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AN APPROACH TO SOLVE A MULTI-INDEX CAPACITATED TRANSPORTATION PROBLEMS

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This study focuses on a particular instance of the capacitated multi-index transportation problem, specifically the four-index variant. It involves transporting various products from multiple sources to multiple destinations using different means of transport chosen depending on reserved charges, while satisfying the constraints associated with each dimension of the model. Although this problem was previously studied by Zitouni and Keraghel (2003), the available methods still suffer from slow convergence and difficulty in handling degeneracy. This work proposes a new procedure that ensures a stable and feasible starting solution. The approach involves introducing dummy allocations with very small values to some null allocations, in order to handle cases of degeneracy. It is based on an extended version of the Vogel's approximation method, where the traditional selection criterion is modified to account for the four dimensions of the problem. This enhancement improves the initial distribution of quantities, reduces ineffective allocations and accelerates convergence. The proposed method maintains full compliance with all problem constraints including supply, demand, capacity and the structural logic of the four-index model. The methodology was evaluated through a comparative numerical study, applying the proposed method to a set of problem instances of varying sizes, and comparing it with Zitouni's approach in terms of convergence speed and number of iterations. The results show that the proposed method achieves a clear advantage in terms of faster convergence required to reach the solution. This highlights the relevance of the approach for complex logistical models and supply chain planning, where efficient solutions must be obtained within limited time. Moreover, the proposed approach lays the foundation for the development of more advanced multi-index optimization algorithms.

Keywords: transportation problem, capacitated transportation problem, multi-index problem, linear programming

Classification: 90C05, 90B06, 03B52

1. INTRODUCTION

Transportation problems have long been a central topic in operations research because of their wide range of practical applications. Since their initial formulation in the 1940s, numerous extensions and improvements have been proposed, establishing them as a cornerstone in optimization theory and practice.

Over the decades, various algorithms were developed to enhance both solution quality

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and computational efficiency. These include the modified distribution method MODI, Vogel's method [16], the Russell method [18], and other heuristics designed to produce effective initial solutions. Research has also expanded to incorporate real-world constraints such as limited vehicle capacity, docking availability, and multimodal logistics systems. In this context, models addressing uncertainty and infrastructure limitations have been explored including those by Dahiya et al. [3], Uddin and Huynh [22] and Ritzinger et al. [17]. Buvaneshwari and Anuradha [1] also addressed the stochastic fuzzy transportation problem with mixed constraints using the Weibull distribution, contributing to improved representation of probabilistic distributions under uncertain environments. Kartli et al. [9] addressed the fully fuzzy transportation problem, where all model parameters are represented by fuzzy numbers in parametric form. They proposed a new heuristic algorithm that transforms the problem into two parametric sub-models to obtain efficient approximate solutions. Meanwhile, Kartli et al. [8], Rani [15], Shivani et al. [20], and Kartli [7] focused on the fixed charge transportation problem FCTP using various approaches. Kartli et al. [8] proposed a heuristic algorithm to minimize total cost, while Rani [15] introduced a nonlinear NPSO algorithm that outperforms genetic and standard PSO variants in terms of quality and efficiency. Shivani et al. [20] developed the CEPSO algorithm based on particle swarm optimization with chaotic maps to achieve a better exploration–exploitation balance, whereas Kartli [7] proposed three hybrid algorithms that improved performance by 4-5%, with higher effectiveness as problem size increased.

To address increasingly complex logistics scenarios, the Three-Index transportation problem 3ITP was introduced, incorporating a third dimension to represent product types or means of transport. Jimenez and Verdegay [6] tackled the fuzzy versions of this problem by converting them into interval problems and applying evolutionary algorithms to handle uncertainty. Shiang-Tai Liu [11] developed an enhanced genetic algorithm for solving multi-objective problems involving fuzzy coefficients. Notably, Pandian and Anuradha [14] proposed efficient algorithmic frameworks to improve accuracy and performance in complex transportation environments.

Building on these developments, Ninh introduced the generalized Multi-Index Transportation Problem nITP in 1979 [12], extending classical models and relying on the potential method. Although Ninh established necessary and sufficient conditions for the existence of solutions, his model did not cover all special cases. Subsequent research further developed the geometric representation of these problems, as shown in the work of Queyranne et al. [13].

More recent contributions have applied sophisticated mathematical tools. Hakim and Zitouni [4] proposed a fuzzy bi-objective approach for solving the multi-index fixed charge transportation problem using an improved fuzzy modeling framework. Sandhiya and Dhanapal [19] addressed the neutrosophic multi-dimensional fixed charge transportation problem, enhancing uncertainty handling in transportation cost data. Kumar and Dhanapal [10] further extended the field by solving a multi-objective, bi-item capacitated transportation problem using fermatean fuzzy logic and multi-choice stochastic constraints under a normal distribution. Singh and Singh [21] provided a comprehensive review of the variations, methods, and applications of multi-index transportation problems.

Among the multi-index transportation models, the four-index transportation problem 4ITP has gained considerable research interest due to its practical relevance in shuttle-type transport systems utilizing fleets of trucks. Based on Ninh's solution existence condition SEC, the model's constraint system was simplified in non-degenerate cases, improving solution efficiency. Zitouni later extended this framework to address the capacitated four-index transportation problem C4ITP [23, 24, 25]. In 2020, Hedid and Zitouni introduced an adaptive algorithm for solving the fuzzy 4ITP, showing greater accuracy and efficiency over classical methods [5]. Most recently, a 2025 study on the "four-dimensional green transportation problem" integrated multiple objectives and product blending in a fermatean fuzzy environment to balance transportation cost, time and emissions [2].

In 2003, Zitouni [24] proposed an algorithm for solving the capacitated four-index transportation problem. This algorithm provides a direct solution with rapid convergence, offering advantages over traditional methods such as the simplex method in linear programming, particularly in terms of reducing computational time and the number of iterations required to reach a solution. Zitouni's method relies on finding an initial distribution close to the optimal solution and then gradually improving it through specific steps without the need to traverse all possible basic solutions, which contributes to speeding up the solution process.

However, despite its advantages, Zitouni's method suffers from certain drawbacks when dealing with the degeneracy problem, which arises when the number of allocations in the initial solution is less than the theoretically required number, i. e., $m+n+p+q-3$. This leads to stagnation in the improvement process or repeated solutions without progress, hindering reaching the optimal solution or slowing down the algorithm's convergence.

Our study aims to improve Zitouni's method, specifically during the initialization phase, by implementing two main modifications. First, we changed the mechanism of selecting initial allocation cells by adopting the principle of Vogel's Approximation Method, which involves calculating the differences between the two lowest costs within each subset of the four-dimensional structure (i. e., across the four dimensions of the problem), and then prioritizing allocation in positions with the highest differences. This modification helps improve the quality of the initial solution, thus reducing the number of subsequent iterations. Second, to address the degeneracy problem, we introduced dummy allocations with very small values (close to zero) in unallocated positions to ensure reaching the necessary number of allocations while maintaining all capacity constraints without affecting the accuracy of the solution.

Zitouni's method aims to solve the capacitated four-index transportation problem by first finding an initial solution and then improving it iteratively. Although the method is characterized by fast convergence and a low number of iterations, it suffers from degeneracy in the initial phase, which reduces efficiency, and the previously proposed solution [25] was not always effective. Our study improves the first phase of the method by adapting Vogel's Approximation to select the initial allocation cells and addressing degeneracy by adding small dummy allocations to ensure a sufficient number of allocations while maintaining solution accuracy. The effectiveness of these improvements has been demonstrated through a comprehensive numerical study, showing reduced computation time and fewer iterations, thereby enhancing the practical applicability of the method in capacitated transportation problems. The data are generated automatically in a random manner, ensuring that the input parameters are feasible and satisfy the necessary conditions to guarantee the existence of a feasible solution. The aim of Zitouni's method is to solve the capacitated four-index transportation problem, starting by finding an initial solution and improving it iteratively. Although it is characterized by fast convergence and fewer iterations, but it suffers from degeneracy in the initial phase, which leads to reduced efficiency. While a solution was proposed to address this issue [25], it was not sufficiently effective in all cases. Our study improves the first phase of Zitouni's method by using an adaptation of the Vogel's approximation to select initial allocation cells. It also tackles degeneracy by adding tiny dummy allocations to ensure enough allocations without compromising accuracy. The proposed improvements include a new mechanism for selecting initial allocation cells, handling degeneracy by completing the initial solution with small dummy allocations, and demonstrating the effectiveness of the method through a comprehensive numerical study, showing reduced computation time and fewer iterations compared to the Zitouni's method. These improvements are supported by extensive numerical analysis, which confirms that the proposed approach reduces the number of required iterations and accelerates solution time compared to Zitouni's method, enhancing its practical applicability in capacitated transportation problems. The data are generated automatically in a random manner, ensuring that the input parameters are feasible and satisfy the necessary conditions to guarantee the existence of a valid solution for the problem.

This paper is organized as follows. In section 2, we present some preliminaries, while in section 3 we give feasibility conditions and the optimality criterion. Section 4 presents the new methodology proposed to improve Zitouni's method. Section 5 provides numerical simulations to demonstrate the effectiveness of our approach and Section 6 concludes with the results and insights drawn from the study.

2. PRELIMINARIES [25]

2.1. Formulation of the capacitated four-index transportation problem

The capacitated four-index transportation problem C4ITP is an advanced extension of classical transportation models. It destinations distributing q distinct products from m sources to n destinations using p transportation means, while considering the unique characteristics of each product. The objective is to minimize the total transportation cost while satisfying constraints related to the supply, the demand, the product characteristics, and the transportation mean capacities. C_{ijkl} denotes the unit transportation cost of the quantity x_{ijkl}. The mathematical formulation of the problem is presented as follows:

$$\text{Minimize}\, Z = \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} c_{ijkl}x_{ijkl} \tag{1}$$

subject to the constraints:

$$\sum_{j=1}^{n}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \alpha_i \quad \text{for all} \quad i = 1, \ldots, m, \tag{2}$$

$$\sum_{i=1}^{m}\sum_{k=1}^{p}\sum_{l=1}^{q} x_{ijkl} = \beta_j \quad \text{for all} \quad j = 1, \dots, n, \tag{3}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{l=1}^{q} x_{ijkl} = \gamma_k \quad \text{for all} \quad k = 1, \dots, p, \tag{4}$$

$$\sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} x_{ijkl} = \delta_l \quad \text{for all} \quad l = 1, \dots, q, \tag{5}$$

$$0 \leq x_{ijkl} \leq d_{ijkl} \qquad \text{for all} \quad (i, j, k, l). \tag{6}$$

Such that the quantities α_i, β_j, γ_k, δ_l, d_{ijkl} and c_{ijkl} are given and are non-negative for all i, j, k and l. Where:

- α_i: The quantity of the supply available at source i,
- β_j: The quantity of the demand required at destination j,
- γ_k: The total capacity available for the transportation mean k,
- δ_l: The total amount to be transported of the product l,
- d_{ijkl}: The maximum amount that can be transported along the route (i, j, k, l),
- m, n, p and q: The number of sources, destinations, transportation means and product types, respectively.

The C4ITP problem can be also reformulated as the following linear programming problem:

$$\min_{x} \; Z = c^\top x \quad \text{s.t.} \quad Ax = b, \; 0 \leq x \leq d.$$

With:

- $\boldsymbol{x} = (x_{1111}, x_{1112}, \dots, x_{mnpq})^\top \in \mathbb{R}^N$,
- $\boldsymbol{c} = (c_{1111}, c_{1112}, \dots, c_{mnpq})^\top \in \mathbb{R}^N$,
- $\boldsymbol{d} = (d_{1111}, d_{1112}, \dots, d_{mnpq})^\top \in \mathbb{R}^N$,
- $\boldsymbol{b} = (\alpha_1, \dots, \alpha_m, \beta_1, \dots, \beta_n, \gamma_1, \dots, \gamma_p, \delta_1, \dots, \delta_q)^\top \in \mathbb{R}^M$,
- $A \in \mathbb{R}^{M \times N}$, where $M = m + n + p + q$ and $N = m\,n\,p\,q$.

In this representation $\boldsymbol{x} = (x_{1111}, x_{1112}, \dots, x_{mnpq})$, we associate to each $(i, j, k, l) \in \{1, \dots, m\} \times \{1, \dots, n\} \times \{1, \dots, p\} \times \{1, \dots, q\}$ a vector $P_{ijkl} \in \mathbb{R}^M$. Only four entries of the vector P_{ijkl} are non-zero; they are located in the rows i, $m + j$, $m + n + k$ and $m + n + p + l$, and have 1 as a common value. Thus the matrix A can be written as $A = \begin{bmatrix} P_{1111} \; P_{1112} \; \cdots \; P_{mnpq} \end{bmatrix}$.

2.2. Definitions and properties

Definition 2.1. A feasible solution x of C4ITP is called *basic solution* if the columns of the submatrix A_x obtained from the coefficient matrix A by keeping only the columns corresponding to the variables x_{ijkl} such that $0 < x_{ijkl} < d_{ijkl}$ are linearly independent.

Definition 2.2. A basic feasible solution is said to be *non-degenerate* if

$$rank(A_x) = rank(A).$$

Throughout the paper, we assume that the following condition holds

$$\sum_{i=1}^{m} \alpha_i = \sum_{j=1}^{n} \beta_j = \sum_{k=1}^{p} \gamma_k = \sum_{l=1}^{q} \delta_l = F. \tag{7}$$

Under this hypothesis, we can show that $rank(A) = M - 3$, where $M = m + n + p + q$.

Remark. Unlike the transportation problem with two indices, the matrix A is not totally unimodular since some of its minors do not belong to $\{-1, 0, 1\}$.

Transportation table. It is useful to present the data of the problem thanks to the following transportation table. It consists of an array of M rows and N columns, three additional rows and an additional column. The entries of these N columns of the first, second, and third additional rows are reserved for the data of the quantities d_{ijkl}, c_{ijkl}, and x_{ijkl}, respectively. The additional column is for the data of quantities α_i, β_j, γ_k and δ_l respectively. Finally, the entry of the array on the line corresponding to $\alpha_{i'}$ and the entries of the column P_{ijkl} are equal to 1 if $i = i'$ and 0 otherwise. Same things for $\beta_{j'}$,$\gamma_{k'}$ and $\delta_{l'}$. We illustrate that by the following table.

d_{1111}	d_{1112}	$\cdots$	d_{mnpq}	
c_{1111}	c_{1112}	$\cdots$	c_{mnpq}	
x_{1111}	x_{1112}	$\cdots$	x_{mnpq}	
1	1	$\cdots$	0	α_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\cdots$	1	α_m
1	1	$\cdots$	0	β_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\cdots$	1	β_n
1	1	$\cdots$	0	γ_1
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\cdots$	1	γ_p
1	0	$\cdots$	0	δ_1
0	1	$\cdots$	0	δ_2
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
0	0	$\cdots$	1	δ_q

Tab. 1. Transportation table.

3. FEASIBILITY CONDITIONS AND OPTIMALITY CRITERION

3.1. Feasibility conditions [25]

We begin by presenting a useful theorem that ensures that the C4ITP problem admits a feasible solution $x = (x_{ijkl})$.

Theorem 3.1.

1. **Necessary condition:** A necessary condition for the feasibility of the problem C4ITP is that the condition (7) and the following conditions

$$\begin{cases} \alpha_i \leq \sum_{j=1}^{n} \sum_{k=1}^{p} \sum_{l=1}^{q} d_{ijkl} & \text{for all} \quad i = 1, \dots, m, \\ \beta_j \leq \sum_{i=1}^{m} \sum_{k=1}^{p} \sum_{l=1}^{q} d_{ijkl} & \text{for all} \quad j = 1, \dots, n, \\ \gamma_k \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{l=1}^{q} d_{ijkl} & \text{for all} \quad k = 1, \dots, p, \\ \delta_l \leq \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} d_{ijkl} & \text{for all} \quad l = 1, \dots, q, \end{cases} \tag{8}$$

are satisfied.

2. **Sufficient condition:** A sufficient condition for the feasibility of the problem C4ITP is that the condition (7) and the following condition

$$\frac{\alpha_i \, \beta_j \, \gamma_k \, \delta_l}{F^3} \leq d_{ijkl}, \quad \forall (i, j, k, l) \tag{9}$$

are satisfied.

3.2. Optimality criterion [25]

In this subsection, we give a theorem that ensures when a feasible solution of the problem C4ITP becomes optimal.

Theorem 3.2.
A feasible solution $x = (x_{ijkl})$ of the problem C4ITP is optimal if and only if there exists a vector $y = (u_1, \dots, u_m, v_1, \dots, v_n, w_1, \dots, w_p, t_1, \dots, t_q)^\top \in \mathbb{R}^M$ such that:

$$\begin{cases} u_i + v_j + w_k + t_l \leq c_{ijkl}, & \text{if} \quad x_{ijkl} = 0, \\ u_i + v_j + w_k + t_l = c_{ijkl}, & \text{if} \quad 0 < x_{ijkl} < d_{ijkl}, \\ u_i + v_j + w_k + t_l \geq c_{ijkl}, & \text{if} \quad x_{ijkl} = d_{ijkl}. \end{cases}$$

4. RESOLUTION OF THE PROBLEM

To obtain the optimal solution to the capacitated four-index transportation problem C4ITP, we begin by determining an initial feasible solution, followed by the optimal solution. It is beneficial to use methods that provide an initial solution very close to the optimal. To improve Zitouni's method, we introduced an adaptation of VAM [16] [in phase 1 of this algorithm] for the problem C4ITP to obtain an initial solution and address the degeneracy issue in the first phase. The newly adapted method, which we denote MAL_{CTP_4}, is described in the following section.

4.1. Description of the proposed method

To solve the capacitated four-index transportation problem C4ITP, we proceed through two phases:

Phase 1: Determining an initial basic feasible solution or declaring that the problem is not feasible
In this phase, most of the modifications to handle the degeneracy problem are incorporated into Zitouni's method. The objective is to determine a basic feasible solution or to declare that the C4ITP problem is not feasible. Full details can be found below in Algorithm 1.

Phase 2: Improving a basic feasible solution
This phase is the same as the one described in Zitouni's method, it aims to improve a basic feasible solution to obtain an optimal solution. See Algorithm 2.

We now present the details of the algorithms:

Algorithm 1:

Input: Costs c_{ijkl}, capacities d_{ijkl}, data $\alpha_i, \beta_j, \gamma_k, \delta_l$.
Output: Optimal solution x_{ijkl} or a not feasible problem.

Phase 1: Finding a basic feasible solution
Initialize all $x_{ijkl} \leftarrow 0$
Define index sets: $I = \{1, \ldots, m\}$, $J = \{1, \ldots, n\}$, $K = \{1, \ldots, p\}$ and $L = \{1, \ldots, q\}$.
First sub-phase 1: Composed of three steps
Step 1: Choice of the cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$ to allocate.

1. Determine the penalties u_i, v_j, w_k and t_l as the difference between the minimum and the second minimum cost for each unsaturated $i \in I$, $j \in J$, $k \in K$, and $l \in L$, respectively.
2. Identify the dimension with the highest penalty among u_i, v_j, w_k and t_l.
3. In the chosen dimension, select the cell (i, j, k, l) with the lowest cost.
4. If there is a tie, select the dimension that contains the cell with the original lowest cost.
5. If there is again a tie, choose from these cells the first one on the left of the transportation table.

Step 2: Assignment to the chosen cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$
Handle degeneracy by assigning small non-zero values (ψ) to cells that originally had zero value, allowing their inclusion in the basis, to give all basic variables of the curent solution. At the end, these ψ values are restored to zero.

Step 3: Stop allocation
Repeat Steps 1 and 2 until all variables x_{ijkl} have been determined. Define $x_B^{(r)}$ as the set of the $m+n+p+q-3$ determined basic variables.

Second sub-phase 1: Apply the same procedure as in Phase 1 of Zitouni's method (Zitouni and Achache [25]).

Phase 2: Improving a basic feasible solution
Follow the steps of Phase 2 of Zitouni's method to improve x until optimality is reached.

We now present step 2:

Algorithm 2: Step 2: Assignment to the cell $(\bar{i},\bar{j},\bar{k},\bar{l})$

Input: Chosen cell $(\bar{i},\bar{j},\bar{k},\bar{l})$ and values $\Omega_1,\Omega_2,\Omega_3,\Omega_4,d_{\bar{i}\bar{j}\bar{k}\bar{l}}$.

Output: Updated Ω_r and x_{ijkl} values.

Let ψ be a small positive real number.
Assign $\alpha_{\bar{i}} \leftarrow \Omega_1$, $\beta_{\bar{j}} \leftarrow \Omega_2$, $\gamma_{\bar{k}} \leftarrow \Omega_3$ and $\delta_{\bar{l}} \leftarrow \Omega_4$
Compute $x_{\bar{i}\bar{j}\bar{k}\bar{l}} \leftarrow \min(\Omega_1,\Omega_2,\Omega_3,\Omega_4,d_{\bar{i}\bar{j}\bar{k}\bar{l}})$

For $r \leftarrow 1$ **to** 4 **do**
if $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \Omega_r$ *and* $\Omega_s \neq \Omega_r$ *for all* $s \neq r$ **then**
$\Omega_r \leftarrow \Omega_r - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$

if $\Omega_1 = 0$ **then**
$x_{\bar{i}jkl} \leftarrow 0$ for all $(j,k,l) \neq (\bar{j},\bar{k},\bar{l})$
if $\Omega_2 = 0$ **then**
$x_{i\bar{j}kl} \leftarrow 0$ for all $(i,k,l) \neq (\bar{i},\bar{k},\bar{l})$
if $\Omega_3 = 0$ **then**
$x_{ij\bar{k}l} \leftarrow 0$ for all $(i,j,l) \neq (\bar{i},\bar{j},\bar{l})$
if $\Omega_4 = 0$ **then**
$x_{ijk\bar{l}} \leftarrow 0$ for all $(i,j,k) \neq (\bar{i},\bar{j},\bar{k})$

if $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \Omega_r = \Omega_s$ *for some* $s \neq r$ *and* $\Omega_t \neq \Omega_r$ *for all* $t \neq r$ *and* $t \neq s$ **then**
Set the values of r and s and set the corresponding values x_{ijkl} to 0 in row r and then
$\Omega_r \leftarrow 0$
$\Omega_s \leftarrow \psi$
$\Omega_t \leftarrow \Omega_t - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$ for all t different from r and s

if $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \Omega_r = \Omega_s = \Omega_t$ *for some* $s \neq r$, $t \neq r$ *and* $s \neq t$, *and* $\Omega_u \neq \Omega_r$ *for all* r, s, t *and* u *pairwise different* **then**
Set the values of r, s and t and set the corresponding values x_{ijkl} to 0 in row r and then
$\Omega_r \leftarrow 0$
$\Omega_s \leftarrow \Omega_t = \psi$
$\Omega_u \leftarrow \Omega_u - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$, for all u different from t, r and s

if $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = \Omega_r = \Omega_s = \Omega_t = \Omega_u$ *for all r, s, t and u pairwise different* **then**
Set the values of r, s, t and u and set the corresponding values x_{ijkl} to 0 in row r and then
$\Omega_r \leftarrow 0$
$\Omega_s = \Omega_t = \Omega_u = \psi$

if $x_{\bar{i}\bar{j}\bar{k}\bar{l}} = d_{\bar{i}\bar{j}\bar{k}\bar{l}} \neq \Omega_r$ **then**
$\Omega_r = \Omega_r - x_{\bar{i}\bar{j}\bar{k}\bar{l}}$, for all $r = 1, \dots, 4$

Remark. When setting r, we assign the value 0 to x_{ijkl} as follows
for $r \leftarrow 1$ **to** *4* **do**
if $r = 1$ **then**
$x_{\bar{i}jkl} = 0$ for all $(j,k,l) \neq (\bar{j},\bar{k},\bar{l})$
if $r = 2$ **then**
$x_{i\bar{j}kl} = 0$ for all $(i,k,l) \neq (\bar{i},\bar{k},\bar{l})$
if $r = 3$ **then**
$x_{ij\bar{k}l} = 0$ for all $(i,j,l) \neq (\bar{i},\bar{j},\bar{l})$
if $r = 4$ **then**
$x_{ijk\bar{l}} = 0$ for all $(i,j,k) \neq (\bar{i},\bar{j},\bar{k})$.

4.1.1. Example

Solve the following C4ITP with: $m = n = p = q = 2$, $\alpha_1 = 40$, $\alpha_2 = 40$, $\beta_1 = 35$, $\beta_2 = 45$, $\gamma_1 = 20$, $\gamma_2 = 60$, $\delta_1 = 40$ *and* $\delta_2 = 40$.

The values of c_{ijkl} and d_{ijkl} are given in the following tables:

(i,j,k,l)	1111	1112	1121	1122	1211	1212	1221	1222
c_{ijkl}	5	4	3	11	21	8	7	1
d_{ijkl}	88	71	86	18	5	14	114	71

(i,j,k,l)	2111	2112	2121	2122	2211	2212	2221	2222
c_{ijkl}	18	4	8	24	36	7	42	18
d_{ijkl}	22	7	120	200	52	66	44	12

We take here ψ is a small positive real number and $x_B^{(r)}$, $x_H^{(r)}$ as the sets of basic and non-basic variables at iteration r, respectively.

First sub-phase 1

Step 1: choice of a cell $(\bar{i},\bar{j},\bar{k},\bar{l})$.

We have $I = \{1,2\}$, $J = \{1,2\}$, $K = \{1,2\}$ and $L = \{1,2\}$.

1.1. Penalties: We have

$$u_1 = c_{1121} - c_{1222} = 3 - 1 = 2$$
$$u_2 = c_{2212} - c_{2112} = 7 - 4 = 3$$
$$v_1 = c_{1112} - c_{1121} = 4 - 3 = 1$$
$$v_2 = c_{1221} - c_{1222} = 7 - 1 = 6$$

$$w_1 = c_{1112} - c_{1111} = 5 - 4 = 1$$
$$w_2 = c_{1121} - c_{1222} = 3 - 1 = 2$$
$$t_1 = c_{1111} - c_{1121} = 5 - 3 = 2$$
$$t_2 = c_{1112} - c_{1222} = 4 - 1 = 3.$$

1.2. The highest penalty is $v_2 = 6$ so $j = 2$.
1.3. The cell with the smallest cost for $j = 2$ is $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) = (1, 2, 2, 2)$.
Step 2: Allocation to the chosen cell

2.1. $\Omega_1 = \alpha_1 = 40, \quad \Omega_2 = \beta_2 = 45, \quad \Omega_3 = \gamma_2 = 60, \quad \Omega_4 = \delta_2 = 40.$
2.2. $x_{1222} = \min(\Omega_1, \Omega_2, \Omega_3, \Omega_4, d_{1222}) = \min(40, 45, 60, 40, 71) = 40.$
2.3. We have $x_{1222} = \Omega_1 = \Omega_4$, we update the supply values as follows:
$\Omega_1 = \Omega_1 - x_{1222} = 0 = \alpha_1$ thus $x_{1jkl} = 0, \ \forall (j, k, l) \neq (2, 2, 2)$,
$\Omega_2 = \Omega_2 - x_{1222} = 45 - 40 = 5 = \beta_2$,
$\Omega_3 = \Omega_3 - x_{1222} = 60 - 40 = 20 = \gamma_2$,
$\Omega_4 = \psi = \delta_2$.
Step 3: We have
$\alpha_1 = 0, \ \beta_1 = 35, \ \gamma_1 = 20, \ \delta_1 = 40, \ \alpha_2 = 45, \ \beta_2 = 5, \ \gamma_2 = 20, \ \delta_2 = \psi.$

First sub-phase 1

Step 1: choice of a cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$

We have $I = \{2\}, \quad J = \{1, 2\}, \quad K = \{1, 2\}$ and $L = \{1, 2\}$.
1.1. Penalties: We have

$$u_2 = c_{2212} - c_{2112} = 7 - 4 = 3$$
$$v_1 = c_{2121} - c_{2112} = 8 - 4 = 4$$
$$v_2 = c_{2222} - c_{2212} = 18 - 7 = 11$$
$$w_1 = c_{2212} - c_{2112} = 7 - 4 = 3$$
$$w_2 = c_{2121} - c_{2222} = 18 - 8 = 10$$
$$t_1 = c_{2111} - c_{2121} = 18 - 8 = 10$$
$$t_2 = c_{2212} - c_{2112} = 7 - 4 = 3.$$

1.2. The highest penalty is $v_2 = 11$ so $j = 2$.
1.3. The cell with the smallest cost for $j = 2$ is $(2, 2, 1, 2)$.
Step 2: Allocation to the chosen cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) = (2, 2, 1, 2)$

2.1. $\Omega_1 = \alpha_2 = 40, \quad \Omega_2 = \beta_2 = 5, \quad \Omega_3 = \gamma_1 = 20, \quad \Omega_4 = \delta_2 = \psi.$
2.2. $x_{2212} = \min(\Omega_1, \Omega_2, \Omega_3, \Omega_4, d_{2212}) = \min(40, 5, 20, \psi, 66) = \psi.$
2.3. We have $x_{2212} = \Omega_4$, (Ω_1, Ω_2, Ω_3 and Ω_4 are pairwise distinct), we update the supply values as follows:
$\Omega_1 = \Omega_1 - x_{2212} = 40 - \psi = \alpha_2$,
$\Omega_2 = \Omega_2 - x_{2212} = 5 - \psi = \beta_2$,
$\Omega_3 = \Omega_3 - x_{2212} = 20 - \psi = \gamma_1$,
$\Omega_4 = \Omega_4 - x_{2212} = \psi - \psi = 0 = \delta_2$ thus $x_{ijk2} = 0, \ \forall (j, k, l) \neq (2, 2, 1)$.
Step 3: We have
$\alpha_1 = 0, \ \beta_1 = 35, \ \gamma_1 = 20 - \psi, \ \delta_1 = 40, \ \alpha_2 = 40 - \psi, \ \beta_2 = 5 - \psi, \ \gamma_2 = 20, \ \delta_2 = 0.$

First sub-phase 1

Step 1: choice of a cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$

We have $I = \{2\}$, $J = \{1, 2\}$, $K = \{1, 2\}$ and $L = \{1\}$.

1.1. Penalties: We have

$$\begin{aligned} u_2 &= c_{2111} - c_{2121} = 18 - 8 = 10 \\ v_1 &= c_{2121} - c_{2112} = 18 - 8 = 10 \\ v_2 &= c_{2221} - c_{2211} = 42 - 36 = 6 \\ w_1 &= c_{2211} - c_{2111} = 36 - 18 = 18 \\ w_2 &= c_{2221} - c_{2121} = 42 - 8 = 34 \\ t_1 &= c_{2111} - c_{2121} = 18 - 8 = 10. \end{aligned}$$

1.2. The highest penalty is $w_2 = 34$ so $k = 2$.

1.3. The cell with the smallest cost for $k = 2$ is $(2, 1, 2, 1)$.

Step 2: Allocation to the chosen cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) = (2, 1, 2, 1)$

2.1. $\Omega_1 = \alpha_2 = 40 - \psi$, $\Omega_2 = \beta_1 = 35$, $\Omega_3 = \gamma_2 = 20$, $\Omega_4 = \delta_1 = 40$.

2.2. $x_{2121} = \min(\Omega_1, \Omega_2, \Omega_3, \Omega_4, d_{2121}) = \min(40 - \psi, 35, 20, 40, 120) = 20$.

2.3. We have $x_{2121} = \Omega_3$, (Ω_1, Ω_2, Ω_3 and Ω_4 are pairwise distinct), we update the supply values as follows:

$\Omega_1 = \Omega_1 - x_{2121} = 40 - \psi - 20 = 20 - \psi = \alpha_2$,
$\Omega_2 = \Omega_2 - x_{2121} = 35 - 20 = 15 = \beta_1$,
$\Omega_3 = \Omega_3 - x_{2121} = 20 - 20 = 0 = \gamma_2$ thus $x_{ij2l} = 0, \ \forall (i, j, l) \neq (2, 1, 1)$,
$\Omega_4 = \Omega_4 - x_{2121} = 40 - 20 = 20 = \delta_1$.

Step 3: We have

$\alpha_1 = 0, \ \beta_1 = 15, \ \gamma_1 = 20 - \psi, \ \delta_1 = 20, \ \alpha_2 = 40 - \psi, \ \beta_2 = 5 - \psi, \ \gamma_2 = 0, \ \delta_2 = 0.$

First sub-phase 1

Step 1: choice of a cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l})$

We have $I = \{2\}$, $J = \{1, 2\}$, $K = \{1\}$ and $L = \{1\}$.

1.1. Penalties: We have

$u_2 = 18$, $v_1 = 0$, $v_2 = 0$, $w_1 = 18$, $t_1 = 18$.

1.2. The highest penalty is $u_2 = 18$ so $i = 2$.

1.3. the cell with the smallest cost for $i = 2$ is $(2, 1, 1, 1)$.

Step 2: Allocation to the chosen cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) = (2, 1, 1, 1)$

2.1. $\alpha_2 = \Omega_1 = 20 - \psi$, $\beta_1 = \Omega_2 = 15$, $\gamma_1 = \Omega_3 = 20 - \psi$, $\delta_1 = \Omega_4 = 20$.

2.2. $x_{2111} = \min(\Omega_1, \Omega_2, \Omega_3, \Omega_4, d_{2111}) = \min(20 - \psi, 15, 20 - \psi, 20, 22) = 15$.

2.3. We have $x_{2111} = \Omega_2 \neq \Omega_1$, $\Omega_2 \neq \Omega_4$ *and* $\Omega_2 \neq \Omega_3$, we update the supply values as follows:

$\Omega_1 = \Omega_1 - x_{2111} = 5 - \psi = \alpha_2$,
$\Omega_2 = \Omega_2 - x_{2111} = 0 = \beta_1$ thus $x_{i1kl} = 0, \ \forall (i, k, l) \neq (2, 1, 1)$,
$\Omega_3 = \Omega_3 - x_{2111} = 5 - \psi = \gamma_1$,
$\Omega_4 = \Omega_4 - x_{2111} = 5 = \delta_1$.

Step 3: We have

$\alpha_1 = 0, \ \beta_1 = 0, \ \gamma_1 = 5 - \psi, \ \delta_1 = 5, \ \alpha_2 = 5 - \psi, \ \beta_2 = 5 - \psi, \ \gamma_2 = 0, \ \delta_2 = 0.$

First sub-phase 1
Step 1: there is only one cell for allocation $(2, 2, 1, 1)$
Step 2: Allocation to the chosen cell $(\bar{i}, \bar{j}, \bar{k}, \bar{l}) = (2, 2, 1, 1)$

2.1. $\alpha_2 = \Omega_1 = 5 - \psi,\ \beta_1 = \Omega_2 = 5 - \psi,\ \gamma_1 = \Omega_3 = 5 - \psi,\ \delta_1 = \Omega_4 = 5.$

2.2. $x_{2211} = \min(\Omega_1, \Omega_2, \Omega_3, \Omega_4, d_{2211}) = \min(5 - \psi, 5 - \psi, 5 - \psi, 5, 36) = 5 - \psi.$

2.3. We update the supply values as follows:

$$\Omega_1 = 0 = \alpha_1, \quad \Omega_2 = 0 = \beta_1, \quad \Omega_3 = 0 = \gamma_1, \quad \Omega_4 = 0 = \delta_1 \quad (\psi = 0).$$

Step 3: We obtained
$\alpha_1 = 0,\ \beta_1 = 0,\ \gamma_1 = 0,\ \delta_1 = 0,\ \alpha_2 = 0,\ \beta_2 = 0,\ \gamma_2 = 0$ and $\delta_2 = 0.$
So the basic variables for this solution are:

$$x_{1222} = 40,\ x_{2212} = \psi,\ x_{2121} = 20,\ x_{2111} = 15,\ x_{2211} = 5 - \psi.$$

Second sub-phase 1:
Setting $\psi = 0$ gives $x_B^{(0)} = \{x_{1222} = 40,\ x_{2212} = 0,\ x_{2121} = 20,\ x_{2111} = 15,\ x_{2211} = 5\}$. Thus, an initial basic feasible solution $x^{(0)} = x_B^{(0)} \cup x_H^{(0)}$ is determined. The associated initial value is $Z^{(0)} = 650$.

After 3 itérations (using phase 2) we obtained the following optimal solution $x^* = x_B^* \cup x_H^*$ with $x_B^* = \{x_{1222} = 20,\ x_{2212} = 20,\ x_{2121} = 20,\ x_{1121} = 15,\ x_{1221} = 5\}$. The associated optimal value is $Z^{(*)} = 400$.

5. COMPARATIVE STUDY

Note that our programs are written in Delphi 10.3.2 and implemented on an i5 machine under the Microsoft Windows environment.

The table below presents the results of numerical tests conducted for 10 different problem sizes. For each problem size, 10 different examples were solved with randomly generated input data. We denote by:

$M \times N$: the size of the problem, where $M = m + n + p + q$ and $N = mnpq$.
$T1$: the average execution time for 10 instances using Zitouni's method.
$T2$: the average execution time of the two phases for 10 instances using the new approach.
$N1$: the number of iterations for 10 instances using Zitouni's method [25].
$N2$: the number of iterations for 10 instances using the new approach.

Example series	**Size** $M \times N$	**T1**	**N1**	**T2**	**N2**
1	8×16	$00:0001$	1	$00:0001$	0
2	22×900	$00:0180$	26	$00:0170$	25
3	30×3136	$00:0781$	55	$00:0734$	48
4	34×5184	$00:2000$	70	$0:1593$	66
5	37×7200	$00:3152$	80	$00:2755$	67
6	38×8100	$00:3881$	87	$00:3885$	86
7	39×8640	$00:4620$	105	$00:4166$	86
8	40×9900	$00:4751$	100	$00:4117$	85
9	40×10000	$00:5773$	106	$00:5417$	98
10	43×13068	$01:0831$	122	$00:8535$	109

Tab. 2. Comparison of the proposed approach with that of Zitouni.

In order to provide a better understanding, the numerical values summarized in Table 2 are further illustrated through a graphical representation in Fig. 1.

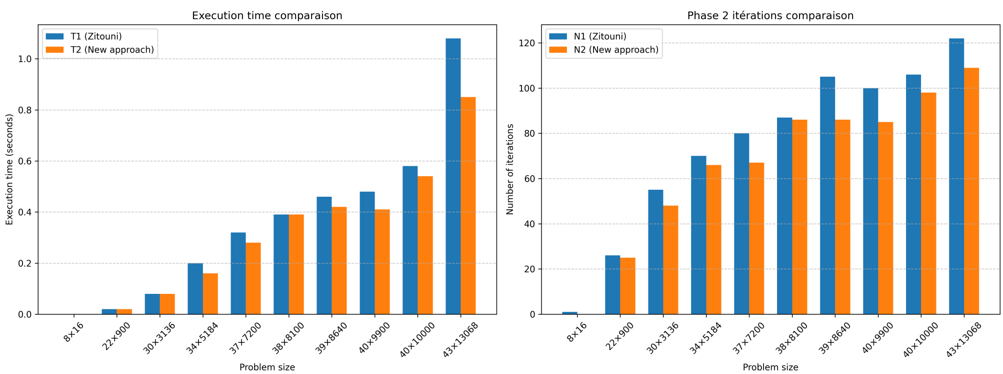

Fig. 1. Comparison of the proposed approach with that of Zitouni.

6. RESULTS AND CONCLUSION

The study addresses the challenges of solving capacitated multi-index transportation problems, which are crucial in optimization and logistics. The table above summarizes the results of our comparative study on a capacitated four-index transportation problem, tested across 10 different examples with various sizes and tested 10 instances for each one of them. The results demonstrate the superiority of the new approach over Zitouni's method. This technique efficiently reached optimal solutions, thus requiring fewer iterations and reducing the computation time. Additionally, it effectively resolved degeneracy issues through a comprehensive framework. The second phase of the algorithm further showcased its robustness, significantly accelerating the process of finding optimal solutions. While the proposed method offers a generalized and scalable mathematical framework with strong computational efficiency, it has been evaluated only on synthetically generated datasets. Real-world applications might present additional complexities such as data uncertainty, dynamic constraints, or stochastic variations. Therefore, future research may focus on extending the approach to address stochastic or fuzzy versions of the problem, as well as developing a user-oriented software tool (based on this method) for an eventual practical application across various industries. Note that this approach is independent of the number of indices. For this reason, it can be extended to solve any capacitated transportation problem under this form with a number of indices greater than four.

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