10 Algebraic structure of the set of phases of oscillatory differential equations (q) in the interval $(-\infty, \infty)$

In this section we investigate the algebraic structure of the set of first phases of oscillatory differential equations (q), with the definition interval $j = (-\infty, \infty)$. The term *phase function* will here always mean a phase function (§ 5.7) *of class* C_3.

We know from §§ 5.5, 5.7, 5.4, that every first phase of a differential equation (q) represents a phase function which in the oscillatory case is unbounded on both sides. We also know (§§ 5.7, 5.4) that conversely every phase function α which is unbounded on both sides in its definition interval j represents a first phase of the oscillatory differential equation (q) constructed according to the formula (5.16). This section will therefore be concerned with the algebraic structure of the set of all phase functions defined in the interval $j = (-\infty, \infty)$ and unbounded on both sides. We shall call this set the *phase set* of the oscillatory differential equations (q) in the interval ($j =$) $(-\infty, \infty)$ or, more briefly, the phase set.

Instead of phase functions we shall speak more briefly of phases, and we shall call the carrier q of an oscillatory differential equation (q) an *oscillatory carrier*. The oscillatory carriers are therefore formed by making use of the formula (5.16) or (5.18), using a phase function α which is unbounded on both sides.

We remark that the power of the set M formed from all oscillatory carriers is equal to the power of the continuum. For, since M is composed of continuous functions, card $M \leqslant \aleph$ and since it contains all carriers formed from arbitrary constants $-k^2$ $(\neq 0)$, we have also card $M \geqslant \aleph$; we thus have card $M = \aleph$.

10.1 The phase group $\mathfrak{G}$

Let G be the phase set of the oscillatory differential equations (q) in the interval $j = (-\infty, \infty)$.

The phase set G obviously includes the identity phase $\phi_0(t) = t$. Moreover, the function $\alpha[\gamma(t)]$, which is the composition of two arbitrary phases α, $\gamma \in G$ is also an element of G, and so is the function α^{-1} inverse to α. We now introduce into the set G a binary operation, which we call multiplication, by means of composition of functions; for arbitrary phases α, $\gamma \in G$ we define the product $\alpha\gamma$ as being the composite function $\alpha[\gamma(t)]$. The set G with this multiplication thus forms a group $\mathfrak{G}$ with the unit element $\phi_0(t)$. We shall call $\mathfrak{G}$ the *phase group*.

The inverse element α^{-1} corresponding to any element $\alpha \in \mathfrak{G}$ represents an increasing or decreasing phase according as α is an increasing or decreasing function. Moreover, the product $\alpha\gamma$ of two elements $\alpha, \gamma \in \mathfrak{G}$ is an increasing function if both phases α, γ increase or decrease and is a decreasing function if one increases and the other decreases.

The set $\mathfrak{N}$ formed from all increasing phases is a normal sub group of $\mathfrak{G}$: $\alpha^{-1}\mathfrak{N}\alpha = \mathfrak{N}$; $\alpha \in \mathfrak{G}$. The factor group $\mathfrak{G}/\mathfrak{N}$ consists of two elements, namely $\mathfrak{N}$ and the class A of all decreasing phases.

10.2 The equivalence relation Q

Our next step is to introduce into the phase group $\mathfrak{G}$ an equivalence relation, as follows: two phases $\alpha, \gamma \in \mathfrak{G}$ are equivalent if they are linked by means of a relationship of the form

$$\tan \gamma(t) = \frac{c_{11} \tan \alpha(t) + c_{12}}{c_{21} \tan \alpha(t) + c_{22}}, \tag{10.1}$$

where the $c_{11}, c_{12}, c_{21}, c_{22}$ are constants with a non-zero determinant, i.e. $|c_{ij}| \neq 0$, and the relation (1) must hold for all values $t \in j$ except for the singular points of the functions $\tan \alpha(t)$, $\tan \gamma(t)$. It is easy to see that the relation determined by (1) in the phase group $\mathfrak{G}$ is reflexive, symmetric and transitive, and consequently is an equivalence relation. We shall denote this relation by Q.

The phase group $\mathfrak{G}$ is therefore split up into a system of equivalence classes mod Q, which we denote by $\bar{Q}$. $\bar{Q}$ is therefore a partition of the phase group $\mathfrak{G}$; every element $\bar{a} \in \bar{Q}$ consists of those phases which are equivalent to each other, while no phases lying in different elements $\bar{a}, \bar{b} \in \bar{Q}$ are equivalent.

Now, two arbitrary phase functions $\alpha, \gamma \in \mathfrak{G}$ represent first phases of appropriate carriers q, p determined by formulae such as (5.18). If α, γ are equivalent, they belong to the same element $\bar{a} \in \bar{Q}$, so there holds a relationship of the form (1) and from this and the theorem of § 1.8 it follows that

$$p(t) = -\{\tan \gamma, t\} = -\left\{\frac{c_{11} \tan \alpha + c_{12}}{c_{21} \tan \alpha + c_{22}}, t\right\} = -\{\tan \alpha, t\} = q(t),$$

so that $p(t) = q(t)$. Conversely (§ 5.17) the relation (1) holds for any two first phases α, γ of a carrier $q(t)$, $t \in j$; follows that the phase functions α, γ are equivalent and so belong to the same element $\bar{a} \in \bar{Q}$. Thus *every element $\bar{a} \in \bar{Q}$ comprises all first phases of one and the same carrier $q(t)$.* Let us associate with every element $\bar{a} \in \bar{Q}$ the corresponding carrier $q(t)$. We then have a simple mapping $\mathscr{A}$ of the partition $\bar{Q}$ onto the set of all oscillatory carriers. The power of the partition $\bar{Q}$ is that of the continuum, card $\bar{Q} = \aleph$.

In connection with this concept of equivalence of phases we note that: if one of two equivalent phases α, γ is elementary, then since α, γ are first phases of the same carrier q, the other phase is also elementary (§ 8.2), i.e. all phases equivalent to an elementary phase are themselves elementary.

10.3 The fundamental subgroup $\mathfrak{E}$

Next we investigate the algebraic structure of the partition $\bar{Q}$.

With this objective, we consider that element $\mathfrak{E} \in \bar{Q}$ which contains the unit element ϕ_0 of $\mathfrak{G}$. This element obviously consists of all phases $\zeta(t)$ which are equivalent to the unit element ϕ_0, that is to say all phases of the form

$$\tan \zeta(t) = \frac{c_{11} \tan t + c_{12}}{c_{21} \tan t + c_{22}}. \tag{10.2}$$

In view of (2), it is clear that the composite function $\zeta_1\zeta_2$ formed from two phases $\zeta_1, \zeta_2 \in \mathfrak{E}$ also belongs to the class $\mathfrak{E}$, and so does the function ζ_1^{-1} inverse to ζ_1. The elements of $\mathfrak{E}$ thus form a subgroup of $\mathfrak{G}$: $\mathfrak{E} \subset \mathfrak{G}$; we call this subgroup $\mathfrak{E}$ the *fundamental subgroup of* $\mathfrak{G}$.

We now show that *the partition $\bar{Q}$ coincides with the right residue class partition $\mathfrak{G}/_r\mathfrak{E}$ of the phase group $\mathfrak{G}$ with respect to $\mathfrak{E}$.*

Let $\bar{a} \in \bar{Q}$ be an arbitrary element and $\alpha \in \bar{a}$ a phase lying in it. We have to show that $\bar{a} = \mathfrak{E}\alpha$.

Now, for every element $\zeta(t) \in \mathfrak{E}$ there holds a formula such as (2). If, in that, we replace t by $\alpha(t)$ then it is clear that $\zeta\alpha$ is equivalent to α. Consequently $\zeta\alpha \in \bar{a}$ and we have $\mathfrak{E}\alpha \subset \bar{a}$.

Moreover, for every element $\gamma \in \bar{a}$ there holds a formula such as (1). If, in this, we replace t by $\alpha^{-1}(t)$, then it is clear that $\gamma\alpha^{-1}$ is equivalent to t. Hence $\gamma\alpha^{-1} \in \mathfrak{E}$, moreover, $\gamma \in \mathfrak{E}\alpha$, and we have $\bar{a} \subset \mathfrak{E}\alpha$. This establishes the fact that $\bar{a} = \mathfrak{E}\alpha$.

By a result from the theory of groups, the power of all right residue classes in a group with respect to a subgroup is always the same. Consequently, the power of all elements of the partition $\bar{Q}$ is always the same, and consequently equal to the power of $\mathfrak{E}$. This latter is obviously equal to $\aleph$, so card $\mathfrak{E} = \aleph$.

Thus, *the power of the set $\bar{a}$ of all first phases of a carrier $q(t)$ is the power of the continuum*: card $\bar{a} = \aleph$.

We remark that the unit element $\phi_0(t) = t \in \mathfrak{E}$ obviously represents an elementary phase. Since all phases equivalent to an elementary phase are themselves equivalent, it follows that:

The fundamental subgroup $\mathfrak{E}$ is comprised only of elementary phases.

We observe that the mapping $\mathscr{A}$ maps the fundamental subgroup $\mathfrak{E}$ onto the oscillatory carrier $q(t) = -1$.

10.4 The subgroup $\mathfrak{H}$ of elementary phases

Let $\mathfrak{H}$ be the set comprising all elementary phases. We wish to show that $\mathfrak{H}$ *is a subgroup of* $\mathfrak{G}$; $\mathfrak{H} \subset \mathfrak{G}$.

Proof. We have already noted that the unit element ϕ_0 is an elementary phase.

Let $\alpha, \gamma \in \mathfrak{G}$ be arbitrary elementary phases. We note first that the composite phase $\alpha\gamma$ is also elementary, since

$$\alpha\gamma(t+\pi) = \alpha(\gamma(t+\pi)) = \alpha(\gamma(t)+\varepsilon_2\pi) = \alpha(\gamma(t)) + \varepsilon_1\varepsilon_2\pi = \alpha\gamma(t) + \operatorname{sgn}(\alpha\gamma)'\pi$$
$$(\varepsilon_1 = \operatorname{sgn}\alpha', \ \varepsilon_2 = \operatorname{sgn}\gamma').$$

Further, the function inverse to α, namely $\bar{\alpha}$ $(= \alpha^{-1})$ is also an elementary phase. For, when $\bar{t} \in j$ we have

$$\bar{\alpha}(\alpha(\bar{t}) + \pi) = \bar{\alpha}(\alpha(\bar{t} + \varepsilon\pi)) = \bar{t} + \varepsilon\pi \quad (\varepsilon = \operatorname{sgn}\alpha' = \operatorname{sgn}\bar{\alpha}'),$$

and from this it follows for $t = \alpha(\bar{t})$, $\bar{t} = \bar{\alpha}(t)$, that

$$\bar{\alpha}(t+\pi) = \bar{\alpha}(t) + \varepsilon\pi.$$

This completes the proof.

We have shown above that the fundamental subgroup $\mathfrak{E}$ consists only of elementary phases. It follows that $\mathfrak{E}$ is a subgroup of $\mathfrak{H}$, consequently

$$\mathfrak{E} \subset \mathfrak{H}. \tag{10.3}$$

Now let $\mathfrak{G}/_r\,\mathfrak{H}$ be the right residue class partition of the group $\mathfrak{G}$ with respect to $\mathfrak{H}$. The elements of this partition are therefore the right residue classes $\mathfrak{H}\alpha$, $\alpha \in \mathfrak{G}$ with respect to the subgroup $\mathfrak{H}$.

From (3) it follows that the partition $\mathfrak{G}/_r\,\mathfrak{H}$ represents a covering of $\mathfrak{G}/_r\mathfrak{E}$; consequently ([81])

$$\mathfrak{G}/_r\mathfrak{H} \geqq \mathfrak{G}/_r\mathfrak{E}. \tag{10.4}$$

Formula (4) asserts that every element $\mathfrak{H}\alpha \in \mathfrak{G}/_r\,\mathfrak{H}$ is the union of some elements of the partition $\mathfrak{G}/_r\mathfrak{E}$. In particular, the subgroup $\mathfrak{H}$ is also the union of some elements of the partition $\mathfrak{G}/_r\mathfrak{E}$. From a known result in group theory, the power of the set of all elements of $\mathfrak{G}/_r\mathfrak{E}$ whose union gives rise to the element $\mathfrak{H}\alpha$, is independent of the choice of this element. In other words, for every choice of the element $\mathfrak{H}\alpha \in \mathfrak{G}/_r\,\mathfrak{H}$ the power of the set of elements of the partition $\mathfrak{G}/_r\mathfrak{E}$ which give the element $\mathfrak{H}\alpha$ by their union, is always the same.

We now consider the subgroup $\mathfrak{H}$ and a further element $\mathfrak{H}\alpha$ of the partition $\mathfrak{G}/_r\,\mathfrak{H}$ of $\mathfrak{G}$. Let $\bar{A}_0$ and $\bar{A}$ be the sets of all elements of the partition $\mathfrak{G}/_r\mathfrak{E}$ whose unions respectively produce the elements $\mathfrak{H}$ and $\mathfrak{H}\alpha$: $\cup\bar{A}_0 = \mathfrak{H}$, $\cup\bar{A} = \mathfrak{H}\alpha$. From the above, the power of the two sets $\bar{A}_0$, $\bar{A}$ are the same: card $\bar{A}_0$ = card $\bar{A}$. By means of the mapping $\mathscr{A}$ the sets $\bar{A}_0$, $\bar{A}$ are mapped onto certain sets $\mathscr{A}\bar{A}_0$, $\mathscr{A}\bar{A}$ of oscillatory carriers. From the definition of $\mathscr{A}$, the sets $\mathscr{A}\bar{A}_0$ and $\mathscr{A}\bar{A}$ consist of just those carriers whose first phases lie in $\mathfrak{H}$ and $\mathfrak{H}\alpha$ respectively. In particular, the set $\mathscr{A}\bar{A}_0$ comprises those carriers of which the first phases are elementary; that is to say, the elementary carriers. From § 7.6 the power of $\mathscr{A}\bar{A}_0$ is equal to that of the continuum: card $\mathscr{A}\bar{A}_0 = \aleph$. Now, the mapping $\mathscr{A}$ is simple; it follows that card $\mathscr{A}\bar{A}_0$ = card $\bar{A}_0$, card $\mathscr{A}\bar{A}$ = card $\bar{A}$. We have therefore

$$\text{card}\,\mathscr{A}\bar{A} = \text{card}\,\bar{A} = \text{card}\,\bar{A}_0 = \text{card}\,\mathscr{A}\bar{A}_0 = \aleph,$$

which gives the following result:

The power of the set of all oscillatory carriers, whose first phases lie in one and the same element $\mathfrak{H}\alpha \in \mathfrak{G}/_r\,\mathfrak{H}$, is the same for all elements of $\mathfrak{G}_r/\,\mathfrak{H}$ and is equal to the power of the continuum.