X containing n elements, $\{x_1, x_2, \ldots x_n\}$. Choose $t_0 > 0$ fixed. Let $\underset{i,j}{min}\, M(x_i, x_j, t_0) = \beta$. Clearly $\beta \in (0, 1)$. Then we can choose $r \in (0, 1)$ such that $\beta > 1 - r$ which implies that $M(x_i, x_j, t_0) > 1 - r$, $\forall x_i, x_j \in A$. Hence A is $\mathscr{F}$-bounded. □

We now introduce the concepts of $\mathscr{F}$-(r, ϵ)-nets and $\mathscr{F}$-total boundedness in a fuzzy $\mathscr{F}$-metric space

Definition 3.7. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space and $A \subseteq X$. For $0 < r < 1$ and $\epsilon > 0$, a subset $B \subseteq X$ is said to be an $\mathscr{F}$-(r, ϵ)-net for the set A if for every $x \in A$ there exists $y \in B$ such that $M(x, y, \epsilon) > 1 - r$.

Definition 3.8. A set A in a fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ is said to be $\mathscr{F}$-totally bounded if for any $\epsilon > 0$ there exists a finite $\mathscr{F}$-(r, ϵ)-net for the set A.

The following result illustrates the distinction between $\mathscr{F}$-boundedness and $\mathscr{F}$-totally boundedness in a fuzzy $\mathscr{F}$-metric space.

Theorem 3.9. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space and $A \subset X$. If A is $\mathscr{F}$-totally bounded then A is $\mathscr{F}$-bounded.

Proof. Since A is $\mathscr{F}$-totally bounded, so for any given $0 < r < 1$ & $\epsilon > 0$, there exists a finite $\mathscr{F}$-(r, ϵ)-net B for A. Since B is finite, B is $\mathscr{F}$-bounded. So there exists $t_0 > 0$ and $r_0 \in (0, 1)$ such that

$$M(y_1, y_2, t_0) > 1 - r_0, \quad \forall y_1, y_2 \in B. \tag{10}$$

Again for each $a \in A$, there exists $b \in B$ such that

$$M(a, b, t_0) > 1 - r. \tag{11}$$

Now choose $a_1, a_2 \in A$ arbitrarily. Then there exist $b_1, b_2 \in B$ such that

$$M(a_1, b_1, t_0) > 1 - r \quad \& \quad M(a_2, b_2, t_0) > 1 - r. \tag{12}$$

Again using ($\mathscr{F}M4$), we have

$$(f(M(a_1, a_2, t)))^\alpha \geq f(M(a_1, b_1, t_0) \star M(b_1, b_2, t_0) \star M(b_2, a_2, t_0)) \tag{13}$$

where $t = 3t_0$ fixed. Now from (10) and (12) we have

$$M(a_1, b_1, t_0) \star M(b_1, b_2, t_0) \star M(b_2, a_2, t_0) > (1 - r) \star (1 - r_0) \star (1 - r). \tag{14}$$

If $M(a_1, b_1, t_0) \star M(b_1, b_2, t_0) \star M(b_2, a_2, t_0) = 1$ then

$$\begin{aligned} &(f(M(a_1, a_2, t)))^\alpha = 1, \quad \forall a_1, a_2 \in A \\ &\textit{or } M(a_1, a_2, t) = 1, \quad \forall a_1, a_2 \in A \quad (\text{using}\,(\mathscr{F}2)) \\ &\textit{or } M(a_1, a_2, t) > (1 - \beta), \quad \forall a_1, a_2 \in A \,\text{and for any } \beta \in (0, 1). \end{aligned}$$

This shows that A is $\mathscr{F}$-bounded.

If $M(a_1, b_1, t_0) \star M(b_1, b_2, t_1) \star M(b_2, a_2, t_0) < 1$ then

$$\begin{aligned} &f\left(M(a_1, b_1, t_0) \star M(b_1, b_2, t_1) \star M(b_2, a_2, t_0)\right) < 1 \\ &\textit{or } f\left((1 - r) \star (1 - r_0) \star (1 - r)\right) < 1 \quad (\text{using}\,(\mathscr{F}1)). \end{aligned}$$

So we may write, $f\left((1 - r) \star (1 - r_0) \star (1 - r)\right) = (1 - \gamma), \quad \text{for some}\, \gamma \in (0, 1)$.

Thus from (13) we have,

$$\begin{aligned} &(f(M(a_1, a_2, t)))^\alpha > (1 - \gamma), \quad \forall a_1, a_2 \in A \\ &\textit{or } f(M(a_1, a_2, t)) > (1 - \gamma)^{\frac{1}{\alpha}} = (1 - \delta)\,(\text{say}), \quad \forall a_1, a_2 \in A, \text{ where } \delta \in (0, 1). \end{aligned} \tag{15}$$

Using ($\mathscr{F}2$), for $\delta \in (0, 1)$, there exists $\mu \in (0, 1)$ such that

$$1 - \mu < t \leq 1 \iff 1 - \delta < f(t) \leq 1.$$

Therefore the relation (15) gives $M(a_1, a_2, t) > (1 - \mu)$, $\forall a_1, a_2 \in A$. Since t is fixed, so it follows that A is $\mathscr{F}$-bounded. □

Remark 3.10. The converse of the above Theorem 3.9 is not necessarily true. For justification, we consider the following example.

Example 3.11. Recall the fuzzy $\mathscr{F}$-metric space of Example 2.8. Consider the subset $A = \{1, 3, 5, 7, 9\}$ of $\mathbb{R}$. Then by Lemma 3.6, A is $\mathscr{F}$-bounded set.
If possible suppose that A is $\mathscr{F}$-totally bounded.
So, for $\epsilon = \frac{\sqrt{2}}{3}$, $r = 1 - \frac{1}{\sqrt{2}}$, there exists a finite $\mathscr{F}$-(r, ϵ)-net, N(say) for A. Thus for $x_i, x_j (i \neq j) \in A$, there exist $y_i, y_j \in N$ such that

$$M(x_i, y_i, \epsilon) > (1 - r) \text{ and } M(x_j, y_j, \epsilon) > (1 - r). \tag{16}$$

Now, using ($\mathscr{F}$1) and (16), we get

$$(f(M(x_i, x_j, 2\epsilon)))^\alpha \geq f(M(x_i, y_j, \epsilon) \star M(y_j, x_j, \epsilon)) > f((1 - r)^2) = \frac{1}{\sqrt{2}}.$$

Again, $(M(x_i, x_j, 2\epsilon))^\alpha = \left(\frac{2\epsilon}{1+2\epsilon}\right)^{\frac{|x_i - x_j|^2}{2}} = (0.49)^{\frac{|x_i - x_j|^2}{2}}$.

If $x \neq y$ then $\frac{|x_i - x_j|^2}{2} \geq 2$. Therefore, $(0.49)^{\frac{|x_i - x_j|^2}{2}} < \frac{1}{\sqrt{2}}$. It is a contradiction.
So, there is no $\mathscr{F}$-(r, ϵ)-net for A. Hence A is not $\mathscr{F}$-totally bounded.

The next theorem characterizes the connection between $\mathscr{F}$-compactness and $\mathscr{F}$-total boundedness.

Theorem 3.12. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space and $A \subset X$. If A is a $\mathscr{F}$-compact set, then A is $\mathscr{F}$-totally bounded.

Proof. Suppose that A is $\mathscr{F}$-compact. We choose $r \in (0, 1)$ and $\epsilon > 0$ arbitrarily. Then by ($\mathscr{F}$2), there exists $0 < \delta < 1$ such that

$$1 - \delta < t \leq 1 \implies 1 - \epsilon < f(t) \leq 1. \tag{17}$$

For $\delta \in (0, 1)$, we can find $r_1 \in (0, 1)$ such that

$$(1 - r_1) \star (1 - r_1) \geq (1 - \delta). \tag{18}$$

Let x_1 be an arbitrary element of X. If $M(a, x_1, \epsilon) > (1 - r)$, $\forall a \in A$, then a finite $\mathscr{F}$-(r, ϵ)-net B exists for A i.e. $B = \{x_1\}$. If not there exists a point $x_2 \in A$ such that $M(x_1, x_2, \epsilon) \leq (1 - r)$.
If for all $a \in A$, $M(a, x_1, \epsilon) > (1 - r)$ or $M(a, x_2, \epsilon) > (1 - r)$ then $B = \{x_1, x_2\}$ is a finite $\mathscr{F}$-(r, ϵ)-net exists for A. Continuing in this way, we obtain points $\{x_1, x_2, \ldots, x_n\}$ where $x_1 \in X$, $x_2, x_3, \cdots x_n \in A$ such that $M(x_i, x_j, \epsilon) \leq (1 - r)$ for $i \neq j$. Here, two cases may arise.

Case-I The procedure stops after k^{th}- steps. Then we obtain points $x_1, x_2, \ldots, x_k$ such that for every $a \in A$ at least one of the inequalities $M(x_i, a, \epsilon) > (1 - r)$, $i = 1, 2, \ldots, k$ holds and then $B = \{x_1, x_2, \ldots, x_k\}$ is a finite $\mathscr{F}$-(r, ϵ)-net for A. Hence A is $\mathscr{F}$-totally bounded.

Case-II The procedure continues infinitely. Then we obtain an infinite sequence $\{x_n\}$ where $x_1 \in X$ and $x_i \in A$ for $i \geq 2$ such that

$$M(x_i, x_j, \epsilon) \leq (1 - r), \quad \text{for}\, i \neq j. \tag{19}$$

Since A is $\mathscr{F}$-compact so there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ which converges to $x \in A$. Then for $\frac{\epsilon}{2} > 0$ and $r_1 \in (0,1)$, there exists $N(\epsilon, r_1) \in \mathbb{N}$ such that

$$M\left(x_{n_i}, x, \frac{\epsilon}{2}\right) > (1 - r_1), \, \forall i \geq N(\epsilon, r_1). \tag{20}$$

Hence we have,

$$M\left(x_{n_i}, x, \frac{\epsilon}{2}\right) \star M\left(x_{n_{i+1}}, x, \frac{\epsilon}{2}\right) > (1 - r_1) \star (1 - r_1) \geq (1 - \delta), \, \forall i \geq N(\epsilon, r_1). \tag{21}$$

Again using ($\mathscr{F}M4$), we have

$$\left(f\left(M\left(x_{n_i}, x_{n_{i+1}}, \epsilon\right)\right)\right)^{\alpha} \geq f\left(M\left(x_{n_i}, x, \frac{\epsilon}{2}\right) \star M\left(x_{n_{i+1}}, x, \frac{\epsilon}{2}\right)\right). \tag{22}$$

Now, on the appliance of (19), (22) & ($\mathscr{F}1$), we get

$$\begin{aligned}
&(f(1-r))^{\alpha} \geq f\left(M\left(x_{n_i}, x, \frac{\epsilon}{2}\right) \star M\left(x_{n_{i+1}}, x, \frac{\epsilon}{2}\right)\right), \, \forall i \geq N(\epsilon, r_1) \\
&or\;\; (f(1-r))^{\alpha} > (1 - \epsilon) \quad (\text{using}\, (17)\, \&\, (21)) \\
&or\;\; (f(1-r))^{\alpha} \geq 1 \quad (\text{since}\, \epsilon > 0\, \text{is arbitrarily chosen}) \\
&or\;\; (f(1-r)) = 1 \\
&or\; r = 0 \quad (\text{using}\, (\mathscr{F}2)\, \text{and}\, \text{Lemma}\, 3.1).
\end{aligned}$$

This contradicts our assumption. Hence, this case is absurd. This completes the proof. □

4. FIXED POINT THEOREM WITH APPLICATION TO A SATELLITE WEB COUPLING PROBLEM

In this section, we present a fixed point theorem in fuzzy $\mathscr{F}$-metric spaces that provides a useful criterion for the existence of a unique fixed point of self-mappings in fuzzy metric spaces, generalizing classical results by incorporating the notion of fuzzy contractions.

4.1. Fixed point theorem and numerical illustration:

In this subsection, we present a fixed point theorem and provide a concrete numerical example to support it. Additionally, a graphical illustration is included to visualize the claim of the numerical example.

Definition 4.1. Let $(X, M, f, \alpha, \star)$ be a fuzzy $\mathscr{F}$-metric space. A mapping $T : X \to X$ is said to be a fuzzy $\mathscr{F}$-ψ-contraction mapping with respect to the function $\psi \in \Psi$ if the following implication holds:

$$0 < M(x, y, t) < 1 \implies M(Tx, Ty, t) \geq \psi(M(x, y, t)), \, \forall x, y \in X \,\&\, t > 0. \tag{23}$$

Theorem 4.2. Let $(X, M, f, \alpha, \star)$ be an $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space and T be a fuzzy $\mathscr{F}$-ψ-contraction mapping with respect to some function $\psi \in \Psi$ such that there exists $y \in X$ satisfying $0 < M(y, Ty, t) < 1$, $\forall t > 0$. Then T has a unique fixed point in X.

Proof. Let $x \in X$ be such that $0 < M(x, Tx, t) < 1$, $\forall t > 0$ and define $x_n = T^n x$, $n \in \mathbb{N} \cup \{0\}$. If $T(x_k) = x_k$, for some $k \in \mathbb{N} \cup \{0\}$ then the proof is done. So suppose $T(x_r) \neq x_r$, for all $r \in \mathbb{N} \cup \{0\}$.

Since T is a fuzzy $\mathscr{F}$-ψ-contraction mapping then for the sequence $\{x_n\}$, we have

$$M(x_{n+2}, x_{n+1}, t) \geq \psi\left(M(x_{n+1}, x_n, t)\right), \quad \forall n \in \mathbb{N} \cup \{0\} \,\&\, t > 0.$$

Repeating the above relation, for any $m \geq 2$, we get

$$\begin{aligned} M(x_{m+1}, x_m, t) &\geq \psi(M(x_m, x_{m-1}, t)) \\ &\geq \psi^2(M(x_{m-1}, x_{m-2}, t)) \geq \cdots \geq \psi^m(M(x_1, x_0, t)), \quad \forall t > 0. \end{aligned}$$

Taking limit as $m \to \infty$ and using Lemma 2.19 in the above relation, we get

$$\begin{aligned} &\lim_{m\to\infty} M(x_{m+1}, x_m, t) \geq \lim_{m\to\infty} \psi^m(M(x_1, x_0, t)) = 1, \quad \forall t > 0 \\ \textit{or } &\lim_{m\to\infty} M(x_{m+1}, x_m, t) = 1, \quad \forall t > 0. \end{aligned} \tag{24}$$

Proceeding in this way, we obtain

$$\lim_{m\to\infty} M(x_{m+i}, x_{m+i-1}, t) = 1, \quad \forall t > 0 \;\&\; i \geq 1. \tag{25}$$

Next, we show that $\{x_r\}$ is an $\mathscr{F}$-Cauchy sequence in $(X, M, f, \alpha, \star)$. Now using the inequality $(\mathscr{F}M4)$, for any $m \in \mathbb{N}$ and $p = 1, 2, 3, \cdots$, we have

$$(f(M(x_{m+p}, x_m, t)))^\alpha \geq f\left(M\left(x_{m+p}, x_{m+p-1}, \frac{t}{p}\right) \star \cdots \star M\left(x_{m+1}, x_m, \frac{t}{p}\right)\right). \tag{26}$$

Applying the relation (25) on (26), we get

$$\begin{aligned} &\lim_{m\to\infty} f\left(M\left(x_{m+p}, x_{m+p-1}, \frac{t}{p}\right) \star \cdots \star M\left(x_{m+1}, x_m, \frac{t}{p}\right)\right) = 1, \forall t > 0 \quad (\text{by } (\mathscr{F}2)) \\ \textit{or } &\lim_{m\to\infty} (f(M(x_{m+p}, x_m, t)))^\alpha \geq 1, \forall t > 0 \quad (\text{by } (26)) \\ \textit{or } &\lim_{m\to\infty} M(x_{m+p}, x_m, t) = 1, \forall t > 0 \quad (\text{by } (\mathscr{F}2)). \end{aligned}$$

This proves that $\{x_m\}$ is an $\mathscr{F}$-Cauchy sequence in X. Since X is $\mathscr{F}$-complete, so $\{x_m\}$ converges to some $u \in X$. Therefore,

$$\lim_{m\to\infty} M(x_m, u, t) = 1, \forall t > 0.$$

Next, we claim that u is a fixed point of T. If possible, suppose there exists $t_0 > 0$ such that

$$M(u, Tu, t_0) < 1. \tag{27}$$

Then using ($\mathscr{F}M4$), we have

$$(f(M(u,Tu,t_0)))^\alpha \geq f\left(M\left(u,x_m,\frac{t_0}{2}\right)\star M\left(x_m,Tu,\frac{t_0}{2}\right)\right),\ \forall m\in\mathbb{N}. \tag{28}$$

Again,

$$\begin{aligned}
&M\left(x_m,Tu,\frac{t_0}{2}\right) \geq \psi\left(M\left(x_{m-1},u,\frac{t_0}{2}\right)\right),\ \forall m\in\mathbb{N}\\
\textit{or}\ &\lim_{m\to\infty} M\left(x_m,Tu,\frac{t_0}{2}\right) \geq \lim_{m\to\infty}\psi\left(M\left(x_{m-1},u,\frac{t_0}{2}\right)\right) = 1 \quad (\text{using Lemma 2.19})\\
\textit{or}\ &\lim_{m\to\infty} M\left(x_m,Tu,\frac{t_0}{2}\right) = 1.
\end{aligned}$$

This yields,

$$\begin{aligned}
&\lim_{m\to\infty} f\left(M\left(u,x_m,\frac{t_0}{2}\right)\star M\left(x_m,Tu,\frac{t_0}{2}\right)\right) = f(1\star 1) = 1 \quad (\text{using}\ (\mathscr{F}2))\\
\textit{or}\ &\lim_{m\to\infty} (f(M(u,Tu,t_0)))^\alpha \geq 1 \quad (\text{using the relation (28)})\\
\textit{or}\ &M(u,Tu,t_0) = 1 \quad (\text{using}\ (\mathscr{F}2)).
\end{aligned}$$

Thus, we arrived at a contradiction to the relation (27). Therefore, u is a fixed point of T. To prove the uniqueness of the fixed point, suppose there exists an element $v\in X$ with $u\neq v$ such that $Tv=v$. Since $v\neq u$, there exists $s>0$ such that $M(u,v,s)<1$. In that case, we get

$$M(u,v,s) = M(Tu,Tv,s) \geq \psi(M(u,v,s)) > M(u,v,s),$$

which is a contradiction. So, T has a unique fixed point in X. □

We present the following example in support of the above theorem.

Example 4.3. Consider the Example 2.8. Here we take $X=[-5,5]$. First, we show that $(X,M,f,\alpha,\star)$ is a $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space. For, consider a $\mathscr{F}$-Cauchy sequence $\{x_n\}$ in $(X,M,f,\alpha,\star)$. Then

$$\begin{aligned}
&\lim_{m,n\to\infty} M(x_n,x_m,t) = \lim_{m,n\to\infty}\left(\frac{t}{t+1}\right)^{|x_n-x_m|^2} = 1, \quad \forall t\in(0,\infty)\\
\textit{or}\ &\lim_{m,n\to\infty} |x_n-x_m| = 0.
\end{aligned}$$

Therefore, $\{x_n\}$ is a Cauchy sequence in X with respect to the usual metric. Since X is complete with respect to the usual metric, there exists $x\in X$ such that $\lim_{n\to\infty}|x_n-x|=0$ which implies

$$\lim_{n\to\infty} M(x_n,x,t) = \lim_{n\to\infty}\left(\frac{t}{1+t}\right)^{|x_n-x|^2} = 1, \quad \forall t\in(0,\infty).$$

This implies $x_n \to x$ as $n \to \infty$ in $(X, M, f, \alpha, \star)$. Hence $(X, M, f, \alpha, \star)$ is a $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space.

Next we define two mappings as $T : X \to X$ by $T(x) = \frac{x}{6}$, $\forall x \in X$ and $\psi : [0, 1] \to [0, 1]$ by $\psi(t) = \sqrt{t}$, $\forall t \in [0, 1]$. Then we have

$$M(Tx, Ty, t) = M\left(\frac{x}{6}, \frac{y}{6}, t\right) = \left(\frac{t}{1+t}\right)^{\frac{|x-y|^2}{36}}$$

$$\text{and} \quad \psi\left(M(x, y, t)\right) = \sqrt{M(x, y, t)} = \left(\frac{t}{1+t}\right)^{\frac{|x-y|^2}{2}}$$

which implies

$$M(Tx, Ty, t) \geq \sqrt{M(x, y, t)} = \psi\left(M(x, y, t)\right).$$

Hence, T satisfies the fuzzy $\mathscr{F}$-ψ-contraction condition (23). So by the above theorem, T has a unique fixed point in X, which is $x = 0$.

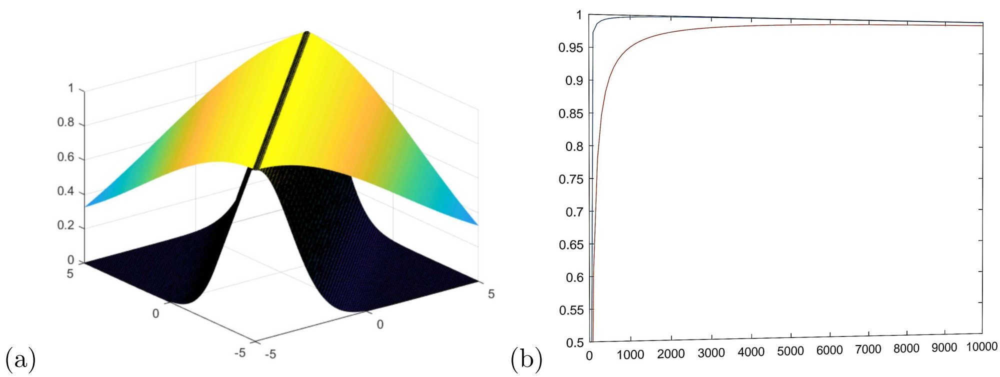


Fig. 1. (a) $M(Tx, Ty, t)$ vs $\psi(M(x, y, t))$ at $t = 2$ (b) $M(Tx, Ty, t)$ vs $\psi(M(x, y, t))$ for $|x - y| = 10$.

The geometric illustration of this example is shown in Figure 1, which shows the variation between $M(Tx, Ty, t)$ and $\psi\left(M(x, y, t)\right)$. In Figure 1$(a)$, the variation is due to a particular fixed value of t and Figure 1(b) shows the variation for fixed values of $|x - y|, x, y \in X$. This visualization highlights the interplay between the mapping T and the function ψ that demonstrates how the inequality $M(Tx, Ty, t) \geq \psi\left(M(x, y, t)\right)$ is sustained under the given conditions. The figure not only supports the theoretical framework of the contraction condition but also provides an intuitive understanding of the relationship between the parameters, thereby making the main result more comprehensible and accessible.

In the following example, we show that the conditions of the theorem are sufficient but not necessary.

Example 4.4. Consider the fuzzy $\mathscr{F}$-metric M of Example 2.8 over $X = \{0, 2, 4, \cdots\}$ and define a mapping $T : X \to X$ by $T(x) = 10x$, $\forall x \in X$.

First, we show that X is complete in $\mathbb{R}$ with respect to the usual metric. For, let $\{x_n\}$ be a $\mathscr{F}$-Cauchy sequence in $(X, M, f, \alpha, \star)$. Then

$$\lim_{m,n\to\infty} M(x_n, x_m, t) = 1 \; i.e. \lim_{m,n\to\infty} |x_n - x_m| = 0.$$

Thus, $\{x_n\}$ is Cauchy sequence X in with respect to usual metric. Since X is closed in $\mathbb{R}$ under the usual metric. So, X is complete, thus there exists $x \in X$ such that

$$\lim_{n\to\infty} |x_n - x| = 0$$

and hence $\lim_{n\to\infty} M(x_n, x, t) = \lim_{n\to\infty} \left(\frac{t}{t+1}\right)^{|x_n - x|^2} = \left(\frac{t}{t+1}\right)^{\lim_{n\to\infty} |x_n - x|^2} = 1.$

Therefore, $(X, M, f, \alpha, \star)$ is $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space.

Next consider the mapping $\psi(t) = \sqrt{t}, \quad \forall t \in [0, 1]$. Then, we get

$$M(Tx, Ty, t) = \left(\frac{t}{t+1}\right)^{|Tx - Ty|^2} = \left(\frac{t}{t+1}\right)^{100|x-y|^2}$$

$$and \; \psi(M(x, y, t)) = \sqrt{M(x, y, t)} = \left(\frac{t}{t+1}\right)^{\frac{|x-y|^2}{2}}$$

which yields

$$or \; M(Tx, Ty, t) \le \psi(M(x, y, t)).$$

This shows that T does not satisfy the contraction condition (23), but it has a fixed point in X which is '0'.

4.2. Application to a satellite web coupling problem:

Inspired by the applications of fixed point techniques to a variety of real-world problems, we employ Theorem 4.2 to address a satellite web coupling boundary value problem [18]. A satellite web coupling can be conceptualized as a thin sheet linking two cylindrical satellites. The analysis of radiation emanating from the web coupling between these satellites gives rise to the following nonlinear boundary value problem (BVP):

$$-\frac{d^2\omega}{dx^2} = \mu\omega^4, \; 0 < t < 1, \; \omega(0) = \omega(1) = 0 \tag{29}$$

where $\omega(x)$ represents the temperature of radiation at any point $x \in [0, 1]$, $\mu = \frac{2al^2K^3}{bh} > 0$ is a dimensionless positive constant, K denotes the constant absolute temperature of both satellites, while heat is radiated from the surface of the web at 0 absolute temperature, l is the distance between the two satellites, a is a positive constant characterizing the radiation properties of the web's surface, the factor 2 accounts for radiation from both the top and bottom surfaces, b is the thermal conductivity, and h is the thickness of the web.

Next, we recall the corresponding integral equation:

$$\omega(x) = \mu \int_0^1 G(x,s)\,\omega^4(s)\,\mathrm{d}s$$

where $G(x,s)$ is the Green's function, defined by

$$G(x,s) = \begin{cases} x\,(1-s), & 0 < x < s \\ s\,(1-x), & s < x < 1. \end{cases}$$

Let $X = C[0,1]$, the class of all real-valued continuous functions defined on $[0,1]$ and define a function $M : X \times X \times (0,\infty) \to [0,1]$ by

$$M(g,h,t) = \left(\frac{t}{t+1}\right)^{\sup_{s\in[0,1]} |g(s)-h(s)|^2}$$

for all $g, h \in X$ and $t > 0$. Then, following the Example 2.8, we can show that M is a fuzzy $\mathscr{F}$-metric on X with $f(x) = x^2$, $\forall x \in [0,1]$, $\alpha = \frac{1}{2}$ and $\star =$ product. Moreover, $(X, M, f, \alpha, \star)$ is a $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space.

Remark 4.5. In this section, the considered fuzzy $\mathscr{F}$-metric M is continuous with respect to the parameter $t > 0$, for all $g, h \in X$. Although such spaces may be derived from an b-metric structure, which is generally discontinuous with respect to its variable, continuity is assumed here to ensure the applicability of the fixed point framework to the satellite web coupling problem.

Theorem 4.6. Consider the $\mathscr{F}$-complete fuzzy $\mathscr{F}$-metric space $(X, M, f, \alpha, \star)$ defined above. Suppose that the BVP admits functions $\omega, v \in X$ such that

$$\sup_{s\in[0,1]} \left|\,\big(\omega^2(s) + v^2(s)\big)\,(\omega(s) + v(s))\,\right| \le \frac{k}{\mu}, \qquad k \in (0,4). \tag{30}$$

Then the satellite web coupling BVP (29) admits a unique solution.

Proof. We define a self-mapping $T : X \to X$ by

$$T(\omega(x)) = \mu \int_0^1 G(x,s)\,\omega^4(s)\,\mathrm{d}s, \quad s \in [0,1]. \tag{31}$$

Then the integral operator T is associated with (29). Clearly, a solution to the satellite web coupling problem (29) corresponds to a fixed point of the self-mapping A.

Now for all $\omega, v \in X$ and $x \in [0,1]$,

$$|T\omega(x) - Tv(x)|^2$$
$$= \mu^2 \left| \int_0^1 (\omega^4(s) - v^4(s))\,G(x,s)\,\mathrm{d}s \right|^2$$

$$= \mu^2 \left| \int_0^1 \left\{ \left(\omega^2(s) + v^2(s)\right) (\omega(s) + v(s)) (\omega(s) - v(s)) \right\} G(x,s)\,\mathrm{d}s \right|^2$$
$$\leq \mu^2 \sup_{s\in[0,1]} \left|(\omega^2(s) + v^2(s))(\omega(s) + v(s))\right|^2 \left| \int_0^1 (\omega(s) - v(s))\, G(x,s)\,\mathrm{d}s \right|^2$$
$$\leq k^2 \sup_{s\in[0,1]} \left|\omega(s) - v(s)\right|^2 \sup_{x\in[0,1]} \left| \int_0^1 G(x,s)\,\mathrm{d}s \right|^2$$
$$= \beta \sup_{s\in[0,1]} |\omega(s) - v(s)|^2 \quad \text{where}\ \beta = \frac{k^2}{64} \in (0,1).$$

This implies

$$\sup_{x\in[0,1]} |T\omega(x) - Tv(x)|^2 \leq \beta \cdot \sup_{s\in[0,1]} |\omega(s) - v(s)|^2$$
$$\text{or} \left(\frac{t}{t+1}\right)^{\sup_{x\in[0,1]} |T\omega(x) - Tv(x)|^2} \geq \left(\frac{t}{t+1}\right)^{\beta\cdot \sup_{x\in[0,1]} |\omega(x) - v(x)|^2}, \quad \forall t > 0.$$

Now consider the mapping $\psi(t) = t^\beta$, $\forall t \in [0,1]$ where $\beta = \frac{k^2}{64} \in (0,1)$. Then the above inequality gives

$$M(T\omega, Tv, t) \geq \psi\left(M(\omega, v, t)\right), \quad \forall t > 0.$$

Thus, the mapping T fulfill the conditions of the Theorem 4.2 and therefore T has a unique fixed point in X. Consequently, the BVP (29) has a solution in X. □

4.3. Numerical illustration:

For numerical illustration of the Theorem 4.6, we consider a numerical example based on the BVP (29) and the corresponding operator $T : X \to X$ defined by (31). First note that, $T(0) = 0$, so $u \equiv 0$ is always a fixed point of T.

Next, we choose nontrivial functions from $X = C[0,1]$ as:

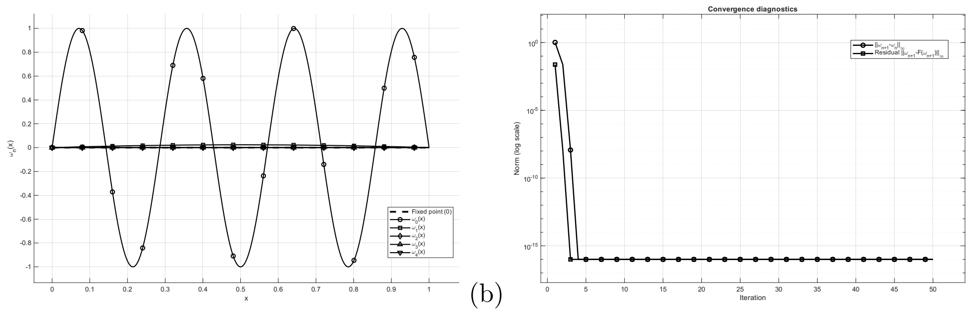
$$\omega(s) = \sin(7\pi s) \quad \& \quad v(s) = s\,(1 - s),\ \forall s \in [0,1].$$

Both functions vanish at $x = 0$ and $x = 1$. A numerical maximization yields

$$M = \sup_{s\in[0,1]} \left|(\omega(s)^2 + v(s)^2)\,(\omega(s) + v(s))\right| \approx 1.2944691606,$$

attained at $s \approx 0.3574$. In particular, we choose $\mu = 0.5$ and $k = 0.7$, so that the hypothesis (30) is satisfied. These serve as benchmark functions showing the admissibility of nontrivial solutions.

Hence, as per the proof of Theorem 4.6, the operator T satisfies the contraction condition (23) with respect to the fuzzy $\mathscr{F}$-metric M and the function ψ defined in Section 4.2. Therefore, T admits a unique fixed point in X. Since $T(0) = 0$, the unique fixed point in X is precisely the zero function.

(a) 

(b)

Fig. 2. (a) Picard iterates ω_n and the zero fixed point (b) Sup-norm error $\|\omega_{n+1} - \omega_n\|_\infty$ vs iteration n. Final step diff $= 0.000e + 00$, final residual $= 0.000e + 00$ after 50 iterations.

Initial Function for Iteration: For the numerical Picard iteration, we choose an initial guess as: $\omega_0(x) = \sin(7\pi x)$.

Picard Iteration Scheme: We discretize the interval $[0, 1]$ using $N = 1000$ equally spaced points and construct the corresponding Green's function matrix G. The Picard iteration is then defined by

$$\omega_{n+1} = \mu\, G\,(\omega_n^4)\,\Delta x, \quad n = 0, 1, 2, \ldots,$$

where Δx is the grid spacing, and the boundary conditions $\omega_{n+1}(0) = \omega_{n+1}(1) = 0$ are enforced at each iteration. The iteration continues until the sup-norm error

$$\|\omega_{n+1} - \omega_n\|_\infty = \max_{1 \le i \le N} |\omega_{n+1}(x_i) - \omega_n(x_i)|$$

is below a specified tolerance.

Results: The iterates $\omega_n(x)$ generated by the Picard scheme converge toward the trivial fixed-point solution $u \equiv 0$ of (31). Figure 2 (a) displays several consecutive iterates $\omega_0(x), \omega_1(x), \omega_2(x), \ldots,$ superimposed with the horizontal axis representing the fixed point. As n increases, the oscillatory profile of the initial function gradually flattens out, and the curves contract uniformly toward the zero axis. Figure 2 (b) further confirms this behavior by showing that the sup-norm errors $\|\omega_n\|_\infty$ decrease monotonically with each iteration.

CONCLUSION

In this article, we study several fundamental properties related to the $\mathscr{F}$-compactness and $\mathscr{F}$-totally boundedness of fuzzy $\mathscr{F}$-metric spaces. Within this framework, we prove a fixed point theorem that not only enriches the existing literature but also demonstrates practical applicability through its relevance to the satellite web coupling problem. To illustrate the sufficiency of the theorem's conditions, we present concrete examples. A geometric illustration of the example is also provided, that shows the variation between $M(Tx, Ty, t)$ and $\psi(M(x, y, t))$ as described in relation (23), and highlights the interplay between the mapping T and the function ψ. The figure supports the theoretical foundation of the contraction condition and provides an intuitive understanding of the relationships among the parameters involved. Furthermore, a numerical illustration is implemented using MATLAB to demonstrate the consistency of numerical results with the theoretical convergence guaranteed by Theorem 4.6, thereby validating the applicability of the BVP (29) within the fuzzy $\mathscr{F}$-metric space framework.

These results lay the foundation for further research that extends to study more generalized spaces or explores additional applications in dynamic systems, optimization, and computational mathematics.

Data availability. Data sharing does not apply to this article.

Conflicts of Interest. The authors declare no conflicts of interest.

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