CHAPTER I

CLASSES AND RELATIONS

(Sections 1 – 5)

This chapter contains the fundamental concepts and theorems of general or abstract set theory. In principle, no previous knowledge is required from the reader. Some remarks and examples are of a purely illustrative character, in which notions and propositions introduced later are freely used; the reader can, of course, ignore all remarks and examples of this kind. On the other hand, familiarity with the basic concepts of set theory would considerably assist the reading of this chapter.

Throughout the book, sets and classes will always be distinguished. Roughly speaking, a class will be any collection of elements, or what is currently called a set. Some classes will be termed sets: namely those which are not "too large", in the sense that they themselves may be elements of other classes. Thus a set is a special type of class.

There are important reasons for introducing the notion of classes as distinct from that of sets. One of these is that the familiar paradoxes of set theory are then eliminated (e.g. those concerning the "set of all sets"). Another, more important, reason is that without the notion of class, the concept of a category (see Section 13), whose importance in various branches of mathematics is increasing, would have to be restricted rather artificially.

In distinguishing between sets and classes, it would be difficult to retain the customary or naive point of view, in which the basic interrelations between sets, the set operations, etc., are taken as intuitively evident. However, a strictly formal axiomatic treatment of set and class theory would hardly be in place here, since it is not necessary for the main object of this book and would require a thorough and precise development of the logical means and procedures used. In fact, an exposition of a considerable part of mathematical logic would be necessary. The method chosen is that of an axiomatic but nonformal development which, however, is only axiomatic with a certain reserve. The fundamental notions, such as "class", "to belong to", are, of course, not defined, but only described, i.e. their meaning is briefly indicated in an intuitive way. Their fundamental properties are formulated as axioms; all theorems are then deduced from them. In this process, ordinary logical means and current language are used, with only such deviations as have become natural in mathematical expression (for instance, the sentence "every x possessing property $\boldsymbol{P}$ satisfies condition $\boldsymbol{C}$" is taken to be true if there is no x at all possessing property $\boldsymbol{P}$). Occasionally, the symbols $\Rightarrow$ (implication) and $\Leftrightarrow$ (logical equivalence) will be used; these are to be taken solely as abbreviations for "if ..., then ..." and "... if and only if ..."; we shall return to these symbols somewhat later. In this sense the method used is similar to that of the axiomatic treatment of geometry in the classical sense.

We have said that our treatment is only axiomatic with a certain reserve; exception may be taken to postulate 1 A.4 which states that to every property there exists a class consisting of precisely those elements which possess the given property. If the concept of "property" is accepted as sufficiently clear, then our development of the foundations of class and set theory may be considered to be axiomatic. We do not analyse this concept since the treatment is sufficiently axiomatic for the purposes of this book. A formal development may be performed on well known lines, though some new problems may appear and possibly a slight modification of the approach adopted heie may be necessary.

After introducing fundamental ideas and defining further concepts (e.g. that of relation, defined as a class of pairs), we obtain in Section 1 a basis for the theory of classes, sets and relations which is, however, still only a fragment since, for example, the axioms of this section are satisfied if there is one class only, the void one.

Section 2 is of an auxiliary character and concerns the so-called basic set-theoretical operations such as union, intersection, difference and symmetric difference.

In Section 3, the axiom of infinity is introduced, its consequences are examined, and the "existence of natural numbers" is proved. The elementary arithmetic of natural numbers is, of course, not treated (only some basic definitions are given explicitly) and is assumed to be known.

The Axiom of Choice is introduced in Section 4 in a rather strong form which differs somewhat from the usual formulation since it will be necessary to "choose" elements from classes and not merely from sets. In contrast to Sections 1 and 2, some proofs are relatively complicated here as well as in Section 3. With Section 4, the axiomatic exposition of set theory is completed. The reader familiar with set theory may pass through these paragraphs quite rapidly; however, we recommend a careful reading of the definitions and conventions, especially those concerning notation, since they will often be used later without reference (moreover, the distinction between classes and sets necessitates some slight but not unimportant changes).

In Section 5, the cartesian product is considered in a rather detailed but essentially current manner.

We shall conclude these introductory remarks with some details concerning the notation.

Only two logical signs will be used: the symbol $\Rightarrow$ for logical implication, and the symbol $\Leftrightarrow$ for logical equivalence; they will be conceived as abbreviations for certain expressions of current language.

Convention. In place of expressions of the form "if $\boldsymbol{A}$, then $\boldsymbol{B}$" or "$\boldsymbol{A}$ implies $\boldsymbol{B}$" (with the same meaning), we shall often write "$\boldsymbol{A} \Rightarrow \boldsymbol{B}$".

The expression "if $\boldsymbol{A}$, then $\boldsymbol{B}$" will, of course, be interpreted in the manner customary in mathematical formulation. Thus, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are statements (i.e. expressions which are either true or not), then the statement "if $\boldsymbol{A}$, then $\boldsymbol{B}$" is true in the following cases: both $\boldsymbol{A}$, $\boldsymbol{B}$ are true, $\boldsymbol{A}$ is not true and $\boldsymbol{B}$ is true, both $\boldsymbol{A}$, $\boldsymbol{B}$ are not true; and the statement in quotes is not true if $\boldsymbol{A}$ is true and $\boldsymbol{B}$ is not.

Convention. In place of expressions of the form "$\boldsymbol{A}$ if and only if $\boldsymbol{B}$" or "$\boldsymbol{A}$ is sufficient and necessary for $\boldsymbol{B}$" (with the same meaning), etc., we shall often write "$\boldsymbol{A} \Leftrightarrow \boldsymbol{B}$".

If $\boldsymbol{A}$, $\boldsymbol{B}$ are statements, then "$\boldsymbol{A}$ if and only if $\boldsymbol{B}$" is true if $\boldsymbol{A}$, $\boldsymbol{B}$ are both true or both not true, and is not true in the remaining cases; in other words, the statement in quotes is true precisely if both "if $\boldsymbol{A}$, then $\boldsymbol{B}$" and "if $\boldsymbol{B}$, then $\boldsymbol{A}$" are true.

We shall use another convention generally adopted in mathematical texts, but seldom stated explicitly; it will be explained now by an example. If we wish to state e.g. that $\sin^2 x + \cos^2 x = 1$ for all numbers x, we often merely write "$\sin^2 x + \cos^2 x = 1$ holds" omitting the clause "for all x". We shall proceed similarly whenever convenient.

Let us point out that all of the above conventions will be used throughout the book without reference.

Finally, it is necessary to emphasise that we do not define the notion of identity and it is considered as intuitively clear. The sign "$=$" will be used exclusively to express that two objects are identical; thus if we write $A = B$, then A and B is the same object, for which two symbols, namely "A" and "B", are being used in the reasoning in question.

1. CLASSES AND SETS

The concept of a set is probably familiar to the reader. This notion is usually introduced by means of a more or less detailed and suggestive description which, however, is not a definition in the proper sense of the word. Thus it may be said that a set is a collection of certain objects, the collection considered as an individual entity. In essence, this is the familiar "definition" of G. Cantor: "Eine Menge ist eine Zusammenfassung bestimmter wohlunterschiedener Objekte unserer Anschauung oder unseres Denkens – welche die Elemente der Menge genannt werden – zu einem Ganzen". We will depart from this tradition in that the concept of class will appear first, and the notion of set will subsequently be defined strictly (the reasons for distinguishing between classes and sets have been given in the Introductory Remarks at the beginning of the present chapter). Parallelling the traditional approach, we give a description of classes, but in essence, the concepts of class and of belonging to are the primary undefined concepts.

A. CLASSES

1 A.1. Description. A *class* is a collection of objects, considered as an individual entity. The objects are said to *belong* to the class considered.

1 A.2. Convention. If x belongs to X, and X is a class, we say that x is an *element of the class* X, or that X *contains* x as an element, and write $x \in X$ or $X \ni x$. If $x \in X$ does not hold, we write $x \notin X$ or x non $\in X$ or $X \not\ni x$ or X non $\ni x$.

If X is a class and every x belonging to X satisfies a given condition $\mathbf{C}$, we shall say that X is a class of elements satisfying $\mathbf{C}$ or that X consists of elements satisfying $\mathbf{C}$. In this sense, as usual, we shall use expressions such as "a class of sets", "a class of positive numbers", and so on.

If X is a class, and $x \in X$ if and only if x is an element satisfying a given condition $\mathbf{C}$ we shall say, as usual, that X consists of all elements x satisfying $\mathbf{C}$ or that X consists precisely of those elements which satisfy $\mathbf{C}$, or else that X is (cf. 1 A.4 and 1 A.7) the class of all elements x satisfying $\mathbf{C}$, or, finally, that X is the class of all x such that $\mathbf{C}$ is satisfied.

Remark. Observe that $x \in X$ means that x belongs to X and X is a class. We do not exclude the possibility that some x belong to some X without X being a class (hence also without x being necessarily an element; cf. 1 A.3). Thus, there are *a priori* no obstacles for introducing, after an extension of the framework of the axiomatic system presented here, "superclasses" to which classes may belong. We do not, however, consider these questions further.

1 A.3. Definition. We say that x is *comprisable* or that x is an *element* if it belongs to some class, i.e. if there is a class X with $x \in X$.

Note that all this does not assert the existence of any element (in fact, we shall be in this situation throughout the present section). For practical reasons it is convenient, but for mathematics rather irrelevant, to consider every sufficiently well-defined materially existing object as an element. In this sense we may then speak of the class of all cats, of the class of all inhabitants of London (at a given instant) and so on.

We remark that a class may, but need not, itself be an element. The classes which do belong to other classes – intuitively speaking, which are not excessively extensive – will be termed sets (see 1 A.9).

1 A.4. Postulate. *If* $\mathbf{P}$ *is a property, then there exists a class consisting precisely of those elements which have the property* $\mathbf{P}$.

As noted in the Introductory Remarks, if the notion of property is taken as sufficiently clear, then Postulate 1 A.4 may be considered to be an axiom. This, together with the Axiom of Extensionality (1 A.7), yield the usual intuitive meaning of the notion of class. In order to by-pass problems of mathematical logic, in this book we shall adopt this point of view even though it is open to several objections.

1 A.5. *There exists a class which does not contain any element.*

The proof follows from Postulate 1 A.4 on taking for $\mathbf{P}$ some property possessed by no element, e.g. the following one: x has property $\mathbf{P}$ if and only if $x \neq x$, i.e. if and only if x is not identical with x. Obviously no x (and thus no element) has property $\mathbf{P}$.

1 A.6. Definition. A class will be termed *void* or *empty* if it contains no element.

If $X = Y$ holds for two classes X, Y (i. e. if the classes X and Y are identical) then of course they contain the same elements, $x \in X$ if and only if $x \in Y$. The following axiom affirms that, conversely, if two classes contain the same elements then these classes are identical; that is, a class is "uniquely determined" by its elements.

1 A.7. Axiom of Extensionality. *If A and B are classes, and if every element of A belongs to B and every element of B belongs to A, then the classes A, B are identical.*

Remark. Observe that the statement obtained from this axiom by dropping the assumption of A, B being classes is not asserted. In other words, the axioms do not exclude *a priori* the possibility that e.g. A is a class, $\mathscr{A}$ is an object which is not a class and, for any x, $x \in A$ if and only if x is an element and x belongs to $\mathscr{A}$.

1 A.8. Theorem. *There exists precisely one void class.*

This is an obvious consequence of the Axiom of Extensionality.

Convention. The void class will be denoted by $\emptyset$.

1 A.9. Definition. A comprisable class, i.e. a class which is also an element, will be called a *set*.

Convention. A set will be also called a *collection*, usually if it consists of sets. Thus we shall, as a rule, speak of a collection of sets instead of a set of sets.

Remarks. 1) Non-comprisable classes, that is classes which are not sets, are sometimes called proper classes. 2) If in Postulate 1 A.4 we take for $\mathbf{P}$ the property of x being equal to itself we obtain that there exists a class which contains all elements whatever. From the Axiom of Extensionality it follows that there is only one such class; it will be termed the *universal class.*

1 A.10. We still do not know whether there exist sets. But it is already possible to show that there exist non-comprisable classes (classes which are not sets); in fact, we can give an example of such a class: the class A of all sets which are not their own elements, i.e. the class of all sets X with $X \notin X$.

Indeed, $X \in A$ if and only if X is a set and $X \notin X$. Suppose that A is a set; then substituting A for X we have $A \in A$ if and only if $A \notin A$ which is a contradiction.

A reader acquainted with informal expositions of set theory may recognize here the familiar argument which, in the naive interpretation, leads to an antinomy; however, in our case we merely obtain the positive statement that a certain class is not a set.

1 A.11. Convention. The symbol $\mathbf{E}\{T \mid \mathbf{R}\}$ will denote the class consisting precisely of those T which are elements and satisfy condition $\mathbf{R}$.

We will assume that the preceding formulation is sufficiently clear for our purposes (a rigorous formulation would necessitate the introduction of notions from mathematical logic). In fact, $\mathbf{E}\{T \mid \mathbf{R}\}$ is merely an abbreviation for an expression in current language, namely "the class of all T such that $\mathbf{R}$ is satisfied". The expressions T and $\mathbf{R}$ may depend, actually or apparently, on further variable "parameters". The symbol $\mathbf{E}\{T \mid \mathbf{R}\}$ generalizes the current use of $\mathbf{E}\{x \mid \mathbf{R}(x)\}$ to denote the set of all elements x which satisfy $\mathbf{R}(x)$.

Note that for every class A we have $A = \mathbf{E}\{x \mid x \in A\}$.

Examples. (A) $\mathbf{E}\{x \mid x$ is a real number, $x > 0\}$ as well as $\mathbf{E}\{x^2 \mid x$ is a real number, $x \neq 0\}$ is the class of all positive real numbers. – (B) $\mathbf{E}\{x \mid x, y$ are real numbers, $x > y\}$ as well as $\mathbf{E}\{x + y \mid x, y$ are real numbers, $x > 0\}$ is the class of all real numbers greater than y (this class depends on y). – (C) $\mathbf{E}\{x \mid x, y$ are real numbers, $x^2y^2 \geqq 0\}$ is the class of all real numbers (the class depends on y only formally). – (D) $\mathbf{E}\{x \mid x, y$ are real numbers, $x^2 + y^2 + 1 = 0\} = \emptyset$ for any real number y. – (E) $\mathbf{E}\{X \mid X$ is a set, $\emptyset \in X\}$ is the class of all classes X which are elements and have the void class as an element, that is of all sets X such that $\emptyset \in X$. Axioms introduced in what follows (1 B.1, 1 E.1) will imply that this class is non-comprisable; on the other hand, after constructing the theory of real numbers within the frame of set theory, we can show that the classes in all the other examples are sets.

1 A.12. Conventions. 1) If a is an element, then the class $\mathbf{E}\{x \mid x = a\}$ will be denoted by (a). Similarly, if $a, b, \ldots, h$ are elements, then the class $\mathbf{E}\{x \mid x = a$ or $x = b$ or ... or $x = h\}$ will be denoted by $(a, b, \ldots, h)$. – 2) A class consisting of precisely one element, i.e. a class of the form (x), will be called a *one-element class* or a *singleton.*

It will follow from the axiom introduced later (1 E.1) that every class containing only a "finite number" of elements (in particular, every one-element class) is a set.

1 A.13. Definition. Let A and B be classes. If every element of A belongs to B, then we write $A \subset B$ or $B \supset A$ and say that A is a *subclass* of B, or that A is *contained* (as a subclass) in B, or that B *contains* A (as a subclass), or that A is a *part* of B. If $A \subset B$ and $A \neq B$ then we say that A is a *proper part* or *proper subclass* of B. If $A \subset B$ and A is a set, then we say that A is a *subset* of the class B.

The following proposition is obvious.

1 A.14. *If A, B are classes, then $A = B$ if and only if both $A \subset B$ and $B \subset A$; in particular, $A \subset A$ for any class A. If A, B, C are classes and $A \subset B$, $B \subset C$, then $A \subset C$.*

B. RELATIONS

In mathematical reasoning (and, in fact, in any systematic reasoning) there constantly occur various relations. Now we will consider the so-called binary relations (relations between two objects, e.g. ... is greater than ..., ... is older than ...) in the extensional aspect; this means, roughly, that we do not distinguish relations, say ϱ and ϱ', such that if two objects are in relation ϱ then they are also in relation ϱ' and conversely.

The current procedure is to introduce relations as certain sets of pairs; thus, to the relation (between reals) "is greater than" there corresponds the set of all couples x, y, where x, y are real numbers and x is greater than y.

The strict definition of relation must therefore be preceded by a definition of pairs (that is, ordered pairs). The notion of a pair may, of course, be reduced to the concept of set, e.g. in such a way that a pair of elements a, b is defined as the set $((a), (a, b))$.

We shall proceed in a different way since, for instance, the approach mentioned above does not permit the formation of pairs of non-comprisable classes, i.e. classes which are not sets. We shall introduce pairs as a new undefined concept and assume appropriate axioms. This can be done in various ways; we adopt axioms which are similar, to a certain degree, to those valid for classes (cf. 1 A.4, 1 A.7). The intuitive content of the concept of a pair is clear; we have to assure in the axioms that every pair determines its left (first) and right (second) member (or "coordinate") and is determined by them, and that a pair can be formed from any x, y (at least if x, y are elements or classes or pairs). There will be two more axioms; they will be discussed in subsequent remarks in some detail since they (as well as the entire approach presented here) are not current in the literature.

1 B.1. Axioms for pairs.

(a) *For any x, y there exists a* pair *z such that x is a* left member *of z and y is a* right member *of z;*

(b) *if a is a left member of c as well as of c' and b is a right member of c as well as of c', then $c = c'$;*

(c) *if both a and a' are left (respectively, right) members of c, then $a = a'$;*

(d) *if a is a left member of c, and b is a right member of c, than c is an element if and only if both a and b are elements;*

(e) *no pair is a class.*

1 B.2. Convention. For every x and y, the pair (uniquely determined, by axioms (a) and (b)) such that x is its left and y is its right member will be denoted by $\langle x, y\rangle$.

Remarks. 1) Observe that, besides the concept of a pair, there are two more undefined notions, namely, of being a left (respectively, right) member of a pair. The first three axioms are more or less self-explanatory (it is worth-while to compare (a) with 1 A.4, (b) with 1 A.7). By Axiom (c) the left (as well as the right) member of any pair is uniquely determined.

2) The members of a pair are not restricted in any manner; a member may be an element, but it might also be e.g. a non-comprisable class. Also, any pair may be a member of a pair, and we admit pairs such as $\langle X, \langle Y, Z\rangle\rangle$ or $\langle\langle X, Y\rangle, Z\rangle$ or $\langle\langle A, B\rangle, \langle C, D\rangle\rangle$, etc. (this agrees with current usage). The axioms do not even exclude the existence of an element α with $\alpha = \langle\alpha, \alpha\rangle$, whereupon also $\alpha = \langle\alpha, \langle\alpha, \alpha\rangle\rangle = \langle\langle\alpha, \alpha\rangle, \alpha\rangle = \langle\langle\alpha, \alpha\rangle, \langle\alpha, \alpha\rangle\rangle = \dots$.

3) Remark to Axiom (d). It is natural to consider a pair to be an element if both of its members are elements. On the other hand, it seems appropriate (and perhaps even necessary) to eliminate situations in which, e.g., $\langle X, a\rangle$ would be an element, X being a non-comprisable class.

4) Remark to Axiom (e). This axiom is intuitively clear, since forming pairs is evidently an entirely different process from that of forming classes. It is not absolutely necessary, however; but it eliminates the unpleasant and complicated situation (which is not excluded by the remaining axioms) in which some pair ξ is also a class – and thus there would be members of ξ as a pair and also elements of ξ as a class.

Now we introduce a notation for some special pairs.

1 B.3. Convention. The pair $\langle a, \langle b, c\rangle\rangle$ will be denoted by $\langle a, b, c\rangle$. In a similar fashion, we denote $\langle a, b, c, d\rangle = \langle a, \langle b, \langle c, d\rangle\rangle\rangle$, $\langle a, b, c, d, e\rangle = \langle a, \langle b, \langle c, \langle d, e\rangle\rangle\rangle\rangle$.

A formal definition of an n-tuple of elements will be given and discussed in Section 3.

Note also that, in general, $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle$ need not hold; e.g. $\langle\emptyset, \emptyset, \emptyset\rangle \neq \langle\langle\emptyset, \emptyset\rangle, \emptyset\rangle$, since $\langle\emptyset, \emptyset, \emptyset\rangle = \langle\emptyset, \langle\emptyset, \emptyset\rangle\rangle$ has for its left (first) member the class $\emptyset$, while $\langle\langle\emptyset, \emptyset\rangle, \emptyset\rangle$ has for the left member the pair $\langle\emptyset, \emptyset\rangle$ which is not a class by Axiom (e).

The next step is the concept of relation (or rather, binary relation) in its extensional aspect.

1 B.4. Definition. A class all the elements of which are pairs will be termed a *relation*.

Remarks. 1) Since relations are classes, we may, of course, speak of comprisable relations (i.e. relations which are sets), of a relation contained in another one, and so on. – 2) A relation in the current ("logical", intensional) sense may be termed a logical relation (the words "relationship", "interrelation" will also be used in this sense).

Examples. (A) An important example of a relation is the class of all pairs of the form $\langle x, x\rangle$ where x is any element. This relation will be termed the *identity* (the *identity relation*) and denoted by J. If X is a given class, then J_X will denote the class of all pairs $\langle x, x\rangle$ with $x \in X$ (the restriction of J to X, cf. 1 B.10). – (B) The relation of belonging to a set, sometimes denoted by $\in$, is the class of all pairs $\langle x, X\rangle$ where $x \in X$ and X is a set. – (C) The relation of inclusion (sometimes denoted by $\subset$) is the class of all pairs $\langle X, Y\rangle$ where X, Y are sets and $X \subset Y$. In all these cases (except J_X, X being a set) it can be shown that the relations in question are non-comprisable. Two more examples of this type are the following: (D) The relation consisting of all pairs $\langle x, \langle x, y\rangle\rangle$, where x, y are elements. – (E) The relation consisting of all pairs $\langle X, Y\rangle$ where X, Y are sets such that for no z, $z \in X$ and $z \in Y$ (the class of all pairs of disjoint sets, see 2.2).

The following relations are sets (this will follow immediately from axioms 1 E.1 as well as, for (H) and (J), from the definition of real numbers). – (F) The relation J_X if X is a set. – (G) The relation consisting of all pairs $\langle X, Y\rangle$ such that $X \subset Y \subset A$, where A is a given set. – (H) The relation $<$ ("is smaller than") between real numbers, i.e. $\mathbf{E}\{\langle x, y\rangle \mid x, y \text{ are real numbers}, x < y\}$. – (J) The relation consisting of all pairs $\langle x, y\rangle$ where x is a number from some given subset A of the reals and $y = f(x)$, where f is a given real-valued function defined on A (this relation is often called the "graph" of f).

Next we introduce some important notions and symbols connected with the concept of relation.

1 B.5. Convention. If ϱ is a relation, then $\langle x, y\rangle \in \varrho$ is sometimes written as $x\varrho y$.

Conversely, given a symbol r and a relation ϱ such that $\langle x, y\rangle \in \varrho$ if and only if x, y are elements and $x\mathrm{r}y$ is true, we shall sometimes use the symbol r to denote ϱ, provided there is no danger of ambiguity. For instance the symbols $\in$ and $\subset$ will denote, respectively, the relations defined in examples (B) and (C) and the symbol $<$ will sometimes be used to denote the relation in example (H) in 1 B.4.

1 B.6. Definition. Let ϱ be a relation. The class of all x such that $x\varrho y$, i.e. that $\langle x, y\rangle \in \varrho$, for some y, will be termed the *domain* of ϱ and denoted by $\mathbf{D}\varrho$; the class of all y such that $x\varrho y$, i.e. that $\langle x, y\rangle \in \varrho$, for some x, will be termed the *range* of ϱ and denoted by $\mathbf{E}\varrho$.

Remark. The use of the letter $\mathbf{E}$ in the symbol for the range of relations may seem somewhat strange. However, it is motivated by the fact that if a relation ϱ is denoted by the symbol $\{U \mid \mathbf{R}\}$ (see below, 1 B.11, 1 B.13), then $\mathbf{E}\{U \mid \mathbf{R}\}$ (see 1 A.11) denotes the range of ϱ. Observe that the letter $\mathbf{E}$ occurs in two roles, which may be quite different, *a priori* – namely in the symbol $\mathbf{E}\varrho$ and in the symbol $\mathbf{E}\{T \mid \mathbf{R}\}$. However, this does not imply any ambiguities as a rule. We do not consider these questions any more, having in mind that the symbols $\mathbf{E}\{T \mid \mathbf{R}\}$ etc. are conceived as mere abbreviations for certain expressions of current language as used in mathematics.

Convention. Let ϱ be a relation; let A, B be classes. We are going to introduce expressions for various combinations of the properties indicated by $\mathbf{D}\varrho \subset A$, $\mathbf{D}\varrho = A$, $\mathbf{E}\varrho \subset B$, $\mathbf{E}\varrho = B$.

If $\mathbf{D}\varrho \subset A$, $\mathbf{E}\varrho \subset B$ we shall say that ϱ is a relation *for A* and *B* or that ϱ is a relation *for A ranging in B*. – If $\mathbf{D}\varrho \subset A$, $\mathbf{E}\varrho = B$, we shall say that ϱ is a relation *for A ranging on B*. – If $\mathbf{D}\varrho = A$, $\mathbf{E}\varrho \subset B$, it will be said that ϱ is a relation *on A ranging in B* or simply that the relation *ϱ is on A into B*. – If $\mathbf{D}\varrho = A$, $\mathbf{E}\varrho = B$, we shall say that ϱ is a relation *on A ranging on B* or simply that *ϱ is on A onto B*.

Finally, if A is a class, $\mathbf{D}\varrho \subset A$, $\mathbf{E}\varrho \subset A$, we shall say that ϱ is a relation *in A* (or *for A*). As already indicated above, if $\mathbf{D}\varrho = A$, $\mathbf{E}\varrho \subset A$ (respectively, $\mathbf{E}\varrho = A$), it is said that ϱ is a relation *on A into A* (respectively, *onto A*).

Remark. We shall not use expressions such as "ϱ is a relation on X" since it is convenient to apply such expressions for special kinds of relations in a specific sense; see e.g. 1 C.5, 1 C.7.

In the examples mentioned above we have the following domains and ranges: $\mathbf{D}\mathsf{J} = \mathbf{E}\mathsf{J}$ is the universal class. $\mathbf{E} \in$ is the class of all non-void sets. – As for $\mathbf{D} \in$, it will follow from 1 E.1 (a) that $\mathbf{D} \in$ is equal to the universal class. – $\mathbf{D} \subset = \mathbf{E} \subset$ is the class of all sets. – In example (D) the domain is the universal class, and the range consists of those pairs which are elements. – In example (E) the domain as well as the range is the class of all sets (since for every set there is another set disjoint with it, e.g. the set $\emptyset$). – Clearly, $\subset$ and $\supset$ are relations on $\mathscr{S}$ onto $\mathscr{S}$, $\mathscr{S}$ denoting the class of all sets, $\in$ is a relation on V, the universal class, ranging on the class of all non-void sets. (As noted at the beginning of this chapter, the examples sometimes anticipate facts proved later.)

1 B.7. Definition. If ϱ is a relation and A is a class, then the class of all y such that $x\varrho y$ for some $x \in A$ will be termed the *image* of A (under ϱ) and denoted by $\varrho[A]$.

If x is an element, then the class $\varrho[(x)]$, i.e. the image of singleton (x), will be called the *fibre* of ϱ at x.

Observe that the fibre of ϱ at x is non-void if and only if $x \in \mathbf{D}\varrho$.

Examples. (A) $\mathsf{J}[A] = A$ for every class A. – (B) The class $\in [A]$ consists of all sets with an element in common with A. – (C) The fibre of the relation $\supset$ (see **1 B.5**) at a set A consists of all sets X with $X \subset A$ and thus is identical with exp A (see **1 E.9**). – (D) If A is the set of all positive reals then $< [A] = A$.

1 B.8. Definition. If ϱ is a relation, then the class of all $\langle x, y\rangle$ with $\langle y, x\rangle \in \varrho$ will be denoted by ϱ^{-1} and termed the *inverse relation* to ϱ or the *inverse* of ϱ.

Examples. (A) $\mathsf{J}^{-1} = \mathsf{J}$, $\subset^{-1} = \supset$. – (B) The relation $\in^{-1}$, denoted as $\ni$ in accordance with **1 B.5**, consists of all $\langle X, x\rangle$ where X is a set, $x \in X$. – (C) If ϱ denotes the relation from **1 B.4**, example (J), and for every $y \in \varrho[A]$, the set $\varrho^{-1}[(y)]$ contains precisely one element, then ϱ^{-1} is the graph (in the usual sense) of the function inverse to f.

1 B.9. Definition. If ϱ is a relation and B is a class, then the class $\varrho^{-1}[B]$ will be called the *inverse image* of B under ϱ. If y is an element, then $\varrho^{-1}[(y)]$, i.e. the inverse image of (y) under ϱ, will be called the *inverse fibre* of ϱ at y.

Example. If B is a class of sets, then $\in^{-1}[B]$ is the class of all elements which belong to some set of B (the union of the class B, see 2.7). The inverse fibre of $\in$ at a set B is the set B itself.

Convention. Occasionally the fibre $\varrho[(x)]$ of ϱ at x will be denoted by $\varrho[x]$ and termed the *image* of x (under ϱ), and similarly for $\varrho^{-1}[(y)]$.

This notation and terminology are, strictly speaking, incorrect, as shown by the following example. Let a, b, c be elements, $A = (a)$, let ϱ consist of the pairs $\langle a, b\rangle$ and $\langle A, c\rangle$. By Definition **1 B.7**, $\varrho[A]$ is (b); however, according to the convention, it might also be $\varrho[(A)]$, i.e. (c). Nevertheless, such situations occur only rarely and, with suitable care, the convention may be applied. Observe that another possibility of ambiguity arises in connection with single-valued relations (cf. **1 D.1**).

1 B.10. Definition. Let ϱ be a relation, and let A, B be classes. The relation consisting of all $\langle x, y\rangle \in \varrho$ such that $x \in A$ will be called the *domain-restriction* of ϱ to A. The relation $\mathbf{E}\{\langle x, y\rangle \mid x\varrho y, y \in B\}$ will be termed the *range-restriction* of ϱ to B. Finally, the relation $\mathbf{E}\{\langle x, y\rangle \mid x\varrho y,\ x \in A,\ y \in B\}$ will be called the *restriction* of ϱ to A and B.

Let σ and ϱ be relations. If σ is a domain-restriction or, respectively, range-restriction, restriction of ϱ, then we shall say that ϱ is a *domain-extension* or, respectively, *range-extension*, *extension* of σ.

Instead of a domain-restriction or a range-restriction, we shall often speak simply of a restriction if it is clear from the context which kind of restrictions is considered, and similarly for extensions. The domain-restriction of ϱ to A will be sometimes denoted by ϱ_A or $\varrho \mid A$ (observe, however, that a symbol like ϱ_A may also mean the restriction of a composition in the sense of Section 6).

Example. Consider the relation $\subset$ (it consists of all pairs $\langle X, Y\rangle$ of sets such that $X \subset Y$). Let $\mathcal{S}$ be the class of all singletons. The range-restriction σ of $\subset$ to $\mathcal{S}$

consists exactly of all $\langle (x), (x) \rangle$ and all $\langle \emptyset, (x) \rangle$. The domain-restriction of σ to $\mathscr{S}$ coincides with $\mathsf{J}_{\mathscr{S}}$.

Before proceeding further, certain abbreviations will be introduced in connection with the notation $\mathbf{E}\{T \mid \mathbf{R}\}$ of 1 A.11.

1 B.11. Convention. The symbol $\{T \to U \mid \mathbf{R}\}$ will denote the class consisting precisely of those pairs $\langle T, U \rangle$ which are elements and satisfy $\mathbf{R}$.

In other words, $\{T \to U \mid \mathbf{R}\}$ is, by definition, equal to $\mathbf{E}\{\langle T, U \rangle \mid \mathbf{R}\}$; thus, the introduction of a new symbol may seem unnecessary; nevertheless, the symbol $\{T \to U \mid \mathbf{R}\}$ has, at least, the advantage of being rather suggestive.

All remarks made after the convention 1 A.11 apply also to the above symbol. We would like to stress that $\{T \to U \mid \mathbf{R}\}$ is merely an abbreviation for the following expression of current language: "the class of all pairs of elements T, U such that $\mathbf{R}$ is satisfied".

Examples. (A) $\{x \to x^2 \mid x \geqq 0\}$ is the set of pairs $\langle x, x^2 \rangle$ for all non-negative x, i.e. the graph of the function x^2 defined for $x \geqq 0$ only. – (B) $\{t^2 \to t^3 \mid t \text{ real}\}$ is the set of all pairs $\langle t^2, t^3 \rangle$ with t real; this is the "curve" consisting of graphs of the two functions $\sqrt{x^3}$ and $-\sqrt{x^3}$ for $x \geqq 0$. – (C) $\{X \to Y \mid X \subset Y\}$ is the inclusion relation introduced in 1 B.4, example (C). – (D) Obviously $\{x \to y \mid x = y\} = \mathsf{J}$, and $\{x \to x \mid x \in A\} = \{x \to y \mid x = y, x \in A\} = \mathsf{J}_A$. – (E) if ϱ is a given relation, then $\{X \to \varrho[X] \mid X \text{ is a class}\}$ consists of all pairs $\langle X, \varrho[X] \rangle$ such that X and $\varrho[X]$ are sets.

1 B.12. In accordance with 1 B.11 and 1 B.6, $\mathbf{E}\{T \to U \mid \mathbf{R}\}$ denotes the range of the relation $\{T \to U \mid \mathbf{R}\}$, i.e. the class of all U such that $\mathbf{R}$ for some T.

Examples. (A) $\mathbf{E}\{x \to x^2 \mid x \text{ is even}\}$ is the set of squares of all even integers. – (B) $\mathbf{E}\{x^3 \to x \mid x^3 \text{ positive integer}\}$ is the set of cube roots of all positive integers. – (C) $\mathbf{E}\{\cos t \to \sin t \mid t \text{ real}, \tan t = 1\}$ is the set $(\frac{1}{2}\sqrt{2}, -\frac{1}{2}\sqrt{2})$. – (D) If ϱ is a given relation, then $\mathbf{E}\{X \to \varrho[X] \mid X \text{ is a set}\}$ consists of all sets of the form $\varrho[X]$, X being a subset of $\mathbf{D}\varrho$.

The expressions $\{T \to U \mid \mathbf{R}\}$ and $\mathbf{E}\{T \to U \mid \mathbf{R}\}$ are often unnecessarily cumbersome, and we will abbreviate them in the following manner.

1 B.13. Convention. $\{T \to U \mid \mathbf{R}\}$ will usually be written as $\{T \to U\}$ if $\mathbf{R}$ is obvious from the context. The notation $\{U \mid \mathbf{R}\}$ is used as abbreviation for $\{T \to U \mid \mathbf{R}\}$ provided T is clear from the context, and $\{U\}$ is written for $\{T \to U \mid \mathbf{R}\}$ if the T, $\mathbf{R}$ are obvious. Similarly for $\mathbf{E}\{T \to U\}$, $\mathbf{E}\{U \mid \mathbf{R}\}$, $\mathbf{E}\{U\}$. Finally, sometimes we shall write simply $\{\mathbf{R}\}$ instead of $\{T \to U \mid \mathbf{R}\}$, $\mathbf{E}\{\mathbf{R}\}$ instead of $\mathbf{E}\{U \mid \mathbf{R}\}$.

Remark. Observe that $\mathbf{E}\{U \mid \mathbf{R}\}$ has the same meaning under convention 1 A.11 as under the present one.

Examples. (A) $\{x \to \sin x \mid x \text{ is real}\}$ will be written as $\{x \to \sin x\}$ or even $\{\sin x\}$ whenever the set of reals is the obvious domain. – (B) $\{\sin x \mid x > 0\}$ is the "graph" (in the current sense) of the function sin defined for positive numbers only. –

(C) Expressions such as $\{y \mid x < y\}$ will be used rarely; nevertheless, the meaning of this expression is clear: it is, of course, an abbreviation for $\{x \to y \mid x < y\}$ which denotes the set of all $\langle x, y \rangle$ such that $x < y$. – (D) $\{x^2 + 2x + 2\}$ denotes, of course, if the domain of x's is clear from the context, the relation assigning to each x the element $x^2 + 2x + 2$; $\mathbf{E}\{x^2 + 2x + 2\}$ is its range, that is the set of all reals $y \geqq 1$ (if x varies over reals).

It is to be noted that abbreviated expressions, e.g. of the form $\{U \mid \mathbf{R}\}$, may be ambiguous in certain cases. For instance $\{S(x, y) \mid \mathbf{R}(x, y)\}$ may mean either $\{\langle x, y \rangle \to S(x, y) \mid \mathbf{R}(x, y)\}$, i.e. the set of pairs $\langle\langle x, y \rangle, S(x, y)\rangle$ satisfying certain conditions, or $\{x \to S(x, y)\}$, i.e. the set of pairs $\langle x, S(x, y)\rangle$ satisfying given conditions, with y variable (indeterminate), etc. Such difficulties can be avoided if abbreviations of the form $\mathbf{E}\{...\}$, $\{...\}$ are chosen, in each individual case, in such a way that the full expression can be restored from the context without ambiguity.

C. PROPERTIES OF RELATIONS

1 C.1. Definition. Let ϱ, σ be relations. Their *relational composite* (or merely *composite*) is the relation denoted by $\varrho \circ \sigma$ and defined as the class of all pairs $\langle x, y \rangle$ such that there is a z with $x\sigma z$, $z\varrho y$, i.e. such that $\langle x, z \rangle \in \sigma$, $\langle z, y \rangle \in \varrho$.

Thus for example the composite of the relations $\{x \to f(x)\}$, $\{y \to g(y)\}$ is the relation $\{x \to g(f(x))\}$; thus we have a nice agreement with the standard definition of composition (superposition) of funtions (in composed mappings, operations, etc., the standard order of consecutively performed operations is from right to left). To some extent, our definition is not entirely natural: it might seem more appropriate to define $\varrho \circ \sigma$ as the class of x, y with $x\varrho z$, $z\sigma y$ for some z, but this would disagree with standard notation for composite mappings.

Examples of composition of relations: (A) $\mathsf{J} \circ \mathsf{J} = \mathsf{J}$, and more generally $\mathsf{J} \circ \varrho = \varrho \circ \mathsf{J} = \varrho$ for every relation ϱ. – (B) The relation $<$ on the set of reals satisfies $< \circ < = <$. – (C) $\supset \circ \subset$ as well as $\subset \circ \supset$ consists of all pairs of sets. – (D) Let $\varrho = \{x \to e^x \mid x \text{ real}\}$; then $\varrho \circ \varrho^{-1} = \mathsf{J}_P$ where P is the set of all positive reals. – (E) Let $\varrho = \{x^2 \to x\}$, $\sigma_1 = \{x \to x\}$, $\sigma_2 = \{x \to -x\}$; then $\sigma_1 \circ \varrho = \sigma_2 \circ \varrho = \varrho$, but $\sigma_1 \neq \sigma_2$. – (F) Obviously $\mathbf{D}(\varrho \circ \sigma) \subset \mathbf{D}\sigma$, $\mathbf{E}(\varrho \circ \sigma) \subset \mathbf{E}\varrho$ for any relations ϱ, σ. Nevertheless, if $\varrho = \{\langle x, y \rangle \to (x, y)\}$ (see 1 A.12), then $\varrho \circ \varrho = \varrho^{-1} \circ \varrho^{-1} = \emptyset$, thus $\mathbf{D}(\varrho \circ \varrho) = \emptyset \neq \mathbf{D}\varrho$, $\mathbf{E}(\varrho \circ \varrho) = \emptyset \neq \mathbf{E}\varrho$.

Next, several important types of relations will be introduced.

1 C.2. Definition. A relation ϱ will be called *symmetric* if $\varrho = \varrho^{-1}$ (i.e. if $x\varrho y \Leftrightarrow y\varrho x$); ϱ will be called *antisymmetric* if never both $x\varrho y$ and $y\varrho x$; ϱ will be called *reflexive* if $\mathbf{D}\varrho = \mathbf{E}\varrho$ and $x\varrho x$ for every $x \in \mathbf{D}\varrho$; ϱ will be called *irreflexive* if $x\varrho x$ for no x.

Examples. The identity J is a reflexive and symmetric relation. The inclusion relation is reflexive but not symmetric. The "inequality" relation $\{x \to y \mid x \neq y\}$ is irreflexive and symmetric. The relation consisting of all pairs of disjoint sets is symmetric but not reflexive. The relation $<$ on the set of reals is irreflexive and antisymmetric.

Using composition of relations, we will define an important concept of a transitive relation; properties of transitive relations will be studied in Section 10.

1 C.3. Definition. A relation ϱ will be called *transitive* if $\varrho \circ \varrho \subset \varrho$, i.e. if $x\varrho y$ and $y\varrho z$ imply $x\varrho z$.

Examples. (A) Identity, inclusion and the relation $<$ on the set of reals are transitive relations. – (B) The relation $\varrho = \{x \to y \mid x, y$ are positive integers, y is divisible by $x\}$ is transitive. – All the relations just listed satisfy $\varrho \circ \varrho = \varrho$. – (C) Consider the relation $\varrho = \{X \to Y \mid X \neq Y, X \subset Y \subset A\}$, where A is a given set; this relation is transitive, but $\varrho \circ \varrho = \varrho$ usually does not hold: taking e.g. $A = (a, b)$, we obtain that $(a) \in \mathbf{D}\varrho$ and $(a) \notin \mathbf{D}(\varrho \circ \varrho)$. – (D) The relation $\{x \to y \mid x, y$ are positive integers, $y = x - k$ for some positive integer $k\}$ is transitive; the relation $\{x \to y \mid x, y$ are positive integers, $y = x - k\}$ where k is a given positive integer is not transitive.

1 C.4. Theorem and definition. *Let ϱ be a relation. Then there exists exactly one transitive relation ϱ^* such that $\varrho \subset \varrho^*$ and if ϱ' is transitive and $\varrho \subset \varrho'$, then $\varrho^* \subset \varrho'$. This relation ϱ^* is called the* smallest transitive relation containing ϱ.

Proof. We shall say that a relation ξ has property $\boldsymbol{T}$ if $\varrho \subset \xi$ and ξ is transitive; clearly if ξ_0 consists of all $\langle x, y\rangle$ such that $x \in \mathbf{D}\varrho$ or $x \in \mathbf{E}\varrho$, and $y \in \mathbf{D}\varrho$ or $y \in \mathbf{E}\varrho$, then ξ_0 has property $\boldsymbol{T}$. Let ϱ^* denote the class of all $\langle x, y\rangle$ belonging to every relation with property $\boldsymbol{T}$. Then ϱ^* is a relation and clearly $\varrho \subset \varrho^*$. If $\langle x, y\rangle \in \varrho^*$, $\langle y, z\rangle \in \varrho^*$, then, for every ξ with property $\boldsymbol{T}$, $\langle x, y\rangle$ as well as $\langle y, z\rangle$ belongs to ξ, and therefore $\langle x, z\rangle \in \xi$, ξ being transitive. This implies $\langle x, z\rangle \in \varrho^*$. If a relation ϱ' is transitive, $\varrho \subset \varrho'$, then ϱ' possesses property $\boldsymbol{T}$, hence $\varrho^* \subset \varrho'$.

Remark. It is easy to see that the above theorem holds if "transitive" is replaced by "transitive reflexive" or "symmetric" or "symmetric transitive" and so on. We formulate explicitly one of such propositions in 1 C.8.

As mentioned above, we shall consider order relations in Section 10; nevertheless, the fundamental definitions and some notation is introduced here for convenience.

1 C.5. Definition. A transitive relation will also be called a *quasi-order*; if σ is a quasi-order and A is a class containing all (consisting of all) x such that either $x \in \mathbf{D}\sigma$ or $x \in \mathbf{E}\sigma$, then we shall say that σ is a *quasi-order in A* (*on A*). A reflexive quasi-order σ such that $x\sigma y$, $y\sigma x$ imply $x = y$ will be called an *order*.

Examples. (A) $\subset$ and $\supset$ are orders on the class of all sets. – (B) $\{\varrho \to \sigma \mid \varrho = \sigma \circ \mu$ for some $\mu\}$ is a quasi-order on the class of all comprisable relations, but it is not an order.

1 C.6. Convention. Let a quasi-order σ in a class A be given. We shall use the following symbols, similar (except for a graphic difference) to those currently used for intervals of reals.

If $a \in A$, then $]\!] a, \rightarrow [\![$ denotes the class of all $x \in A$ such that $a\sigma x$ but neither $x\sigma a$ nor $x = a$; $[\![a, \rightarrow [\![$ denotes the class of all $x \in A$ such that $a\sigma x$ or $x = a$; $]\!] \leftarrow, a [\![$ denotes the class of all $x \in A$ such that $x\sigma a$ but neither $a\sigma x$ nor $x = a$; finally, $]\!] \leftarrow, a]\!]$ denotes the class of all $x \in A$ such that $x\sigma a$ or $x = a$. (Observe that if σ is an order, then $]\!] a, \rightarrow [\![= \mathbf{E}\{a\sigma x, a \neq x\}$, $[\![a, \rightarrow [\![= \mathbf{E}\{a\sigma x\}$, etc.)

If $a \in A$, $b \in A$, then consistently with the above: $]\!] a, b [\![$ consists of all elements belonging both to $]\!] a, \rightarrow [\![$ and $]\!] \leftarrow, b [\![$; $]\!] a, b]\!]$ of all those belonging to both $]\!] a, \rightarrow [\![$ and $]\!] \leftarrow, b]\!]$; the symbols $[\![a, b [\![$, $]\!] a, b [\![$ are defined similarly.

Example. Consider the order relation $\subset$. Then if A is a set, $[\![A, A]\!] = (A)$, $[\![A, A [\![= \emptyset$, etc. For any set B, $]\!] \leftarrow, B]\!]$ consists of all subsets of B, and $]\!] \leftarrow, B [\![$ consists of all proper subsets.

An important type of relation which will soon be indispensable for further developments is an equivalence relation.

1 C.7. Definition. A relation ϱ will be termed an *equivalence* if it is reflexive, symmetric and transitive, i.e. if $x\varrho y$ implies $y\varrho x$, $x\varrho y$ and $y\varrho z$ imply $x\varrho z$, $\mathbf{D}\varrho = \mathbf{E}\varrho$ and $x\varrho x$ for $x \in \mathbf{D}\varrho$.

We shall say that ϱ is an *equivalence on* A, A being a class, if ϱ is an equivalence and $\mathbf{D}\varrho = A$. An *equivalence in* A is, in accordance with 1 B.6, an equivalence ϱ with $\mathbf{D}\varrho \subset A$.

Examples. (A) The identity is an equivalence relation. — (B) $\{\langle x, y\rangle \rightarrow \rightarrow \langle u, v\rangle \mid x, y, u, v$ are positive integers, $xv = yu\}$ is an equivalence. — (C) The inclusion relation is not an equivalence since it is not symmetric. — (D) If ϱ is a given relation, then $\{\varrho[X] = \varrho[Y]\}$ is an equivalence on the class of all sets.

Remark. If ϱ is an equivalence, then any two fibres of ϱ either are identical or have no elements in common. Conversely, if a symmetric relation satisfies this condition, then it is an equivalence.

1 C.8. Theorem and definition. *Let ϱ be a relation. Then there exists exactly one equivalence ϱ^* such that $\varrho \subset \varrho^*$ and if ϱ' is an equivalence, $\varrho \subset \varrho'$, then $\varrho^* \subset \varrho'$. This equivalence ϱ^* is called the* smallest equivalence containing ϱ.

The proof is quite analogous to that of 1 C.4 and is left to the reader.

Relations have been introduced to express the notion of (logical) relationship in its extensional aspect. They are classes of pairs and every relation for classes A and B is a part of the class of all pairs $\langle x, y\rangle$ where $x \in A$, $y \in B$. It is convenient to introduce this class, denoted by $A \times B$, at the present stage; it will be examined in detail later (in Section 5). The terminology in the following definitions and conventions is drawn from the analogy between $A \times B$ and, say, the plane expressed as a "product" of two replicas of the real line.

1 C.9. Definition. If X and Y are classes, then the class of all pairs $\langle x, y\rangle$ where $x \in X$, $y \in Y$ will be termed the *pair-product* (or simply *product*) of X and Y, and denoted by $X \times Y$.

Convention. If X is a class, then $X \times X$ is sometimes called the *square* of X, and the class of all $\langle x, x\rangle$ where $x \in X$, i.e. the relation J_X, is called the *diagonal of the square* of X.

Example. If a is an element, $X = (a)$ and Y is the class of all sets containing a, then $X \times Y$ is identical with the relation $\in$ (see 1 B.4, example (B)) restricted to (a).

Remark. Clearly $X \times Y$ is "the largest relation" for X and Y; more precisely, ϱ is a relation for X and Y if and only if $\varrho \subset X \times Y$.

D. SINGLE-VALUED RELATIONS

We now define single-valued relations. These are currently termed mappings; however, we will define mappings in another manner (see Section 7) and single-valued relations will then be the so-called graphs of mappings.

1 D.1. Definition. We shall say that a relation ϱ is *single-valued* at an element x if there exists exactly one y such that $\langle x, y\rangle \in \varrho$. If ϱ is single-valued at x, then the (uniquely determined) element y such that $\langle x, y\rangle \in \varrho$ will be denoted by ϱx and called the *value* of ϱ at x (if ϱ is not single-valued at x, then, of course, "ϱx" has no meaning).

If a relation φ is single-valued at every $x \in \mathbf{D}\varphi$, (that is, if $x \in \mathbf{D}\varphi$, $\langle x, y_1\rangle \in \varphi$, $\langle x, y_2\rangle \in \varphi$ imply $y_1 = y_2$), then φ will be called *single-valued* (or also a *mapping relation*).

Conventions. 1) Given a symbol f and a single-valued relation φ such that $y = \mathrm{f}x$ if and only if $y = \varphi x$, we shall sometimes use f to denote φ (cf. 1 B.5). – 2) If ϱ is a relation, A and B are classes, $A \subset \mathbf{D}\varrho$ and ϱ_A is single-valued, we shall say that *ϱ maps A into B* (*onto B*) if $\varrho[A] \subset B$ (respectively, $\varrho[A] = B$). – 3) If φ is a single-valued relation, $x \in \mathbf{D}\varphi$, $y = \varphi x$, we shall say that *φ assigns y to x*.

Examples. (A) The relation $\ni\, = \,\in^{-1}$ (see 1 B.8), whose domain is the class of all non-void sets, is single-valued at a set X if and only if X is a singleton; in this case we have $x = \ni X$ where $X = (x)$; in the other cases, "$\ni X$" has no meaning. – (B) Both relations J and J_X (for any class X) are single-valued. – (C) The relations $\{\langle x, y\rangle \to x\}$ and $\{\langle x, y\rangle \to y\}$ are single-valued, but the inverse relations $\{x \to \langle x, y\rangle\}$, $\{y \to \langle x, y\rangle\}$ are not. – (D) By 1 E.1, the class $\supset [(A)]$, i.e. the class of all sets $X \subset A$, is a set whenever A is a set. It is easy to see that $\{X \to \supset [(X)]\}$ is a single-valued relation, and its inverse is also single-valued.

Remark. If ϱ is single-valued and $\sigma \subset \varrho$, then σ is single-valued. If ϱ, σ are single-valued, then $\varrho \circ \sigma$ is also single-valued.

1 D.2. According to our definitions, there is an essential difference between $\varrho[(x)]$ and ϱx. The class $\varrho[(x)]$, the image of the singleton (x), is defined for any relation ϱ and any element x; it is non-void precisely if $x \in \mathbf{D}\varrho$, and in general contains more than one element. On the other hand, "ϱx" is defined only if ϱ is single-valued at x, and denotes a certain element. If ϱx is defined, then necessarily $\varrho[(x)] = (\varrho x)$. However, for simplicity, and in accordance with current usage, we formulate the following convention, to be applied in unambiguous cases.

Convention. If φ is a single-valued relation and $x \in \mathbf{D}\varphi$, then the element φx, the value of φ at x, will sometimes be called the *image* of x under φ. Also, if ϱ is an arbitrary relation, and x is an element, the class $\mathbf{E}\{y \mid \langle x, y\rangle \in \varrho\}$ will be denoted and termed as follows: $\varrho[(x)]$, $\varrho[x]$, the fibre of ϱ at x, the image of (x) under ϱ.

1 D.3. Definition. A relation ϱ is said to be a *fibering relation* if $\langle x_1, y\rangle \in \varrho$, $\langle x_2, y\rangle \in \varrho$ implies $x_1 = x_2$ (in other words, if ϱ^{-1} is single-valued).

The motivation of this concept is deferred to Section 7 (see, in particular, 7 C.8).

1 D.4. *Let ϱ be a relation. Then there exists a fibering relation ψ and a single-valued relation φ such that $\varrho = \varphi \circ \psi$.*

Proof. Put $\psi = \{x \to \langle x, y\rangle \mid \langle x, y\rangle \in \varrho\}$, $\varphi = \{\langle x, y\rangle \to y \mid \langle x, y\rangle \in \varrho\}$.

1 D.5. Definition. Let ϱ be a relation. If $A \subset \mathbf{D}\varrho$ and σ is a single-valued relation with $\mathbf{D}\sigma = A$ such that $\sigma \subset \varrho$, then σ is called a *section* of ϱ upon (over) A. If $B \subset \mathbf{E}\varrho$ and τ is a single-valued relation with $\mathbf{D}\tau = B$ such that $\tau \subset \varrho^{-1}$ (i.e. if τ is a section of ϱ^{-1} upon B), then τ is called a *cross-section* of ϱ upon (over) B.

Observe, that, for a single-valued ϱ, a section of ϱ upon A coincides with the restriction of ϱ to A; in other words, the restriction (more precisely: the domain-restriction) of a single-valued ϱ to a class X coincides with the (unique) section of ϱ upon the class of all those $x \in \mathbf{D}\varrho$ which belong to X.

Examples. (A) Let X, Y be classes. Then every single-valued relation for X and Y is a section of $X \times Y$, and conversely. – (B) Using the terminology and notation to be introduced in Section 3, we have the following example: a section (upon N) of the natural order $\leqq$ is an infinite sequence $\{n_k\}$ of natural numbers such that $n_k \geqq k$, and a cross-section (upon N) of this order is a sequence $\{n_k\}$ with $n_k \leqq k$.

Remark. If ϱ is single-valued, then σ is a cross-section of ϱ upon $B \subset \mathbf{E}\varrho$ if and only if σ is a singleton and $\varrho \circ \sigma = \mathsf{J}_B$. If ϱ is a fibering relation, then σ is a section of ϱ upon $A \subset \mathbf{D}\varrho$ if and only if $\varrho^{-1} \circ \sigma = \mathsf{J}_A$.

1 D.6. Definition. We shall say that ϱ is a *one-to-one relation* if both ϱ and ϱ^{-1} are single-valued relations.

Convention. If φ is a one-to-one relation with domain A and range B we shall also say that φ is *bijective on A onto B* (or that φ is *bijective for A and B*). If φ is bijective on A onto itself, we shall say that φ is *bijective for A*.

Examples. (A) The relation $\{x \to (x)\}$ is bijective for the universal class and the class of all singletons. – (B) The relation $\{\langle x, y\rangle \to \langle x, y\rangle\}$ is bijective for the class of all those pairs which are elements. – (C) The relation $\{x \to x^2 \mid x \text{ a real number}\}$ is single-valued but it is not bijective; its restriction to non-negative reals is bijective.

Remark. It is easy to prove that a relation ϱ is one-to-one if and only if there exists a σ such that $\varrho \circ \sigma$ and $\sigma \circ \varrho$ are identity relations, $\varrho \circ \sigma = \mathsf{J}_{\mathsf{E}\varrho}$, $\sigma \circ \varrho = \mathsf{J}_{\mathsf{D}\varrho}$. In fact, we have $\sigma = \varrho^{-1}$.

1 D.7. Definition. A *permuting relation* is a one-to-one relation φ such that $\mathsf{D}\varphi = \mathsf{E}\varphi$.

For instance, the relation in the above example (B) is a permuting one.

1 D.8. Before proceeding further (to the axioms for sets), we shall first consider another aspect of the concept of a single-valued relation, leading e.g. to the notion of a sequence.

Suppose that to every element of a class there is assigned an element from a second class and it is the latter which is of primary importance in a given context, whereas the former one is, in a certain sense, merely auxiliary. In point of fact, we have a single-valued relation which is, however, considered from a different point of view. In such a situation, it is useful to introduce new terms for concepts already defined, and even to introduce new concepts for properties which come to the forefront in connection with the different point of view adopted.

1 D.9. Definition. A single-valued relation will sometimes be called an *indexed class*; a comprisable indexed class (i.e. a single-valued relation which is a set) will sometimes be called a *family*. If $\mathscr{X}$ is an indexed class (and particularly if $\mathscr{X}$ is a family), then $\mathsf{D}\mathscr{X}$ will be called its *domain of indexes* (or simply *domain*), or its *class* (*set*) *of indexes*, and every element from $\mathsf{D}\mathscr{X}$ will be called an *index* of the indexed class $\mathscr{X}$; elements from $\mathsf{E}\mathscr{X}$ will be called *members* (or *values*) of the indexed class (or family) $\mathscr{X}$. If $\mathscr{X}$ is an indexed class and $\mathscr{Y} \subset \mathscr{X}$, then we shall say that $\mathscr{Y}$ is an *indexed subclass* of $\mathscr{X}$ or (provided $\mathscr{Y}$ is a set) that $\mathscr{Y}$ is a *subfamily* of $\mathscr{X}$. (Note that the related terms subnet, subsequence are used in a substantially different sense, see 15 B.17.)

Remarks. 1) We stress once more that an indexed class is a single-valued relation, a family is a (single-valued) comprisable relation; hence all notions introduced for single-valued relations are meaningful for indexed classes (in particular, for families). – 2) An indexed class is a class, a family is a set, but there is a substantial difference between elements and members of an indexed class (family). Thus take the indexed class $\mathscr{A} = \{X \to X \times X\}$, i.e. the class of all pairs $\langle X, X \times X \rangle$ where X is a set; then every set is an index, every set of the form $X \times X$, X a set, is a member of $\mathscr{A}$, but the elements of $\mathscr{A}$ are pairs $\langle X, X \times X \rangle$ and these, of course, are not sets. – 3) A notation of the following form will usually be used for indexed classes (in accordance with 1 B.11 and 1 B.13): $\{x_a \mid a \in A\}$. When using such a notation, we will always assume, unless the contrary is stated, that A is the domain of indexes, i.e. that "x_a" is meaningful for every $a \in A$.

Examples. (A) $\{X \to \mathsf{J}_X \mid X \text{ is a set}\}$ is an indexed class but not a family. – (B) The relation $\{n \to A_n \mid n \text{ is a positive integer, } A_n \text{ is the set of all integers greater than } n\}$ is a family. – (C) Let F be the set of all real-valued functions which have a derivative in the interval $]\!]\, 0, 1 \,[\![$; then $\{f \to f' \mid f \in F\}$ is a family.

1 D.10. Definition. An indexed class $\{x_a \mid a \in A\}$ is called *constant with value* z if it is non-void and $a \in A$ implies $x_a = z$. It is called *constant* if there exists an element z such that $a \in A \Rightarrow x_a = z$.

Since indexed classes and single-valued relations are the same objects considered from different points of view, we have, in fact, already defined a *constant relation* (*with value* z). For convenience, we state explicitly: a single-valued relation ϱ is called *constant* if $\langle x, y\rangle \in \varrho$, $\langle x', y'\rangle \in \varrho$ implies $y = y'$.

It is clear that $\emptyset$ is a constant relation and that a relation $\varrho \neq \emptyset$ is constant if and only if $\mathbf{E}\varrho$ is a singleton.

E. SETS

The concept of a set has been already defined (a set is a comprisable class, i.e. a class X which belongs to some class $\mathscr{Y}$). However, we have no means as yet to prove any useful propositions for sets. Thus, in fact, the notion of a set has to be properly introduced. This will now be done.

1 E.1. Axioms for sets.

(a) *If x is an element then (x) is a set.*

(b) *Let ϱ be a relation and let X be a set. If, for every x such that $x \in X$, $x \in \mathbf{D}\varrho$, the class $\varrho[(x)]$ is a set, then the class $\varrho[X]$ is a set.*

(c) *If X is a set, then the class of all subsets of X is a set.*

1 E.2. It may be apparent that these axioms correspond both to intuitive ideas (if sets are regarded as "not excessively large" classes) and also to the practice of mathematics in branches where set theory is applied. Indeed, a one-element class is assuredly "small enough" to warrant being a set; Axiom (b) requires, roughly, that the union (cf. Section 2) of a collection of sets again be a set – and this is intuitively obvious and current practice. That the class of all subsets of a given set is again a set, i.e. that the operation of taking all subsets remains within the scope of "not excessively large" classes, may not be as clear intuitively; but it is in complete agreement with procedures standard in modern mathematics.

1 E.3. Remarks. (1) Note that the axioms listed still do not guarantee the existence of sets (for instance, it can be easily shown that the axioms are satisfied when there are no elements, and only one class, the void class). Also note that the axioms are satisfied (assuming the standard form of class and set theory is available) if the only admitted sets are all the finite classes (finite in the usual sense). – (2) On the other hand, we may already show that, if there exist any elements, then the void class $\emptyset$ is a set; this follows from Axioms (a) and (b); if x is an element, $\varrho = \emptyset$, $X = (x)$, then $\emptyset = \varrho[X]$, hence $\emptyset$ is a set. – It can be even shown (if we suppose that there exist some elements) that there exist "infinitely many" (in an intuitive sense) sets. Namely, it is intuitively clear that the sets $\emptyset$, $(\emptyset)$, $((\emptyset))$, $(((\emptyset)))$, ... are mutually distinct. For, evidently, $\emptyset \neq (\emptyset)$. Therefore, $(\emptyset) \neq ((\emptyset))$ and, clearly, $\emptyset \neq ((\emptyset))$; thus,

$\emptyset, (\emptyset), ((\emptyset))$ are different, and we may proceed with this argument indefinitely. Observe, however, that this reasoning as such constitutes no proof as yet. – (3) From the Axioms for sets it is also easily deduced that any actually given class with a "finite number" of elements is a set. Let us illustrate this for, say, a class (a, b) containing two elements. The existence of such a class implies, by the previous argument, that $\emptyset$ is a set. Let ϱ be the relation consisting of the pairs $\langle \emptyset, a \rangle$ and $\langle (\emptyset), b \rangle$. The class $X = (\emptyset, (\emptyset))$ is a set by Axiom (c), since it consists of all subsets of the set $(\emptyset)$. Since $\varrho[(\emptyset)] = (a)$, $\varrho[((\emptyset))] = (b)$, we conclude, by Axiom (b), that $\varrho[X] = (a, b)$ is a set.

Next we will prove some simple consequences of the axioms. If φ is a single-valued relation, then for every $x \in \mathbf{D}\varphi$ the class $\varphi[(x)]$ contains precisely one element; thus from Axioms (a) and (b) there follows

1 E.4. Theorem. *If φ is a single-valued relation and X a set, then $\varphi[X]$ is a set.*

1 E.5. Theorem. *If X is a set and $Y \subset X$, then Y is a set.*

Proof. Let X be a set, $Y \subset X$. According to 1 E.4, the class $\mathsf{J}_Y[X]$ is a set. Obviously $\mathsf{J}_Y[X] = Y$.

Remark. In 1 A.10, we have exhibited a non-comprisable class. This implies, by the above proposition, since every class is a part of the universal class, that the universal class is non-comprisable.

1 E.6. Theorem. *If X, Y are sets, then $X \times Y$ is a set.*

Proof. For any given element z, the relation $\{y \to \langle z, y \rangle\}$ is single-valued, hence if Y is a set, then $(z) \times Y$ is a set. Put $\varrho = \{x \to \langle x, y \rangle \mid x \in X, y \in Y\}$. For any $x \in X$, $\varrho[(x)] = (x) \times Y$ is a set, and therefore, by axiom (b), $X \times Y = \varrho[X]$ is a set.

1 E.7. Theorem. *A relation ϱ is comprisable (i.e. is a set) if and only if $\mathbf{D}\varrho$ and $\mathbf{E}\varrho$ are both comprisable.*

Proof. Let the relation ϱ be a set. Since $\pi = \{\langle x, y \rangle \to x\}$ is single-valued, $\pi[\varrho] = \mathbf{D}\varrho$ is a set by 1 E.4. Similarly, using $\{\langle x, y \rangle \to y\}$, we obtain that $\mathbf{E}\varrho$ is a set. Conversely, if $\mathbf{D}\varrho$, $\mathbf{E}\varrho$ are sets, then according to 1 E.6, $\mathbf{D}\varrho \times \mathbf{E}\varrho$ is a set; since $\varrho \subset \mathbf{D}\varrho \times \mathbf{E}\varrho$, ϱ is a set by 1 E.5.

We shall now consider briefly two important kinds of classes of sets.

1 E.8. Definition. Let X be a set, Y a class. Then the class of all single-valued relations on X into Y (i.e. of single-valued relations φ such that $\mathbf{D}\varphi = X$, $\mathbf{E}\varphi \subset Y$) will be denoted by Y^X.

This notation is motivated e.g. by the fact that if X, Y have a finite number m, respectively n of elements, then Y^X has n^m elements (cf. Section 3).

Observe that $Y^\emptyset = (\emptyset)$ for any class Y.

1 E.9. Definition. If X is a class, then the class of all subsets of X will be termed the *potency class* of X or the *exponential* of the class X, and will be denoted by $\exp X$.

The motivation of the terms "potency class" and "exponential" lies in the fact that if X has a finite number m of elements, then $\exp X$ has 2^m elements, as well as in the fact (see immediately below) that (provided X is a set) $\exp X$ is "equivalent" to T^X, T being a two-element set.

1 E.10. *Let* $T = (a, b)$, $a \neq b$. *Let* X *be a set. The relation* $\{\varphi \to \varphi^{-1}[a] \mid \varphi \in T^X\}$ *is bijective for* T^X *and* $\exp X$.

Convention. The above relation will be called a *canonical relation* for T^X and $\exp X$. Observe that $\{\varphi \to \varphi^{-1}[b]\}$ will also be termed canonical for T^X and $\exp X$; thus there are exactly two canonical relations for T^X and $\exp X$.

1 E.11. *The class* $\exp X$ *is a set if and only if* X *is a set. If* X *is a non-void set, and* Y *is a class, then* Y^X *is a set if and only if* Y *is a set.*

Proof. For the first assertion, "if" is the axiom 1 E.1, (c) and "only if" is obtained from the fact that $\{x \to (x) \mid x \in X\}$ is one-to one on X into $\exp X$. As for the second assertion, "if" follows, by 1 E.5, 1 E.6, from $Y^X \subset \exp(X \times Y)$, and "only if" follows from the following fact: if α is the relation assigning to every y the constant relation $\varphi_y \in Y^X$ with $\mathbf{E}\varphi_y = (y)$, then α is one-to-one on Y into Y^X.

Some notions concerning indexed classes (in particular, families) of sets will now be introduced.

1 E.12. Definition. A family $\{X_a \mid a \in A\}$ will be called a *cover* if every X_a is a set (this definition of cover will be extended later, see 12 A.1). In a more detailed manner, we shall say that a family of sets $\mathscr{X} = \{X_a \mid a \in A\}$ *covers a set* Z if every $z \in Z$ belongs to X_a for some a, and that it is a *cover of a set* Y if every $y \in Y$ belongs to some X_a and $X_a \subset Y$ for every $a \in A$.

We shall consider covers in detail in Section 12 stating here only some simple facts.

Examples. (A) Let ϱ be a relation. Then $\hat{\varrho} = \{x \to \varrho[(x)] \mid x \in \mathbf{D}\varrho\}$ is a cover if and only if ϱ is comprisable; in this case $\hat{\varrho}$ is a cover of $\mathbf{E}\varrho$. — (B) Let a quasi-order σ on a set A be given; then $\{]\!] \leftarrow, x]\!] \mid x \in A\}$ is a cover of A whereas $\{]\!] \leftarrow, x [\![\mid x \in A\}$ is a cover, but, in general, not a cover of A.

1 E.13. *For any comprisable relation* ϱ, *let* $\hat{\varrho}$ *denote the relation* $\{x \to \varrho[(x)] \mid x \in \mathbf{D}\varrho\}$. *Then* $\{\varrho \to \hat{\varrho}\}$ *is a one-to-one relation on the class of all comprisable relations onto the class of all those families whose every member is a non-empty set. If* X *is a set, and* Y *is a class, then* $\{\varrho \to \hat{\varrho}\}$ *maps* $\exp(X \times Y)$ *onto* $(\exp' Y)^X$ *where* $\exp' Y$ *denotes the class of all non-empty sets* $S \subset Y$.

Proof. We shall first prove that $\{\varrho \to \hat{\varrho}\}$ is one-to-one. Suppose that $\hat{\sigma} = \hat{\varrho}$. Since $\mathbf{D}\hat{\varrho} = \mathbf{D}\varrho$, $\mathbf{D}\hat{\sigma} = \mathbf{D}\sigma$, we have $\mathbf{D}\varrho = \mathbf{D}\sigma$, and for any $x \in \mathbf{D}\varrho$ we obtain $\varrho[(x)] = \sigma[(x)]$; this implies $\varrho = \sigma$. If τ is a cover every member of which is a non-empty set, then put $\varrho = \{x \to y \mid y \in \tau x\}$; clearly $\mathbf{D}\varrho = \mathbf{D}\tau$ (since every τx is non-void), and $x \in \mathbf{D}\varrho$ implies $\varrho[(x)] = \tau x$; thus $\hat{\varrho} = \tau$. The rest of the proof is left to the reader.

Convention. If ϱ is a comprisable relation, then $\{\varrho[(x)] \mid x \in \mathbf{D}\varrho\}$ will be called the *cover associated* with ϱ and will be occasionally denoted by $\hat{\varrho}$. In general, if ϱ is a relation all fibres of which are comprisable, we shall refer to $\{\varrho[(x)] \mid x \in \mathbf{D}\varrho\}$ as the indexed class of fibres of ϱ, also using occasionally the symbol $\hat{\varrho}$.

If, in addition, ϱ is an equivalence and $A \subset \mathbf{D}\varrho$, then the class $\mathbf{E}\{\varrho[(x)] \mid x \in A\}$ will be denoted by A/ϱ and the relation $\hat{\varrho}_A$, which maps A onto A/ϱ, will be called the *natural* (*canonical*) *relation* for A and A/ϱ. Observe that similar symbols are sometimes used in a different sense; namely, if φ is a single-valued relation or a mapping (see 7 B.10) and $A \subset \mathbf{D}|\varphi|$ (see 7 A.1, 7 B.1), then A/φ may denote the set $\varphi[A]$.

Examples. (A) The relation $\hat{\jmath}$ assigns to every element x the singleton (x). – (B) The indexed class of fibres of $\supset$ is the relation $\{X \to \exp X\}$; if A is a set, then the cover associated with $\supset_{\exp A}$ is equal to $\{X \to \exp X \mid X \subset A\}$.

We add one further definition which is loosely related to covers.

1 E.14. Definition. Let ϱ be a relation. If a class $\mathscr{A}$ consists of subsets (not necessarily all) of $\mathbf{D}\varrho$, then the single-valued relation σ which assigns $\varrho[X]$ to every $X \in \mathscr{A}$ will be called a *canonical expansion* of ϱ to a class of sets (namely to $\mathscr{A}$).

Clearly, if ϱ is a comprisable relation, and ϱ' is its expansion to the class of all singletons $(x) \subset \mathbf{D}\varrho$, then $\varrho' \circ \hat{\jmath} = \hat{\varrho}$.

2. UNION AND INTERSECTION

This short section deals with basic class (conventionally set) theoretical operations, namely with union, intersection, difference and symmetric difference. No profound results are proved here, since the only aim of this section is to introduce the concepts and notation necessary for the further development of our axiomatic system which will be used throughout the book.

As for the union (and similarly for the intersection), three definitions are given to cover various situations. First we define the union of a "finite number" of classes (more precisely, of two classes, and then, step by step, of three, four, etc.). Then the union (or the intersection) of an indexed class (in particular, of a family) of sets is defined; evidently, this does not include the case of a "finite number" of non-comprisable classes, as we cannot speak of an "indexed class of non-comprisable classes". Finally, we define the union and the intersection of a class of sets.

2.1. Definition. Let A, B be classes. The class of all elements which belong either to A or to B is called the *union* of the classes A and B, and is denoted by $A \cup B$; the class of all elements which belong to both A and B is called the *intersection* of the classes A and B, and is denoted by $A \cap B$; the class of elements which belong to A but not to B is called the *difference* of the classes A and B, and is denoted by $A - B$; the class $(A - B) \cup (B - A)$ is called the *symmetric difference* of the classes A and B, and is denoted by $A \div B$. If $A \supset B$, then $A - B$ is also called the *complement* of the class B *relative to* A (or, if A is clear from the context, simply the *complement* of B).

Using the notation introduced in Section 1 we have $A \cup B = \mathbf{E}\{x \mid x \in A \text{ or } x \in B\}$, $A \cap B = \mathbf{E}\{x \mid x \in A \text{ and } x \in B\}$, $A - B = \mathbf{E}\{x \mid x \in A,\ x \notin B\}$, $A \div B = \mathbf{E}\{x \mid \text{either } x \in A,\ x \notin B \text{ or } x \notin A,\ x \in B\}$.

The difference $A - B$ is often denoted by $A \setminus B$ or $A \dot{-} B$. Such a notation is useful in considerations involving algebraical operations; since they do not occur too often in this book, we use mainly the symbol "$-$" for the difference of classes.

Remark. Since relations are classes, we may, of course, speak of the union, intersection, difference and symmetric difference of two relations (and similarly for three, four, etc. relations), as well as of the union and intersection of an indexed class of (comprisable) relations, etc.

Examples. (A) If there is given a quasi-order (i.e. a transitive relation, see 1 C.5) on a class A, then (see 1 C.6) the intersection of $]\!] \leftarrow, b [\![$ and $]\!] a, \rightarrow [\![$ is equal to $]\!] a, b [\![$ for any $a \in A$, $b \in A$. — (B) If ϱ denotes the inclusion relation, i.e. if $\varrho = \{X \rightarrow Y \mid X \subset Y\}$, then $\varrho \cap \varrho^{-1}$ is equal to the identity relation restricted to the class of all sets; if $\mathscr{X}$ is a class of sets, then $\varrho_{\mathscr{X}} \cup (\varrho_{\mathscr{X}})^{-1} = \mathscr{X} \times \mathscr{X}$ if and only if the class $\mathscr{X}$ is monotone (see 3 B.1). — (C) If ϱ is a relation, then $\varrho \cup \varrho^{-1}$ is the "smallest symmetric relation containing ϱ" (cf. 1 C.4, remark).

2.2. Definition. We shall say that classes A and B are *disjoint* (or *do not meet*) if their intersection $A \cap B$ is void; if classes A and B are not disjoint, we shall say that they *meet* or *intersect.*

2.3. Theorem. *Let A, B, C be classes. Then the following hold*:

(a) $A \cup B = B \cup A$, $A \cap B = B \cap A$.
(b) $A \cup (B \cup C) = (A \cup B) \cup C$, $A \cap (B \cap C) = (A \cap B) \cap C$.
(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
(d) $A \cup A = A \cap A = A$.
(e) $A - (B \cup C) = (A - B) \cap (A - C)$, $A - (B \cap C) = (A - B) \cup (A - C)$.
(f) $A \div B = B \div A$.
(g) $(A \div B) \div C = A \div (B \div C)$.
(h) $A \cap (B \div C) = (A \cap B) \div (A \cap C)$.

The proof follows directly from the definitions of the set operations.

The equalities in the above theorem will be used without reference, as well as some other basic facts which we do not mention explicitly (such as $A = (A - B) \cup \cup (A \cap B)$, $A \subset A \cup B$ etc.).

As the union and the intersection are commutative and associative (as expressed by (a), (b) above), the following notation will be used, as usual.

Convention. If A, B, C are classes, then instead of $A \cup (B \cup C)$ or $(A \cup B) \cup C$ the symbol $A \cup B \cup C$ is written, and similarly for $A \cap B \cap C$, $A \cup B \cup C \cup D$, $A \cap B \cap C \cap D$, etc. The class $A \cup B \cup C$ is called the union of the classes A, B, C, etc.

2.4. *If ϱ is a relation, A, B are classes, then $\varrho[A \cup B] = \varrho[A] \cup \varrho[B]$; $\varrho[A \cap B] \subset \subset \varrho[A] \cap \varrho[B]$, $\varrho[A] - \varrho[B] \subset \varrho[A - B]$; if ϱ is a fibering relation (in particular, if ϱ is one-to-one), then $\varrho[A \cap B] = \varrho[A] \cap \varrho[B]$, $\varrho[A] - \varrho[B] = \varrho[A - B]$. If ϱ, σ are relations, then $\mathbf{D}(\varrho \cup \sigma) = \mathbf{D}\varrho \cup \mathbf{D}\sigma$, $\mathbf{E}(\varrho \cup \sigma) = \mathbf{E}\varrho \cup \mathbf{E}\sigma$, $\mathbf{D}(\varrho \cap \sigma) \subset \subset \mathbf{D}\varrho \cap \mathbf{D}\sigma$, $\mathbf{E}(\varrho \cap \sigma) \subset \mathbf{E}\varrho \cap \mathbf{E}\sigma$.*

These equalities and inclusions, the proof of which is immediate, will be also used without reference.

Remark. The equalities $\varrho[A \cap B] = \varrho[A] \cap \varrho[B]$, $\varrho[A] - \varrho[B] = \varrho[A - B]$, $\mathbf{D}(\varrho \cap \sigma) = \mathbf{D}\varrho \cap \mathbf{D}\sigma$, $\mathbf{E}(\varrho \cap \sigma) = \mathbf{E}\varrho \cap \mathbf{E}\sigma$ do not hold in general.

We turn now the definition of the union and the intersection of an indexed class of sets.

2.5. Definition. Let $\mathscr{X} = \{X_\alpha \mid \alpha \in A\}$ be an indexed class of sets (this means that all X_α are sets). Then the class of all elements which belong to some X_α, that is the class $\mathsf{E}\{x \mid x \in X_\alpha$ for some $\alpha \in A\}$, is called the *union of the indexed class* $\mathscr{X}$ and denoted by $\bigcup \mathscr{X}$. The class of all elements which belong to each X_α, that is the class $\mathsf{E}\{x \mid x \in X_\alpha$ for each $\alpha \in A\}$, is called the *intersection of the indexed class* $\mathscr{X}$ and is denoted by $\bigcap \mathscr{X}$.

2.6. Convention. Instead of $\bigcup\{X_\alpha \mid \alpha \in A\}$ or $\bigcup \mathscr{X}$ we often write $\bigcup\limits_{\alpha \in A} X_\alpha$ or merely $\bigcup\limits_{\alpha} X_\alpha$ (sometimes even $\bigcup X_\alpha$) if it is evident from the context which class of indices is considered. We proceed similarly in more complicated cases. For example, instead of $\bigcup\{X_{\langle\alpha,\beta\rangle} \mid \alpha\varrho\beta\}$, where ϱ is a relation, we write $\bigcup\{X_{\alpha,\beta} \mid \alpha\varrho\beta\}$ or $\bigcup\limits_{\substack{\alpha,\beta\\ \alpha\varrho\beta}} X_{\alpha,\beta}$ or (provided it is clear from the context that only α, β satisfying $\alpha\varrho\beta$ are considered) more briefly $\bigcup\limits_{\alpha,\beta} X_{\alpha,\beta}$ (or even $\bigcup X_{\alpha,\beta}$); instead of $\bigcup\{\bigcup\{X_{\alpha,\beta} \mid \alpha \in A\} \mid \beta \in B\}$ we write $\bigcup\limits_{\beta \in B}\{\bigcup\limits_{\alpha \in A} X_{\alpha,\beta}\}$ or only $\bigcup\limits_{\beta}\bigcup\limits_{\alpha} X_{\alpha,\beta}$ etc. Instead of the union of the indexed class $\{X_\alpha \mid \alpha \in A\}$ we speak of the union of sets X_α where α runs through A. Analogous conventions also hold for intersections.

Finally, we define the union and the intersection of a class of sets.

2.7. Definition. If $\mathscr{S}$ is a class of sets (that is, a class each element of which is a set), then the class of all x such that $x \in X$ for some $X \in \mathscr{S}$, is called the *union of the class* $\mathscr{S}$ and is denoted by $\bigcup \mathscr{S}$. The class of all x such that $x \in X$ for each $X \in \mathscr{S}$ is called the *intersection of the class* $\mathscr{S}$ and is denoted by $\bigcap \mathscr{S}$.

If $\mathscr{S}$ is a class of sets, then, obviously, $\bigcup \mathscr{S} = \bigcup \mathsf{J}_{\mathscr{S}}$, $\bigcap \mathscr{S} = \bigcap \mathsf{J}_{\mathscr{S}}$ (by 1 B.4, example (A), $\mathsf{J}_{\mathscr{S}}$ denotes the indexed class $\{X \mid X \in \mathscr{S}\}$); thus, definition 2.7 can be reduced to definition 2.5. Let us mention that there is no formal collision between the two definitions since (except for $\emptyset$) no indexed class of sets is a class of sets (the elements of the former being pairs, those of the latter sets and hence not pairs).

2.8. Remark. We observe that the method which can be used for obtaining definition 2.7 from 2.5 has a general character. It can be described as follows: if $\boldsymbol{P}$ is, for example, some property defined for indexed classes of sets, then we can also define the same property for classes of sets, namely in this way: a class A is said to possess a property $\boldsymbol{P}$ if and only if the indexed class $\{x \to x \mid x \in A\}$, that is J_A, has the property $\boldsymbol{P}$.

2.9. Definition. An indexed class of sets $\{X_a \mid a \in A\}$ is called *disjoint* if, for any $a \in A$, $b \in A$, $a \neq b$, the intersection $X_a \cap X_b$ is void. A class of sets $\mathscr{X}$ is called *disjoint* if the indexed class $\{X \mid X \in \mathscr{X}\}$ is disjoint, i.e. if $X \cap Y = \emptyset$ whenever $X \in \mathscr{X}$, $Y \in \mathscr{X}$, $X \neq Y$.

Remark. Observe that if an indexed class $\{X_a \mid a \in A\}$ is disjoint, then the class $\mathsf{E}\{X_a \mid a \in A\}$ of all X_a is disjoint; the converse assertion does not hold, of course.

Example. Let ϱ be a relation such that all inverse fibres $\varrho^{-1}[(x)]$ are sets. Then ϱ is single-valued if and only if the indexed class $\{\varrho^{-1}[(x)]\}$ is disjoint.

2.10. Remark. Let A, B be sets. If α, β are two different elements, and $\mathscr{X}$ is the relation which assigns A to α, B to β, then $\mathscr{X}$ is a family of sets, and, clearly, $A \cup B = \bigcup \mathscr{X}$, $A \cap B = \bigcap \mathscr{X}$ (in particular, we may put $\mathscr{X} = \mathrm{J}_{(A,B)}$). Therefore it is not necessary to introduce the union or the intersection of two sets as a special notion; however, the union and the intersection of two classes (as introduced at the beginning of this section) cannot be expressed in this way.

We now give a few examples concerning unions and intersections. – (A) If ϱ is a relation such that every fibre $\varrho[(x)]$ is a set, then $\mathbf{E}\varrho = \bigcup \varrho[(x)]$. – (B) If σ is an order (see 1 C.5) on a non-void set A, then $\bigcap \{]\!] \leftarrow, x [\![\mid x \in A\} = \emptyset$; $\bigcap \{]\!] \leftarrow, x]\!] \mid x \in A\}$ is void if A contains no "least" (relative to σ) element, and is equal to (a) if a is such an element. – (C) The intersection of the void class, i.e. $\bigcap \emptyset$, is equal to the universal class. Indeed, by definition 2.7, $\bigcap \emptyset$ consists of all elements x such that $x \in X$ for each $X \in \emptyset$; evidently, every x possesses the property that $X \in \emptyset \Rightarrow x \in X$, since the left-hand side of the implication never holds.

2.11. Very often we can restrict our consideration to elements or subclasses, etc., of a fixed class E. With such cases in view, it is convenient to introduce some "relative" notions and notation.

Convention. The complement of X relative to E (see 2.1) will sometimes be denoted by X^*, provided the class E is clear from the context. – Note that symbols like X^* will also be used for various other purposes.

Remark. The class $V - X$, where V is the universal class, is sometimes called the "absolute complement" of X.

Definition. If $\mathscr{X} = \{X_\alpha \mid \alpha \in A\}$ is an indexed class of sets such that $X_\alpha \subset E$ for each $\alpha \in A$, the class of all $x \in E$ such that $x \in X_\alpha$ for each $\alpha \in A$ is called the *intersection* of $\mathscr{X}$ *relative to* E and is denoted by $\bigcap_E \mathscr{X}$ or $\bigcap_E \{X_\alpha \mid \alpha \in A\}$ and so on. The relative intersection $\bigcap_E \mathscr{S}$ of a class of sets is defined, of course, as $\bigcap_E \mathrm{J}_{\mathscr{S}}$. If it is clear from the context that only subclasses of a given E are considered, then we simply speak about the intersection (instead of the relative intersection), and write $\bigcap \mathscr{X}$ instead of $\bigcap_E \mathscr{X}$.

Remarks. 1) Clearly, $\bigcap_E \emptyset = E$ whereas $\bigcap \emptyset$ is the universal class. If considerations are limited to subsets of a fixed E, then this is the only difference between $\bigcap$ and $\bigcap_E$. – 2) The reader will easily see that there is no reason to introduce the notion of a relative union.

2.12. Theorem. *Let $A \subset E$, $B \subset E$. Then*

(a) $(A \cup B)^* = A^* \cap B^*$, $(A \cap B)^* = A^* \cup B^*$.

(b) $(A^*)^* = A$.

(c) $A - B = B^* - A^*$.

(d) $A \cap E = A$, $A \cup E = E$, $A \cup A^* = E$, $A \cap A^* = \emptyset$.

The easy proof may be omitted.

Remark. The above equalities (a) are called de Morgan's formulae (in a special form).

We are now going to prove an assertion concerning the associativity and the commutativity — in a very wide sense — of the union and intersection. First, however, we have to prove that the union of a family of sets is a set.

2.13. Theorem. *The union of a family (or of a collection) of sets is a set.*

Proof. Let $\{X_a \mid a \in A\}$ be a family of sets. Put $\langle a, x\rangle \in \varrho$ if and only if $a \in A$, $x \in X_a$. Then $\varrho[(a)] = X_a$ is a set for every $a \in A$. Thus, by 1 E.1(b), $\bigcup_{a\in A} X_a = \varrho[A]$ is a set.

Remarks. 1) The intersection of a non-void class of sets is a set since $\bigcap_{a\in A} X_a \subset X_a$ for each $a \in A$. — 2) The union of a non-comprisable class of sets is non-comprisable, for we have, for any class of sets $\mathscr{X}$, $\mathscr{X} \subset \exp \bigcup\mathscr{X}$; hence if $\mathscr{X}$ is non-comprisable, then so is $\exp \bigcup\mathscr{X}$ and therefore $\bigcup\mathscr{X}$.

2.14. Theorem. *If $\{X_a \mid a \in A\}$, $\{A_b \mid b \in B\}$ are indexed classes of sets, $A = \bigcup A_b$, then $\bigcup_{b\in B} (\bigcup_{a\in A_b} X_a) = \bigcup_{a\in A} X_a$, $\bigcap_{b\in B} (\bigcap_{a\in A_b} X_a) = \bigcap_{a\in A} X_a$. If $\{X_a \mid a \in A\}$ is an indexed class of sets, φ is a permuting relation on A, then $\bigcup_{a\in A} X_{\varphi a} = \bigcup_{a\in A} X_a$, $\bigcap_{a\in A} X_{\varphi a} = \bigcap_{a\in A} X_a$.*

Let us prove only the first assertion for $\bigcup$, the rest being similar or trivial. If $x \in \bigcup_{b\in B} (\bigcup_{a\in A_b} X_a)$, then, for some $b \in B$, $x \in \bigcup_{a\in A_b} X_a$, hence $x \in X_a$ for some $a \in A_b \subset A$, and thus $x \in \bigcup_{a\in A} X_a$. If $x \in \bigcup_{a\in A} X_a$, then $x \in X_a$ for some $a \in A$; since $A = \bigcup_{a\in B} A_b$, we have, for some $b \in B$, $a \in A_b$ and therefore $x \in \bigcup_{a\in A_b} A_a$.

Remarks. 1) The above theorem can also be formulated as follows: If $\{X_a \mid a \in A\}$, $\{A_b \mid b \in B\}$ are indexed classes of sets, $A = \bigcup_{b\in B} A_b$, φ is a permuting relation on A, $Y_a = X_{\varphi a}$, then $\bigcup_{b\in B} (\bigcup_{a\in A_b} Y_a) = \bigcup_{a\in A} X_a$, and similarly for the intersection. -- 2) We defer to Section 5 the discussion of the mutual "distributivity" of the intersection and the union since the proof of a general theorem on this "distributivity" involves the Axiom of Choice (see Section 4).

2.15. *If ϱ is a relation, then $\varrho[\bigcup X_a] = \bigcup \varrho[X_a]$, and $\varrho[\bigcap X_a] \subset \bigcap \varrho[X_a]$.* — This is clear.

2.16. Theorem. *Let E be a set. Let $X_a \subset E$ for each $a \in A$. Then $E - \bigcap_E X_a = \bigcup(E - X_a)$, or, in short, $(\bigcap_E X_a)^* = \bigcup X_a^*$; similarly, $(\bigcup X_a)^* = \bigcap_E X_a^*$.*

These formulae are often called "de Morgan's formulae" (in the general form). We shall prove, for example, the first formula. Clearly, $x \in E - \bigcap_E X_a$, if and only if the following holds: $x \in E$ and it is not true that $x \in X_a$ for every a; but this is equivalent to the assertion that, for some a, $x \in E - X_a$ which, of course, implies and is implied by $x \in \bigcup(E - X_a)$. – Observe that for $\{X_a\} = \emptyset$, the formulae in question reduce to $\emptyset = \emptyset$, $E = E$.

Definition 2.5 (respectively, 2.7) associates with every family or collection of sets their union. If we denote the relation $\{\mathscr{X} \to \bigcup\mathscr{X}\}$ by $\bigcup$, then, for any collection $\mathbf{A}$ of families or collections of sets, we have $\bigcup[\mathbf{A}] = \mathbf{E}\{\bigcup\mathscr{X} \mid \mathscr{X} \in \mathbf{A}\}$ (see 1 B.7); in a somewhat similar way, we may define $A \cup [\mathscr{B}]$, $[\mathscr{A}] \cup [\mathscr{B}]$, etc. We define explicitly, beginning with the case of a set and a class of sets, and of two classes of sets.

2.17. Definition. If A is a set, $\mathscr{B}$ a class of sets, then $A \cup [\mathscr{B}]$ (or $[\mathscr{B}] \cup A$) denotes the class of all $A \cup B$, $B \in \mathscr{B}$, and similarly for the intersection, $A - [\mathscr{B}]$ denotes the class of all $A - B$, $B \in \mathscr{B}$, $[\mathscr{B}] - A$ that of all $B - A$, $B \in \mathscr{B}$. If $\mathscr{A}$, $\mathscr{B}$ are classes of sets, then $[\mathscr{A}] \cup [\mathscr{B}]$ denotes the class of all $A \cup B$, $[\mathscr{A}] \cap [\mathscr{B}]$ that of all $A \cap B$, and $[\mathscr{A}] - [\mathscr{B}]$ that of all $A - B$, where always $A \in \mathscr{A}$, $B \in \mathscr{B}$.

Remarks. 1) Evidently, $A \cup [\mathscr{B}] = [(A)] \cup [\mathscr{B}]$, etc. – 2) We must carefully distinguish between $\mathscr{A} \cup \mathscr{B}$ and $[\mathscr{A}] \cup [\mathscr{B}]$ (and similarly for the intersection and the difference). If $\mathscr{A}$, $\mathscr{B}$ are classes of sets, then $\mathscr{A} \cup \mathscr{B}$ consists of all $X \in \mathscr{A}$ and all $Y \in \mathscr{B}$ whereas $[\mathscr{A}] \cup [\mathscr{B}]$ consists of all $X \cup Y$ where $X \in \mathscr{A}$, $Y \in \mathscr{B}$.

Examples. (A) If $\mathscr{A}$, $\mathscr{B}$ are collections of sets, then $[\mathscr{A}] \cup [\mathscr{B}]$ is also a collection of sets (and similarly in the other cases considered) for, clearly, $\sigma = \{\langle X, Y\rangle \to \to X \cup Y\}$ is single-valued and $[\mathscr{A}] \cup [\mathscr{B}] = \sigma[\mathscr{A} \times \mathscr{B}]$. – (B) If $\mathscr{A}$ consists of all singletons, then $\mathscr{A} \cup \mathscr{A} = \mathscr{A}$ whereas $[\mathscr{A}] \cup [\mathscr{A}]$ consists of all singletons and all two-element sets. – (C) If A and B are disjoint classes, $\mathscr{A} = \exp A$, $\mathscr{B} = \exp B$, then $[\mathscr{A}] \cup [\mathscr{B}]$ is equal to $\exp(A \cup B)$ and, evidently, different from $\mathscr{A} \cup \mathscr{B}$, provided neither $A = \emptyset$ nor $B = \emptyset$. – (D) If A is a set, $\mathscr{W}$ is the class of all sets, then $A \cup [\mathscr{W}]$ is the class of all sets $X \supset A$, and $A \cap [\mathscr{W}]$ is equal to $\exp A$.

2.18. Definition. If $\mathbf{A}$ is a class of families (or of collections) of sets, then $\bigcup[\mathbf{A}]$ denotes the class $\mathbf{E}\{\bigcup\mathscr{X} \mid \mathscr{X} \in \mathbf{A}\}$ and $\bigcap[\mathbf{A}]$ denotes the class $\mathbf{E}\{\bigcap\mathscr{X} \mid \mathscr{X} \in \mathbf{A}\}$.

Example. If $\mathscr{A}$ is a class of sets, B is a set, then $\mathscr{A}^B$ is the class of all families $\{X_b \mid b \in B\}$, $X_b \in \mathscr{A}$, and $\bigcup[\mathscr{A}^B]$ is the class of all sets which may be expressed as the union of a family of sets from $\mathscr{A}$, indexed by elements from B.

Remark. Observe that special care is necessary when using the notation introduced in 2.17 and 2.18. For instance, if $\mathscr{A}$, $\mathscr{B}$ are collections of sets, then $\mathscr{A} \cup \mathscr{B}$ is a collection of sets, $\bigcup[(\mathscr{A}, \mathscr{B})]$, as defined above, is equal to $(\bigcup\mathscr{A}, \bigcup\mathscr{B})$, and distinct from $[\mathscr{A}] \cup [\mathscr{B}]$.

3. INFINITE SETS

Infinite sets are one of the most fundamental objects and tools of modern mathematics. The existence of infinite sets, however, is not yet guaranteed in our axiomatic construction; we have not even defined the notion of an infinite set. Let us remark that the axioms introduced so far do not at all guarantee the existence of non-void classes since they are certainly satisfied if there exist no elements at all and only one class, namely the void class. If we assume the existence of at least one element, then it can already be shown that there exist arbitrarily large finite (in the usual sense) sets, but the existence of infinite sets cannot be proved.

We shall proceed in such a way that, after giving a definition of finite and infinite sets, we shall introduce an axiom (the so-called Axiom of Infinity) requiring the existence of infinite sets. From this axiom there will be deduced (theorem 3 C.2) the existence of a set endowed with a certain relation which has properties well known for natural numbers. This permits the proper introduction of natural numbers (see 3 D.1). We do not, of course, examine their properties in detail; after the introduction of natural numbers, only some fundamental definitions and theorems are given, and the arithmetic of natural integers is assumed to be known in what follows.

We are now going to introduce several notions and propositions which will be needed for the formulation of the Axiom of Infinity; we shall discuss them here in rather general terms, having in view their application in later sections.

A. EQUIPOLLENT CLASSES

3 A.1. Definition. We shall say that a class A is *equipollent* with a class B if there exists a relation f bijective for A and B (i.e. a one-to-one relation f with domain A and range B).

Examples. (A) Any two one-point sets are equipollent; any two two-point sets are equipollent, etc. – (B) The set of all natural numbers is equipollent with the set of all even natural numbers (e.g. the relation $\{n \to 2n\}$ is bijective for these sets). – (C) The universal class is equipollent with the class of all singletons.

3 A.2. *Let A, B, C be any classes. Then*

(a) *A is equipollent with A;*

(b) *if A is equipollent with B, then B is equipollent with A;*

(c) *if A is equipollent with B and B is equipollent with C, then A is equipollent with C.*

Proof. (a) The relation J_A is bijective on A onto A. – (b) If A is equipollent with B, let f be bijective for A and B; clearly, f^{-1} is bijective for B and A. – (c) If A is equipollent with B and B is equipollent with C, then there exists a relation f bijective for A and B and a relation g bijective for B and C; clearly, $g \circ f$ is bijective for A and C.

It follows from the above proposition that we can say that two classes A, B or three classes A, B, C are mutually equipollent. The following theorem is a self-evident consequence of **3** A.2.

3 A.3. *The relation $\{X \to X \mid X, Y$ are sets, X is equipollent with $Y\}$ is an equivalence on the class of all sets.*

3 A.4. *A class equipollent with a set is a set.*

This follows at once from **1** E.4.

3 A.5. *Let A_1, A_2, B_1, B_2 be any classes. Let $A_1 \cap A_2 = \emptyset$, $B_1 \cap B_2 = \emptyset$. Let A_1 be equipollent with B_1 and let A_2 be equipollent with B_2. Then $A_1 \cup A_2$ is equipollent with $B_1 \cup B_2$.*

Proof. Since A_1 and B_1, A_2 and B_2 are equipollent, there exist relations f_1, f_2 such that f_i is bijective for A_i and B_i, $i = 1, 2$. Clearly, $f_1 \cup f_2$ is a relation bijective for $A_1 \cup A_2$ and $B_1 \cup B_2$.

We shall not need immediately the following two important propositions (the second of them follows at once from the first one), but it is suitable to introduce them at this place.

3 A.6. Theorem. *If A is a set and if φ is a single-valued relation for A and* exp A *(this means that $\mathbf{D}\varphi \subset A$, $\mathbf{E}\varphi \subset \exp A$), then $\mathbf{E}\varphi \neq \exp A$.*

Proof. Let B be the class of all $x \in A$ such that $x \notin \varphi x$. Since $B \subset A$ and A is a set, B is a set too, hence $B \in \exp A$. It is sufficient to prove that $B \notin \mathbf{E}\varphi$. Suppose that, on the contrary, $B \in \mathbf{E}\varphi$. Then there exists an element $b \in A$ such that $B = \varphi b$. Now, by definition of B, we have $x \in B \Leftrightarrow x \notin \varphi x$ for any $x \in A$; in particular, $b \in B \Leftrightarrow$ $\Leftrightarrow b \notin \varphi b$, which is a contradiction since $\varphi b = B$.

3 A.7. Corollary. *If X is a set, then X and* exp X *are not equipollent.*

Remarks. An example can be easily given where X is a proper class, φ is a single-valued relation on X, $\mathbf{E}\varphi = \exp X$. It is sufficient e.g. to take for X the universal class and to put $\varphi x = x$ if x is a set, $\varphi x = \emptyset$ if x is an element but not a set. – It is obvious that every class X is equipollent with a subclass of $\exp X$ (consider the relation $\{x \to (x) \mid x \in X\}$).

We now proceed to define the notion of a finite class (it will turn out later that every finite class is a set).

The seemingly most simple procedure would be to define a finite set as a set which has a finite number (let us say n) of elements, i.e. as a set which (for a certain n) is equipollent with the set $(1, 2, \ldots, n)$; but this would require among other things the introduction of natural numbers.

Another possibility, for example, is to define finite sets, in a certain sense, inductively, i.e. starting from one-element sets and adding further elements step by step (one element at each step); the appropriate definition would be worded something like this: Let us say that a class $\mathscr{X}$ of sets has the property $\boldsymbol{F}$ if (1) $\emptyset \in \mathscr{X}$, (2) if $A \in \mathscr{X}$ and a is an element, then $A \cup (a) \in \mathscr{X}$; we shall call a set A "finite" if it belongs to each class which has the property $\boldsymbol{F}$.

This approach, however, will not be used in spite of the fact that it has several considerable advantages. In the axiomatic system presented here it will be more appropriate to use a somewhat less natural method, namely to define first the concept of infinite classes by means of one of its characteristic properties. This property is that no finite class is equipollent with any of its proper subclasses, while on the other hand every infinite class X is equipollent with some subclass $Y \neq X$ (i.e. there exists a bijective relation φ on X onto Y, $Y \subset X$, $Y \neq X$); if X is e.g. the set of natural numbers, then we may put $\varphi = \{x \to x + 1\}$. We shall take this property as a basis for the definition of infinite and finite classes.

3 A.8. Definition. A class X will be called *infinite* if it is equipollent with a proper subclass. A class will be called *finite* if it is equipollent with no proper subclass.

3 A.9. *A subclass of a finite class is finite.*

Proof. Let Y be finite, $X \subset Y$. Suppose that X is infinite and let us derive a contradiction. Since X is infinite, there exists a one-to-one relation φ such that $\mathbf{D}\varphi = X$, $\mathbf{E}\varphi \subset X$, $\mathbf{E}\varphi \neq X$. Then the relation $\psi = \varphi \cup \mathrm{J}_{Y-X}$ is bijective for Y and $\mathbf{E}\varphi \cup \cup (Y - X)$, and this last set is distinct from Y. Hence Y is infinite which is a contradiction.

3 A.10. *A class equipollent with a finite (infinite) class is finite (infinite).*

The simple proof is left to the reader.

The following proposition is obvious.

3 A.11. *The void class is finite.*

3 A.12. *If the universal class V is not void (i.e. if elements do exist), then it is infinite.*

Proof. The relation $f = \{x \to (x)\}$ is obviously one-to-one, $\mathbf{D}f = V$, $\mathbf{E}f$ is the class of all singletons. If there exists an element a, then it is easily seen that $\exp (a) = (\emptyset, (a))$ is a two-element set, hence belongs to V but not to $\mathbf{E}f$.

3 A.13. *If A is a finite class and a is an element, then the class $A \cup (a)$ is finite.*

Proof. If $a \in A$, then the proposition is obvious. Let $a \notin A$ and suppose that $A \cup (a)$ is infinite; it will be proved that then A is also infinite. Because $A \cup (a)$ is infinite, there exists a one-to-one relation f such that $\mathbf{D}f = A \cup (a)$, $\mathbf{E}f \subset A \cup (a)$, $\mathbf{E}f \neq A \cup (a)$. If $a \notin \mathbf{E}f$ or $a = fa$, then the restriction of f maps A onto a proper subclass, hence A is infinite. Therefore let $fa = c$, $a = fb$ where $b \in A$, $c \in A$. Let us define a single-valued relation f' whose domain is A thus: $f'x = fx$ for $x \in A - (b)$, $f'b = c$. Then f' is obviously one-to-one, $\mathbf{E}f' \subset A$. Since $\mathbf{E}f = \mathbf{E}f' \cup (a)$, $\mathbf{E}f \neq A \cup \cup (a)$, we have $\mathbf{E}f' \neq A$. Consequently A is an infinite class.

Remark. From the above propositions it follows that sets "finite" in the sense mentioned in the remarks preceding the definition 3 A.8 are finite in the sense of our definition. We shall return to further propositions concerning finite classes later.

B. MONOTONE CLASSES

It is now possible to state the basic axiom postulating the existence of infinite sets. First, however, we shall prove two propositions which are not yet based on the axiom of infinity but will be needed for deriving the consequences of this axiom. These propositions will prepare the way for the proof of the existence of a set which will "essentially" be the set of natural numbers.

Now the procedure leading to the introduction of natural numbers will be more fully described. First it will be proved (Theorem 3 B.4) that, given any non-void class of sets $\mathscr{A}$ and a single-valued relation φ on $\mathscr{A}$ into $\mathscr{A}$ such that always $\varphi X \subset X$, the class $\mathscr{A}$ contains, as a subclass, a collection (i.e. a set of sets) $\mathscr{B}$ which together with the relation φ (or rather its restriction to $\mathscr{B}$) already satisfies, in a certain sense, the principle of mathematical induction; namely it contains a certain given set A and together with X always contains φX as well, and at the same time it is (for given A) the minimal system with these properties.

Then (theorem 3 B.5) it will be proved that, whenever any system $\mathscr{B}$ of sets together with a single-valued relation φ on $\mathscr{B}$ satisfy (in the above sense) the principle of mathematical induction, then the system $\mathscr{B}$ has even further important properties: in particular, it is "well-ordered" by means of inverse inclusion, and every $X \in \mathscr{B}$ is either "immediately followed" by φX or is "the last element" (i.e. the smallest set) in $\mathscr{B}$. Such a system $\mathscr{B}$ consists, roughly speaking, of sets A, φA, $\varphi(\varphi A)$, ..., and is either finite or infinite according to whether there does or does not exist the "last" element. From an intuitive point of view, $\mathscr{B}$ is, essentially, either the set of all natural numbers or the set $(0, 1, \ldots, n)$ where n is a natural number (note that zero is taken to be a natural number); the set A has the role of zero, φ the role of the relation of succeeding, the inverse inclusion $\supset$ that of the order $\leqq$.

We must, of course, first ascertain that such a collection $\mathscr{B}$ exists, more precisely that there exist $\mathscr{B}$, A and φ with the properties mentioned. An axiom requiring the existence of a non-void class (and hence also the existence of sets) would be sufficient for this purpose.

Even under this assumption, however, the possibility that each collection $\mathscr{B}$ is finite cannot be excluded before the introduction of the axiom of infinity. It is only this axiom which ascertains the existence of such $\mathscr{B}$, A and φ that there is no smallest set in $\mathscr{B}$, and thus makes it possible to prove the existence of a set which is "essentially" the set of all natural numbers. This is carried out in theorem 3 C.2.

It is clear that the set $\mathscr{B}$ from theorem 3 C.2 satisfies the so-called Peano's axioms which are usually taken as a basis for the theory of natural numbers. In theorem 3 C.5 it will then be proved (without using the Axiom of Infinity) that the set $\mathscr{B}$ from 3 C.2 is uniquely determined "up to an isomorphism". Roughly speaking, any such set can therefore be chosen to serve as the set of natural numbers.

The choice of one of such sets will be performed formally by introducing natural numbers axiomatically; this means that "the set of natural numbers" N, "zero" 0 and "the successor relation" s will be undefined objects, satisfying Peano's axioms. Theorem 3 C.2 guarantees that such N, 0 and s indeed exist; in a more sophisticated way it can be said that we have constructed a model for Peano's axioms in the framework of the theory of sets and proved that they are not in contradiction with the axioms of this theory.

The theory of natural numbers is, of course, not built up in this book in the sense that known properties of natural numbers are derived from the axioms for natural numbers; immediately after the introduction of natural numbers and the statement of some basic definitions the elementary arithmetic of natural numbers will be assumed to be known. Let us again remark for completeness that integers and rational numbers will be introduced in Section 8, and real numbers in Section 10.

We shall now proceed to the implementation of the indicated procedure by introducing two important definitions (which in this place have a rather auxiliary character).

3 B.1. Definition. Let $\mathscr{X}$ be a class of sets. It will be said that $\mathscr{X}$ is *monotone* if for any $A \in \mathscr{X}$, $B \in \mathscr{X}$ either $A \supset B$ or $A \subset B$.

At this stage, besides trivial examples such as the system of sets $\emptyset$, (a), (a, b), only illustrative ones (using notions to be introduced later) can be given, for instance the collection of all intervals $[\![0, x]\!]$ of the set of reals.

3 B.2. Definition. Let $\mathscr{X}$ be a class of sets. It will be said that M is the *largest* (*smallest*) *set in* $\mathscr{X}$ if $M \in \mathscr{X}$ and $X \subset M$ (respectively, $X \supset M$) for every $X \in \mathscr{X}$.

Remark. If there exists, in a class of sets, a largest set, then it is obviously uniquely determined, i.e. each class of sets has at most one largest set. Similarly for the smallest set.

Examples. (A) If X is a set, then X is the largest and $\emptyset$ is the smallest set in $\exp X$. – (B) If A is an infinite set, then the system of all finite sets $X \subset A$ contains no largest set. – (C) If ϱ is a comprisable relation, then there exists the smallest set in the class of all (comprisable) transitive relations σ with $\sigma \supset \varrho$.

The following concept will not be used in this section, but it is closely related to, though substantially different from, the notion of the largest (smallest) set.

3 B.3. Definition. Let $\mathscr{X}$ be a class of sets. We shall say that $M \in \mathscr{X}$ is *maximal* (*minimal*) *in* $\mathscr{X}$ if there is no $M_1 \in \mathscr{X}$ with $M_1 \supset M$, $M_1 \neq M$ (respectively, with $M_1 \subset M$, $M_1 \neq M$).

Examples. (A) In a class of singletons, every set is maximal and minimal, and similarly for any disjoint class of non-void sets. – (B) If A is an infinite set, then the system of all finite subsets contains no maximal set.

Remarks. 1) Clearly, if M is the largest (smallest) set in a class $\mathscr{X}$, then M is a maximal (minimal) set. – 2) As we have seen, there may exist many maximal (minimal) sets in a class of sets. The largest set is always the only maximal one; on the other hand, if there exists exactly one maximal set, it may happen that there is no largest set. – 3) In a monotone class, the concepts of the largest (smallest) set and a maximal (minimal) set coincide.

Now we shall proceed to the theorems discussed above.

3 B.4. Theorem. *Let $\mathscr{A}$ be a non-void class of sets. Let $A \in \mathscr{A}$, let φ be a single-valued relation with domain $\mathscr{A}$ ranging in $\mathscr{A}$, and let $\varphi X \subset X$ for each $X \in \mathscr{A}$. Then there exists exactly one class $\mathscr{B}$ of sets such that*

(a$_0$) $A \in \mathscr{B}$, (a$_1$) $X \in \mathscr{B} \Rightarrow \varphi X \in \mathscr{B}$;

(b) *If $\mathscr{C} \subset \mathscr{B}$, $A \in \mathscr{C}$ and $\varphi X \in \mathscr{C}$ whenever $X \in \mathscr{C}$, then $\mathscr{C} = \mathscr{B}$.*

This class $\mathscr{B}$ is a collection and A is the largest set in $\mathscr{B}$.

Proof. If will be said that a class $\mathscr{X} \subset \mathscr{A}$ has property $\boldsymbol{S}$ if (1) $A \in \mathscr{X}$, (2) $X \in \in \mathscr{X} \Rightarrow \varphi X \in \mathscr{X}$. Let $\mathscr{B}$ be the class of all $X \in \mathscr{A}$ which belong to each class that has property $\boldsymbol{S}$. It is obvious that $\mathscr{B}$ has property $\boldsymbol{S}$ and therefore properties (a$_0$), (a$_1$). If $\mathscr{C} \subset \mathscr{A}$ contains A and satisfies the condition $X \in \mathscr{C} \Rightarrow \varphi X \in \mathscr{C}$, then $\mathscr{C}$ has property $\boldsymbol{S}$ and therefore necessarily $\mathscr{C} \supset \mathscr{B}$. If, moreover, even $\mathscr{C} \subset \mathscr{B}$, then $\mathscr{C} = \mathscr{B}$. Consequently $\mathscr{B}$ also has property (b).

It will now be shown that there exists exactly one such class $\mathscr{B}$. If $\mathscr{B}'$ is another class satisfying conditions (a$_0$), (a$_1$), (b), then obviously the class $\mathscr{B} \cap \mathscr{B}'$ also has properties (a$_0$), (a$_1$); of course, $\mathscr{B} \cap \mathscr{B}' \subset \mathscr{B}$, $\mathscr{B} \cap \mathscr{B}' \subset \mathscr{B}'$, so that according to (b) it follows that on the one hand $\mathscr{B} \cap \mathscr{B}' = \mathscr{B}$, on the other hand $\mathscr{B} \cap \mathscr{B}' = \mathscr{B}'$, so that $\mathscr{B} = \mathscr{B}'$.

Now it only remains to show that $\mathscr{B}$ is a set, and therefore a collection of sets, and that A is the largest set in $\mathscr{B}$. If $\mathscr{C}$ denotes the class of all sets $X \in \mathscr{B}$ such that $X \subset A$, then it follows from property (b) that $\mathscr{C} = \mathscr{B}$. Therefore $X \subset A$ for each $X \in \mathscr{B}$, so that $\mathscr{B} \subset \exp A$. As A is a set, $\mathscr{B}$ is also a set.

3 B.5. Theorem. *Let $\mathscr{B}$ be a collection of sets, let A be a set and let φ be a single-valued relation with domain $\mathscr{B}$ such that $X \in \mathscr{B} \Rightarrow \varphi X \subset X$. Let conditions* (a$_0$), (a$_1$), (b) *from theorem* 3 B.4 *be satisfied. Then*

(m) *If $X \in \mathscr{B}$, $Y \in \mathscr{B}$, then either $X = Y$ or $X \subset \varphi Y$* or *$Y \subset \varphi X$ (in particular, the collection $\mathscr{B}$ is monotone);*

(w) *If $\mathscr{C} \subset \mathscr{B}$, $\mathscr{C} \neq \emptyset$, then $\mathscr{C}$ contains the largest set;*

(o) *the relation of the inverse inclusion on* $\mathscr{B}$, *i.e. the relation* $\{X \to Y \mid X \in \mathscr{B}, Y \in \mathscr{B}, X \supset Y\}$ *is the smallest transitive reflexive relation on* $\mathscr{B}$ *containing (as a subset) the relation* φ;

(s) *if* $X \in \mathscr{B}$, *then* $X \supset \varphi X$, $X \neq \varphi X$ *except if* X *is the smallest set in* $\mathscr{B}$; *if* $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \supset Y$, $X \neq Y$, *then* $\varphi X \supset \varphi Y$, $\varphi X \neq \varphi Y$ *except in the case when* $\varphi X = Y$ *and* Y *is the smallest set in* $\mathscr{B}$.

Proof. I. First we shall prove assertion (m). Let us introduce this notation: if $X \in \mathscr{B}$, $Y \in \mathscr{B}$, then $X \varrho Y$ means that either $X \subset Y$ or $\varphi X \supset Y$. It will be proved that

(*) if $X \varrho Y$, $Y \varrho X$, then $X \varrho (\varphi Y)$.

Indeed, let $X \varrho Y$, $Y \varrho X$. If $\varphi X \supset \varphi Y$, then obviously $X \varrho(\varphi Y)$. If not, then neither $X = Y$, nor (since $\varphi Y \subset Y$) $\varphi X \supset Y$, so that in view of $X \varrho Y$ we have $X \subset Y$, $X \neq Y$, hence $Y \subset X$ does not hold. This, together with $Y \varrho X$, implies $\varphi Y \supset X$, hence $X \varrho(\varphi Y)$.

Now let $\mathscr{B}'$ be the class of all $Y \in \mathscr{B}$ such that $X \varrho Y$ for any $X \in \mathscr{B}$. Moreover let $\mathscr{B}''$ be the class of all $Z \in \mathscr{B}$ such that $Y \varrho Z$ for any $Y \in \mathscr{B}'$. It will be shown that $\mathscr{B}' = \mathscr{B}'' = \mathscr{B}$. By 3B.4, $X \subset A$ for any $X \in \mathscr{B}$, therefore also $X \varrho A$ for any $X \in \mathscr{B}$. It follows that $A \in \mathscr{B}'$, $A \in \mathscr{B}''$. If $Y \in \mathscr{B}'$, $Z \in \mathscr{B}''$, then by the definition of $\mathscr{B}'$ we have $Z \varrho Y$ and by the definition of $\mathscr{B}''$ we have $Y \varrho Z$; thus according to (*), $Y \varrho(\varphi Z)$. It follows that $Z \in \mathscr{B}'' \Rightarrow \varphi Z \in \mathscr{B}''$. In view of condition (b) (where $\mathscr{B}''$ is taken for $\mathscr{C}$) we have therefore $\mathscr{B}'' = \mathscr{B}$. Now, for arbitrary $Z \in \mathscr{B}$, $Y \in \mathscr{B}'$ we have (since $Z \in \mathscr{B} = \mathscr{B}''$) $Z \varrho Y$ and $Y \varrho Z$ from which, by assertion (*), it follows that $Z \varrho(\varphi Y)$. This means that $Y \in \mathscr{B}' \Rightarrow \varphi Y \in \mathscr{B}'$. In view of (b) we obtain $\mathscr{B}' = \mathscr{B}$. Therefore, for any $X \in \mathscr{B}$, $Y \in \mathscr{B}$ we have $X \varrho Y$, i.e. either $X \subset Y$ or $\varphi X \supset Y$. (This already implies that $\mathscr{B}$ is monotone, because $X \supset Y$ whenever $\varphi X \supset Y$.)

To conclude the proof of the assertion (m), suppose that $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \neq Y$ and $Y \subset \varphi X$ does not hold. Then necessarily $X \subset Y$, but not $Y \subset X$. Since we have just proved (with X, Y interchanged) that either $Y \subset X$ or $\varphi Y \supset X$, we obtain $X \subset \varphi Y$.

II. Next we shall prove that (w) holds. Let $\mathscr{M}$ be the class of all $Y \in \mathscr{B}$ that have this property: if $\mathscr{X} \subset \mathscr{B}$ and there exists $X \in \mathscr{X}$ such that $X \supset Y$, then $\mathscr{X}$ contains the largest set. We shall show that $\mathscr{M} = \mathscr{B}$. As $A \supset Y$ for each $Y \in \mathscr{B}$ it is evident that $A \in \mathscr{M}$. Let $Y \in \mathscr{M}$; we are going to prove that $\varphi Y \in \mathscr{M}$. Let $\mathscr{X} \subset \mathscr{B}$ and suppose that there exist $X \in \mathscr{X}$ such that $X \supset \varphi Y$. If there exists even a set $Z \in \mathscr{X}$ such that $Z \supset Y$, then $\mathscr{X}$ surely has the largest set because $Y \in \mathscr{M}$. On the other hand, if $Y \subset Z$ for no $Z \subset \mathscr{X}$, then according to part I of the proof we must have $\varphi Y \supset Z$ for each $Z \in \mathscr{X}$, thus, in particular, $X \subset \varphi Y$, hence $X = \varphi Y$. It follows that $X = \varphi Y$ is the largest set in $\mathscr{X}$. We have proved that $Y \in \mathscr{M} \Rightarrow \varphi Y \in \mathscr{M}$. Hence indeed $\mathscr{M} = \mathscr{B}$.

Let now $\mathscr{C} \subset \mathscr{B}$, $\mathscr{C} \neq \emptyset$. Then there exists a $C \in \mathscr{C}$. Of course we have $C \in \mathscr{M} = \mathscr{B}$. As $\mathscr{C} \subset \mathscr{B}$ and there is a set (namely C itself) in $\mathscr{C}$ containing C, there exists the largest set in $\mathscr{C}$.

III. In order to prove (o) we have to show that if ϱ is a transitive reflexive relation with $\mathbf{D}\varrho = \mathscr{B}$ (hence $\mathbf{E}\varrho = \mathscr{B}$) and $\langle X, \varphi X\rangle \in \varrho$ for each $X \in \mathscr{B}$, then $X \supset Y \Rightarrow \langle X, Y\rangle \in \varrho$ for each $X \in \mathscr{B}$, $Y \in \mathscr{B}$. Let $\mathscr{Y}$ be the system of all $Y \in \mathscr{B}$ such that $X \supset Y \Rightarrow \langle X, Y\rangle \in \varrho$ for each $X \in \mathscr{B}$; we have to prove that $\mathscr{Y} = \mathscr{B}$. As $X \subset A$ for each $X \in \mathscr{B}$ we have $X \supset A \Rightarrow X = A$; as ϱ is reflexive we have $\langle A, A\rangle \in \varrho$. Hence $A \in \mathscr{Y}$. We shall show that $Y \in \mathscr{Y} \Rightarrow \varphi Y \in \mathscr{Y}$. Let $Y \in \mathscr{Y}$ and let $X \supset \varphi Y$. If $X = \varphi Y$, then of course $\langle X, \varphi Y\rangle = \langle X, X\rangle \in \varrho$. Therefore let $X \supset \varphi Y$, $X \neq \varphi Y$; then it cannot happen that $X \subset \varphi Y$, so that by (m) either $X = Y$ or $Y \subset \varphi X$ and hence in any case $X \supset Y$. As $Y \in \mathscr{Y}$ we have $\langle X, Y\rangle \in \varrho$; further we have $\langle Y, \varphi Y\rangle \in \varrho$ and because ϱ is transitive we again have $\langle X, \varphi Y\rangle \in \varrho$. Consequently, in any case $X \supset \varphi Y \Rightarrow \langle X, \varphi Y\rangle \in \varrho$, so that $\varphi Y \in \mathscr{Y}$. Indeed, by property (b) $\mathscr{Y} = \mathscr{B}$.

IV. Suppose that $X_0 \in \mathscr{B}$, $\varphi X_0 = X_0$. Put $\mathscr{C} = \mathbf{E}\{X \mid X \in \mathscr{B},\ X \supset X_0\}$. Clearly, $A \in \mathscr{C}$. If $X \in \mathscr{C}$, then $X \supset X_0$; by (m) either $X = X_0$ or $X \subset \varphi X_0 = X_0$ or $X_0 \subset \varphi X$. In the first two cases, we get $X = X_0$, hence $\varphi X = \varphi X_0 = X_0$ and therefore $\varphi X \supset X_0$, in the third case, $\varphi X \supset X_0$ also. Thus $X \in \mathscr{C} \Rightarrow \varphi X \in \mathscr{C}$ from which $\mathscr{C} = \mathscr{B}$ follows. Hence $X \supset X_0$ for every $X \in \mathscr{B}$. – The proof of the second assertion in (s) follows directly from (m) using the first one, and may be left to the reader.

Remark. Let us notice that, as we have shown in part III of the foregoing proof, for any $X \in \mathscr{B}$, $Y \in \mathscr{B}$ we have $X \supset \varphi Y$, $X \neq \varphi Y \Rightarrow X \supset Y$. This of course means that, for arbitrary $Y \in \mathscr{B}$, there exists no $Z \in \mathscr{B}$ with $Y \neq Z \neq \varphi Y$, $Y \supset Z \supset \varphi Y$. If $X \in \mathscr{B}$, $X \neq \varphi X$ (i.e. if X is not the smallest set in $\mathscr{B}$), then φX is the largest set in the collection of $Z \in \mathscr{B}$ such that $Z \subset X$, $Z \neq X$; hence φX "follows X immediately" in the ordering of the set $\mathscr{B}$ by inverse inclusion.

C. AXIOM OF INFINITY

We are now going to introduce the Axiom of Infinity; its meaning has already been explained in this section.

3 C.1. Axiom of Infinity. *There exists an infinite set.*

3 C.2. Theorem. *There exists a collection of sets $\mathscr{B}$, a set A, and a single-valued relation σ with domain $\mathscr{B}$ such that*

(a_0) $A \in \mathscr{B}$; (a_1) $X \in \mathscr{B} \Rightarrow \sigma X \in \mathscr{B}$;
(b) *If* $\mathscr{C} \subset \mathscr{B}$, $A \in \mathscr{C}$ *and* $X \in \mathscr{C} \Rightarrow \sigma X \in \mathscr{C}$, *then* $\mathscr{C} = \mathscr{B}$;
(c_0) *if* $X \in \mathscr{B}$, *then* $\sigma X \neq A$; (c_1) *if* $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \neq Y$, *then* $\sigma X \neq \sigma Y$;
(d) *if* $X \in \mathscr{B}$, *then* $\sigma X \subset X$, $\sigma X \neq X$.

Such a collection $\mathscr{B}$ is infinite.

Proof. Let A be an infinite set. According to Definition 3 A.8, there exists a one-to-one relation ψ with the domain A such that $\mathbf{E}\psi \subset A$, $\mathbf{E}\psi = \psi[A] \neq A$. Let us put $\mathscr{A} = \exp A$; for $X \in \mathscr{A}$ (i.e. for $X \subset A$) let us put $\varphi X = \psi[X]$ if $\psi[X] \subset$

$\subset X$ and $\varphi X = \emptyset$ in the remaining cases. Then the assumptions of theorem 3 B.4 are satisfied, so that there exists a collection $\mathscr{B} \subset \mathscr{A}$ with properties (a_0), (a_1), (b) indicated in 3 B.4. Now, let $\mathscr{C}$ be the collection of all those $X \in \mathscr{B}$ for which $\psi[X] \subset X$. Obviously $A \in \mathscr{C}$. If $Y \in \mathscr{C}$, then $\varphi Y = \psi[Y] \subset Y$, hence $\psi[\varphi Y] \subset \psi[Y] = \varphi Y$; therefore we have $Y \in \mathscr{C} \Rightarrow \varphi Y \in \mathscr{C}$. By property (b) from 3 B.4 it follows that $\mathscr{C} = \mathscr{B}$ and hence $\psi[X] \subset X$ for each $X \in \mathscr{B}$.

Now, let σ denote the restriction of the relation φ to $\mathscr{B}$; it is clear that $\mathscr{B}$ and σ satisfy the assumption from 3 B.5. If we prove that there is no smallest set in $\mathscr{B}$, then assertions $(a_0) - (d)$ of the present theorem will follow from 3 B.5.

Let $\mathscr{C}$ be the set of those $X \in \mathscr{B}$ for which $\sigma X \neq X$; we have to prove that $\mathscr{C} = \mathscr{B}$. Because $\psi[A] \neq A$ we have $A \in \mathscr{C}$. Further, $\{X \to \psi[X]\}$ is a one-to-one relation. If we had $\psi[\psi[X]] = \psi[X]$, then we would also have $\psi[X] = X$, from which it would follow that $\varphi X \neq X \Rightarrow \varphi(\varphi X) \neq \varphi X$, hence that $X \in \mathscr{C} \Rightarrow \varphi X \in \mathscr{C}$. By property (b) in Theorem 3 B.4 indeed $\mathscr{C} = \mathscr{B}$.

Finally, the collection $\mathscr{B}$ is an infinite set because σ is a one-to-one relation and $\mathbf{D}\sigma = \mathscr{B}$ but $\mathbf{E}\sigma \neq \mathscr{B}$ because $A \notin \mathbf{E}\sigma$.

3 C.3. Remark. As noted previously in this section, the system $\mathscr{B}$ in theorem 3 C.2, from an intuitive point of view, is "essentially" the set of natural numbers.

The natural numbers are frequently characterized by means of Peano's axioms; the basic notions are: a set N ("the set of natural numbers"), an element 0 ("number zero"), and a single-valued relation s, the so-called "relation of following" or the "successor relation" (sx is "the number following x"); the axioms are usually given in various equivalent formulations closely resembling conditions (a), (b), (c), (d) from 3 C.2. We shall choose one of these formulations and introduce, for brevity, the following

3 C.4. Convention. If N is a set, 0 is an element, s is a single-valued relation with domain N and

(a_0) $0 \in N$; (a_1) $x \in N \Rightarrow sx \in N$;

(b) if for any set $M \subset N$ we have $0 \in M$ and $x \in M \Rightarrow sx \in M$, then $M = N$;

(c_0) $sx = 0$ for no $x \in N$; (c_1) for $x \in N$, $y \in N$, $x \neq y$ we have $sx \neq sy$,

then we shall say that Peano's axioms are satisfied by N, 0, s (more precisely, by the triple $\langle N, 0, s\rangle$).

The main meaning of theorem 3 C.2 lies in asserting that there exist $\mathscr{B}$, A, σ satisfying Peano's axioms; moreover, $\mathscr{B}$ is a collection of sets, A is a set, and the relation σ has some special features, namely $\sigma X \subset X$ holds for every $X \in \mathscr{B}$ (this is not implied by Peano's axioms), and the smallest order containing σ coincides with the relation of inverse inclusion. This situation offers certain advantages, for simultaneously with the construction of a set which is essentially the set of natural numbers, its "natural" order is obtained.

Let us remark in addition that it would not, of course, be sufficient to replace, in Peano's axioms, axiom (c_1) by the axiom $x \in N \Rightarrow sx \neq x$ which would correspond to the property $\sigma X \neq X$ from 3 C.2; such a weakened system of axioms is satisfied e.g. by putting $N = (0, 1, 2)$, $s0 = 1$, $s1 = 2$, $s2 = 1$. On the other hand, it is, of course, possible to derive the property $x \in N \Rightarrow sx \neq x$ from Peano's axioms. The proof may be left to the reader.

We shall now show that any two triples $\langle N_1, 0_1, s_1 \rangle$, $\langle N_2, 0_2, s_2 \rangle$ satisfying Peano's axioms are "isomorphic" in a sense described below.

We are emphasizing that although the axiom of infinity is necessary to the proof of the existence of a set satisfying Peano's axioms, in the proof of theorem 3 C.5 below, i.e. of the assertion that any two such sets (if they exist) are "isomorphic", no use is made of the axiom of infinity.

3 C.5. Theorem. *Let Peano's axioms be satisfied by N_1, 0_1, s_1 as well as by N_2, 0_2, s_2. Then there exists exactly one bijective relation f for N_1 and N_2 such that $f0_1 = 0_2$ and $f \circ s_1 = s_2 \circ f$ (i.e. $f(s_1 x) = s_2(fx)$ for every $x \in N_1$).*

Proof. Let us say that a set $F \subset N_1 \times N_2$ has property $\mathbf{S}$ if $\langle 0_1, 0_2 \rangle \in F$ and $\langle x, y \rangle \in F \Rightarrow \langle s_1 x, s_2 y \rangle \in F$. Let f be the class of all those $\langle x, y \rangle \in N_1 \times N_2$ which belong to each set $F \subset N_1 \times N_2$ which has property $\mathbf{S}$ (i.e. f is the intersection of the collection of all sets F which have property $\mathbf{S}$). We shall prove that the relation f has the needed properties. It is obvious that f has property $\mathbf{S}$, i.e. $\langle 0_1, 0_2 \rangle \in f$ and $\langle x, y \rangle \in f \Rightarrow \langle s_1 x, s_2 y \rangle \in f$. Hence $0_1 \in \mathbf{D}f$; if $x \in \mathbf{D}f$, then for a suitable y, $\langle x, y \rangle \in f$, therefore $\langle s_1 x, s_2 y \rangle \in f$ so that $s_1 x \in \mathbf{D}f$; it follows by property (b) in 3 C.4 that $\mathbf{D}f = N_1$. Similarly it may be shown that $\mathbf{E}f = N_2$.

Now we shall prove that f is a single-valued relation. Let us denote by M the set of all $x \in N_1$ such that $\langle x, y \rangle \in f$, $\langle x, y' \rangle \in f \Rightarrow y = y'$; we shall show that $M = N_1$. Suppose that $0_1 \notin M$ (and let us derive a contradiction); then there exists $z \in N_2$, $z \neq 0_2$ such that $\langle 0_1, z \rangle \in f$. Let us consider the set $f' = f - (\langle 0_1, z \rangle)$. This set has property $\mathbf{S}$. Indeed, it is obvious that $\langle 0_1, 0_2 \rangle \in f'$; furthermore, if $\langle x, y \rangle \in f'$, then $\langle s_1 x, s_2 y \rangle \in f$ and because, by property (c_0), we have $s_1 x \neq 0_1$, we also have $\langle s_1 x, s_2 y \rangle \in$ $\in f'$. This is a contradiction because $\langle 0_1, z \rangle \in f$ and therefore $\langle 0_1, z \rangle$ must belong to every set with property $\mathbf{S}$. Hence we have ascertained that $0_1 \in M$. Now, let $x \in M$; we want to prove that $s_1 x \in M$. Suppose that, on the contrary, $s_1 x \notin M$. Let $y \in N_2$ be such that $\langle x, y \rangle \in f$; let $z \in N_2$, $\langle s_1 x, z \rangle \in f$, $z \neq s_2 y$ (the existence of such a z follows from the assumption $s_1 x \notin M$). Let us put $f'' = f - (\langle s_1 x, z \rangle)$; we assert that f'' has property $\mathbf{S}$. According to (c_0), $s_1 x \notin M$, we have $s_1 x \neq 0_1$ and therefore $\langle 0_1, 0_2 \rangle \in f''$. Further $\langle u, v \rangle \in f'' \Rightarrow \langle s_1 u, s_2 v \rangle \in f''$; indeed, should it happen that $\langle u, v \rangle \in f''$, $\langle s_1 u, s_2 v \rangle \notin f''$, then in view of $\langle s_1 u, s_2 v \rangle \in f$ we should have $\langle s_1 u, s_2 v \rangle =$ $= \langle s_1 x, z \rangle$, hence $s_1 u = s_1 x$, $s_2 v = z$. According to the property (c_1) it would follow that $u = x$ and therefore (since $x \in M$) also $v = y$; hence $z = s_2 y$ which constitutes a contradiction. We obtain $M = N_1$, so that f is single-valued. The reader will easily see that quite similarly $f^{-1} = \bigcap F^{-1}$ is also single-valued.

Now, let $x \in N_1$. Because $\mathbf{D}f = N_1$, there exists a y such that $\langle x, y \rangle \in f$. Since f

has property **S**, we have $\langle s_1x, s_2y\rangle \in f$. Since f is one-to-one, we can write $y = fx$, $s_2y = f(s_1x)$; hence $s_2(fx) = f(s_1x)$.

Now suppose that g is bijective for N_1 and N_2, $g0_1 = 0_2$ and $g \circ s_1 = s_2 \circ g$. Then $\langle 0_1, 0_2\rangle \in g$; if $y = gx$, then, as a consequence of $g(s_1x) = s_2(gx)$, $\langle s_1x, s_2y\rangle = = \langle s_1x, g(s_1x)\rangle \in g$. Thus g has property **S**, hence $f \subset g$ which implies, g being single-valued, $f = g$.

3 C.6. *If N_1, 0_1, s_1 satisfy Peano's axioms, then the smallest reflexive transitive relation containing s_1 (cf.* 1 C.4) *is an order (see* 1 C.5). *Moreover, if N_2, 0_2, s_2 satisfy Peano's axioms and f is a bijective relation for N_1 and N_2 satisfying the conditions from* 3 C.5, *then, for any $x \in N_1$, $y \in N_1$, $\langle x, y\rangle \in t_1$ if and only if $\langle fx, fy\rangle \in t_2$ where t_i is the smallest reflexive transitive relation containing s_i.*

Proof. It is convenient to prove the second assertion first. Denote by t_1' the set of all $\langle x, y\rangle \in N_1 \times N_1$ such that $\langle fx, fy\rangle \in t_2$. It is easy to see that t_1' is a reflexive transitive relation, $t_1' \supset s_1$ (since $\langle fx, f(s_1x)\rangle = \langle fx, s_2(fx)\rangle$). From this it follows that $t_1' \supset t_1$; thus $\langle x, y\rangle \in t_1 \Rightarrow \langle fx, fy\rangle \in t_2$. The proof that $\langle fx, fy\rangle \in t_2 \Rightarrow \langle x, y\rangle \in \in t_1$ is quite analogous. – To prove that t_1 is an order, consider $\mathscr{B}$, A and σ indicated in 3 C.2. Then, using assertion (o) from 3 B.5 and the assertion just proved, and denoting by f the "isomorphism" relation for N_1 and $\mathscr{B}$, we get that, for any $x \in N_1$, $y \in N_1$, $\langle x, y\rangle \in t_1$ if and only if $fx \supset fy$. This proves that t_1 is an order (for if $\langle x, y\rangle \in t_1$, $\langle y, x\rangle \in t_1$, then $fx \supset fy$, $fy \subset fx$, hence $fx = fy$, $x = y$).

Remark. Before passing to natural numbers we point out that if N, 0, s satisfy Peano's axioms, then every $x \in N$, except 0, is a successor; or, more precisely, $s[N] = = N - (0)$. This follows at once from property (b) if we take as M the set consisting of 0 and all sx.

D. NATURAL NUMBERS

We have proved in the preceding theorems that there exists an "essentially unique" triple $\langle N, 0, s\rangle$ satisfying Peano's axioms. Natural numbers are essentially determined by these conditions and the only question is to fix in some way one such triple. It would be possible to exhibit it effectively, in a sense. It is, for example, possible to state precisely and to prove a proposition which – roughly speaking – asserts that the set N of all elements $\emptyset$, $(\emptyset)$, $((\emptyset))$, ..., the element $\emptyset$ and relation $\{x \to (x) \mid \mid x \in N\}$ satisfy Peano's axioms; we would, therefore, declare as natural numbers the elements $\emptyset, (\emptyset), ((\emptyset)), \ldots$, which would be, of course, denoted $0, 1, 2, \ldots$. We shall use another procedure which in a certain sense better expresses the fact that for our purpose the only things that matter are the mutual relations between natural numbers and not the nature of these numbers as individual elements; for we shall choose the triple $\langle N, 0, s\rangle$ "fixed but arbitrary". This can be performed conveniently in such a way that we shall state properties known from Peano's axioms as axioms for certain three fixed elements denoted N, 0, s.

3 D.1. Defining axioms for natural numbers.

(0) N *is a set*, 0 *is an element*, s *is a single-valued relation with domain* N;
(a_0) $0 \in N$; (a_1) *if* $x \in N$, *then* $sx \in N$;
(b) *if* $M \subset N$, $0 \in M$, *and* $x \in M \Rightarrow sx \in M$, *then* $M = N$;
(c_0) $x \in N \Rightarrow sx \neq 0$; ($c_1$) *if* $x \in N$, $y \in N$, $x \neq y$, *then* $sx \neq sy$.

Every $x \in N$ will be called a *natural number* and N will be called the *set of natural numbers*, s will be called the *relation of following* or the *successor relation* for natural numbers; if $x \in N$, then sx will be called the *successor* of the number x.

Convention. We shall denote sn also by $n + 1$ (see also 3 E.1, remark).

Remark. We observe once more that, by the procedure we have chosen, N (i.e. the set of natural numbers) is uniquely determined; it is a certain fixed set about whose properties we are, however, unable to say more then what follows from the axioms.

We shall now define the ordering for natural numbers using proposition 3 C.6 which means, essentially, that the ordering from the set "model" $\mathscr{B}$, constructed in theorem 3 C.2, is "transferred" to natural numbers.

3 D.2. Definition. The smallest reflexive transitive relation in N containing s (which is an order by 3 C.6) will be called the *natural order* on N. If $\langle x, y\rangle$ belongs to this natural order, then we shall say that x is not greater than y (in symbols, $x \leqq y$). Instead of $x \leqq y$, $x \neq y$, we write $x < y$. Other well known symbols and expressions such as $x \geqq y$, $x > y$, x is greater than y, etc., are defined in the familiar way.

Remark. The following statements follow at once from theorems contained in the preceding subsection: if $x \in N$, $y \in N$, then either $x < y$ or $x = y$ or $x > y$; $0 \leqq x$ for every $x \in N$; if $M \subset N$, $M \neq \emptyset$, then there exists a (unique) $a \in M$ such that $a \leqq x$ for every $x \in M$; if $x < y$, then $sx < sy$.

3 D.3. Definition. If $p \in N$, then the set of all natural numbers less then p will be denoted by N_p. Each family whose domain is N will be called a *sequence* (infinite), and each family whose domain (the set of indices) is, for a suitable natural p, equal to N_p, will be called a *finite sequence* (of *length* p).

An infinite sequence $\{a_k \mid k \in N\}$ is often written as $\{a_k\}_{k=0}^{\infty}$. For a finite sequence $\{a_k \mid k \in N_{p+1}\}$ the symbol $\{a_k\}_{k=0}^{p}$ is frequently used and, sometimes, if $a_0, \ldots, a_p$ are actually given, also the symbol $\{a_0, \ldots, a_p\}$. Thus, e.g. $\{a\} = (\langle 0, a\rangle)$, $\{a, b\} = (\langle 0, a\rangle, \langle s0, b\rangle)$.

3 D.4. If A is class, p a natural number, then the class of all finite sequences of length p, whose members belong to A, will be frequently denoted by A^p (instead of the symbol A^{N_p} as introduced in 1 E.8).

We have, of course, $A^0 = (\emptyset)$. The set of all (infinite) sequences with values in A is denoted, of course, by A^N according to 1 E.8.

Remark. We shall occasionally use the symbol A^p, where A is a class, $p \in N$, to denote a class A^P where P is a set equipollent with N_p.

3 D.5. As is well-known, in the whole of mathematics a very important part is played by the so-called principle of mathematical induction which is usually formulated something like this:

($*$) Let $\boldsymbol{P}$ be a property. Suppose that 0 has property $\boldsymbol{P}$, and that for each $n \in \mathsf{N}$ the following holds: if n has property $\boldsymbol{P}$, then the successor $\mathrm{s}n$ of the number n also has property $\boldsymbol{P}$. Then every $n \in \mathsf{N}$ has property $\boldsymbol{P}$.

(If, as usual, we write $\boldsymbol{P}(n)$ to express that n has property $\boldsymbol{P}$, then this assertion may be written briefly as follows: if $\boldsymbol{P}(0)$ and, for any $n \in \mathsf{N}$, $\boldsymbol{P}(n)$ implies $\boldsymbol{P}(\mathrm{s}n)$, then $\boldsymbol{P}(n)$ for every $n \in \mathsf{N}$.)

The formulation ($*$) belongs rather to logic since, as it stands, it bears on a variable (indefinite) property. A mathematical expression of the principle of mathematical induction can be stated thus:

($**$) Let $M \subset \mathsf{N}$ be a set, let $0 \in M$, and let $n \in M \Rightarrow \mathrm{s}n \in M$ for any $n \in \mathsf{N}$. Then $M = \mathsf{N}$.

This is, however, our axiom (b) for natural numbers; it is also evident that the formulations ($*$), ($**$) are equivalent. Therefore, we shall also use the principle of mathematical induction in the form ($*$).

Remark. We can, of course, state the principle of mathematical induction for the sets N_p, for instance thus: Let 0 have the property $\boldsymbol{P}$ and for $n \in \mathsf{N}_p$ let the following hold: if n has the property $\boldsymbol{P}$, then $\mathrm{s}n$ has the property $\boldsymbol{P}$; then each $n \in \mathsf{N}_{p+1}$ has the property $\boldsymbol{P}$. – We omit the proof which can be performed e.g. by means of property $\boldsymbol{P}'$ defined as follows: $n \in \mathsf{N}$ has property $\boldsymbol{P}'$ if and only if either $n \in \mathsf{N}_{p+1}$ and n has property $\boldsymbol{P}$, or $n \notin \mathsf{N}_{p+1}$.

Let us now proceed to the so-called principle of construction by mathematical induction or, more precisely, to the principle of recursive construction. We shall state (and prove) here only one theorem (and its corollary) connected with this principle. For a further theorem based on the Axiom of Choice see Section 4.

Intuitively, the question is this: there are given an element a_0 and a rule by means of which a_{n+1} is uniquely determined whenever $a_0, a_1, \ldots, a_n$ are known; it is asserted that this so-called recursive rule uniquely determines an infinite sequence $a_0, a_1, a_2, \ldots$.

3 D.6. Theorem. *Let A be a class. Let φ be a single-valued relation with $\mathbf{E}\varphi \subset A$, and for each $n \in \mathsf{N}$ let $A^n \subset \mathbf{D}\varphi$. Then there exists exactly one sequence $\{a_n \mid n \in \mathsf{N}\}$ such that, for every natural n, $a_n = \varphi\{a_k \mid k \in \mathsf{N}_n\}$.*

Proof. Let us denote by $\mathscr{S}$ the class of all finite sequences $\{a_k \mid k \in \mathsf{N}_p\}$ with values in A and such that $a_k = \varphi\{a_i \mid i \in \mathsf{N}_k\}$ for $k \in \mathsf{N}_p$. In the first place we shall prove that $T_p = \mathscr{S} \cap A^p$ is a singleton for each $p \in \mathsf{N}$. Suppose that, on the contrary, the set of those $p \in \mathsf{N}$ which do not possess this property, is non-void; let h be its least element. Surely $h = 0$ is false because evidently $\mathscr{S} \cap A^0 = (\emptyset)$. Then there exists (see 3 C.6, remark), an element $j \in \mathsf{N}$ such that $h = \mathrm{s}j$. Then T_j is a singleton. Let $\{b_i \mid i \in \mathsf{N}_j\} \in T_j$. Let us put $b_j = \varphi\{b_i \mid i \in \mathsf{N}_j\}$; then obviously $\{b_i \mid i \in \mathsf{N}_h\} \in \mathscr{S}$, hence $\mathscr{S} \cap$

$\cap A^h \neq \emptyset$. Let also $\{c_i \mid i \in \mathsf{N}_h\} \in \mathscr{S}$. Then obviously $\beta = \{b_i \mid i \in \mathsf{N}_j\} \in \mathscr{S}$, $\gamma = \{c_i \mid i \in \mathsf{N}_j\} \in \mathscr{S}$ and $b_j = \varphi\beta$, $c_j = \varphi\gamma$. Since T_j is, by our assumption, a singleton, we have $\beta = \gamma$, hence $b_j = c_j$. This is a contradiction.

We have proved that, for each $p \in \mathsf{N}$, T_p is a singleton. Put $T_p = (\alpha_p)$, $\alpha_p = \{a_{p,k} \mid k \in \mathsf{N}_p\}$. Now, let $p \in \mathsf{N}$, $q \in \mathsf{N}$, $p \leqq q$. We have $T_q = (\{a_{q,k} \mid k \in \mathsf{N}_q\})$. By the definition of $\mathscr{S}$ we have, obviously, $\{a_{q,k} \mid k \in \mathsf{N}_p\} \in T_p$; because T_p is a one-element set, we have, therefore, $\{a_{q,k} \mid k \in \mathsf{N}_p\} = \{a_{p,k} \mid k \in \mathsf{N}_p\}$, so that $\alpha_p \subset \alpha_q$. Hence for any $k < p$, $k < q$, $a_{p,k} = a_{q,k}$ holds.

For any $n \in \mathsf{N}$ let us now put $a_n = a_{n+1,n}$. In view of what we have just proved we also have $a_n = a_{q,n}$ for any $q > n$. For each $n \in \mathsf{N}$ we have, therefore, $\{a_k \mid k \in \mathsf{N}_n\} \in T_n$, since $a_k = a_{k+1,k} = a_{n,k}$ for each $k \in \mathsf{N}_n$. By the definition of $\mathscr{S}$ we have, therefore, $a_n = \varphi\{a_k \mid k \in \mathsf{N}_n\}$ for each $n \in \mathsf{N}$.

It remains to show that if $\{b_n \mid n \in \mathsf{N}\}$ is any sequence such that $b_n = \varphi\{b_k \mid k \in \mathsf{N}_n\}$ for each n, then $a_n = b_n$ for all $n \in \mathsf{N}$. Let such a sequence be given. Then $\{b_k \mid k \in \mathsf{N}_n\} \in T_n$ for each $n \in N$. Since $\{a_k \mid k \in \mathsf{N}_n\} \in T_n$ and T_n is a singleton, we have $a_k = b_k$ for each $k \in \mathsf{N}_n$; because this holds for each $n \in \mathsf{N}$, the proof is finished.

3 D.7. Corollary. *Let A be a class. Let $a \in A$. Let φ be a single-valued relation on A into A. Then there exists exactly one sequence $\{a_k \mid k \in \mathsf{N}\}$ such that $a_0 = a$ and $a_{n+1} = \varphi a_n$ for $n \in \mathsf{N}$.*

For the proof it is sufficient to put $\varphi^*\{a_k \mid k \in \mathsf{N}_{n+1}\} = \varphi a_n$ for $\{a_k \mid k \in \mathsf{N}_{n+1}\} \in A^{n+1}$, $\varphi^*\emptyset = a$ and to use the preceding theorem on the relation φ^*.

Example. Let $k \neq 0$ be a given natural number. Let the set A consist of all pairs of the form $\langle n, p\rangle$ where $n \in \mathsf{N}$, $p \in (0, 1)$. Let $\psi 1 = 0$, $\psi 0 = 1$. Let us define on A a single-valued relation φ thus: $\varphi\langle n, p\rangle = \langle sn, p\rangle$ in the case $sn \neq k$, $\varphi\langle n, p\rangle = \langle sn, \psi p\rangle$ in the case $sn = k$. Then according to the above corollary there exists a sequence $\{a_i \mid i \in \mathsf{N}\}$ such that $a_0 = \langle 0, 1\rangle$, $a_{n+1} = \varphi a_n$. If we put $n\varrho k$ if and only if a_n has 1 as its second member, then $\varrho = \{n \to k \mid n < k\}$.

E. OPERATIONS ON NATURAL NUMBERS

We have introduced the natural numbers in an exact way but we have not yet defined the basic operations with these numbers – addition, multiplication, etc. A systematic development of the theory of natural numbers would not suit the character of this book and would considerably lengthen our exposition. Therefore we shall only indicate how addition, multiplication and powers can be introduced; the proofs of the theorems on which this introduction is based will not be carried out, and we shall indicate the proof in the case of addition only as an example. All current concepts and theorems of the arithmetic of the natural numbers will be assumed to be known.

3 E.1. Theorem and definition. *There exists exactly one single-valued relation on* $\mathsf{N} \times \mathsf{N}$ *with range in* N *such that if we denote by* $x + y$ *the element assigned by this relation to* $\langle x, y \rangle$, *then we have, for any* $m, n \in \mathsf{N}$,

(1) $m + 0 = m$;

(2) $m + \mathrm{s}n = \mathrm{s}(m + n)$.

This relation will be called addition *(for natural numbers) and it will be said that* $x + y$ *is the* sum *of the numbers* x *and* y.

We shall briefly indicate the proof of this theorem. In the first place let us choose a fixed $m \in \mathsf{N}$. According to 3 D.7 there exists exactly one sequence $A_m = \{a_{m,k} \mid k \in \mathsf{N}\}$ such that $a_{m,0} = m$, $a_{m,\mathrm{s}n} = \mathrm{s}a_{m,n}$. Let us put $m + n = a_{m,n}$. If, furthermore, $m \in \mathsf{N}$, $n \in \mathsf{N}$ are arbitrary, then we put again $m + n = a_{m,n}$ where $a_{m,n}$ belongs to a uniquely determined A_m. Thus we have defined a relation (let us denote it by $+$) with the required properties. Finally let f be a single-valued relation on N with similar properties, i.e. $f\langle m, 0\rangle = m$, $f\langle m, \mathrm{s}n\rangle = \mathrm{s}f\langle m, n\rangle$; we shall show that $f = +$. Suppose that, on the contrary, there are $m \in \mathsf{N}$, $n \in \mathsf{N}$ so that $f\langle m, n\rangle \neq m + n$. Then there exists the smallest n in N which has this property. Obviously $n \neq 0$, so that there exists a $p \in \mathsf{N}$ such that $n = \mathrm{s}p$. But $f\langle m, p\rangle = m + p$, so that $f\langle m, n\rangle = f\langle m, \mathrm{s}p\rangle = \mathrm{s}f\langle m, p\rangle = \mathrm{s}(m + p) = m + \mathrm{s}p = m + n$, which is a contradiction.

Remark. If, as usual, we put $1 = \mathrm{s}0$, then for each $m \in N$ we have $\mathrm{s}m = m + 1$, in accordance with the notation introduced in 3 D.1, convention. The symbol s will be used only occasionally in the following and we shall usually write $m + 1$ instead of $\mathrm{s}m$.

3 E.2. Theorem and definition. *There exists exactly one single-valued relation on* $\mathsf{N} \times \mathsf{N}$ *with values in* N *such that, denoting by* $x \,.\, y$ *(or more concisely* xy*) the element assigned to* $\langle x, y\rangle$ *by this relation, we have, for any* $m \in \mathsf{N}$, $n \in \mathsf{N}$,

(1) $m \,.\, 0 = 0$;

(2) $m \,.\, (n + 1) = m \,.\, n + m$.

This relation will be called multiplication *(for natural numbers) and it will be said that* xy *is the* product *of the numbers* x *and* y.

3 E.3. Theorem and definition. *There exists exactly one single-valued relation on* $\mathsf{N} \times \mathsf{N}$ *with values in* N *such that, denoting by* x^y *the element assigned to* $\langle x, y\rangle$ *by this relation, we have, for any* $m \in \mathsf{N}$, $n \in \mathsf{N}$,

(1) $m^0 = 1$;

(2) $m^{n+1} = m^n \,.\, m$.

It will be said that x^y *is the* power *with the* base x *and* exponent y.

We shall now show that there is a very close relationship between the operations (addition, multiplication, exponentiation) on natural numbers described above and some of the operations in set theory.

First, we give some lemmas.

3 E.4. *Every* N_p, $p \in \mathsf{N}$, *is a finite set.*

This follows at once, by induction, from 3 A.13.

Remark. We have no means as yet to prove that every finite class is equipollent with some N_p.

3 E.5. *If* $p \in \mathsf{N}$, $q \in \mathsf{N}$, $p \neq q$, *then* N_p *and* N_q *are not equipollent.*

Proof. We have either $p < q$ or $q < p$. Consider the first case. Then clearly N_p is a proper subclass of N_q; hence by 3 E.4, N_q and N_p are not equipollent (otherwise N_q would be infinite).

3 E.6. Definition. If A is a class, we shall say that A *has p elements* if A is equipollent with N_p.

Remark. By 3 E.5, for any class A there is at most one p such that A has p elements.

3 E.7. Definition. If a class A has n elements, $n \in \mathsf{N}$, we shall call n the *cardinality* or *power* or *number of elements* of the class A, and denote it by card A. – If A is a class and there is an $n \in \mathsf{N}$ such that A has n elements, we shall say that A has a *finite number of elements.*

Remark. Every set which has a finite number of elements is finite (by 3 E.4). However, we are not able, at this stage, to prove the converse (see 4 C.9).

3 E.8. Theorem. *Let* A, B *be classes. Suppose that* A *has* a *elements,* B *has* b *elements. Then* $A \cup B$ *has* $a + b$ *elements provided* A *and* B *are disjoint;* $A \times B$ *has* ab *elements,* A^B *has* a^b *elements and* $\exp A$ *has* 2^a *elements.*

Proof. I. Clearly, $\varphi = \{x \to x + b \mid x \in \mathsf{N}_a\}$ maps N_a onto $\mathsf{N}_{a+b} - \mathsf{N}_b$. Let f be bijective on A onto N_a, g be bijective on B onto N_b. Then $(\varphi \circ f) \cup g$ is bijective on $A \cup B$ onto N_{a+b}. – II. To prove the second assertion it is sufficient to show that $(*)$ $\mathsf{N}_a \times \mathsf{N}_b$ is equipollent with N_{ab}. Choose arbitrarily $a \in \mathsf{N}$; the assertion $(*)$ is evident for $b = 0$. Suppose that it holds for a certain b. Evidently, $\mathsf{N}_a \times \mathsf{N}_{b+1} = (\mathsf{N}_a \times \mathsf{N}_b) \cup (\mathsf{N}_a \times (b))$, $\mathsf{N}_{a(b+1)} = \mathsf{N}_{ab} \cup (\mathsf{N}_{a(b+1)} - \mathsf{N}_{ab})$. Now, $\mathsf{N}_{a(b+1)} - \mathsf{N}_{ab}$ is clearly equipollent with N_a. This, together with the assumption on b shows that $(*)$ holds for $b + 1$. This proves the theorem. – III. Let us write, in this proof, $\mathsf{N}(x)$ instead of N_x. We are going to prove that $(**)$ $(\mathsf{N}(a))^{\mathsf{N}(b)}$ is equipollent with $\mathsf{N}(a^b)$. Let a be fixed. Clearly, $(**)$ is true if $b = 0$. Suppose that it holds for a certain b. Then clearly $(\mathsf{N}(a))^{\mathsf{N}(b+1)}$ is equipollent with $\mathsf{N}(a)^{\mathsf{N}(b)} \times \mathsf{N}(a)$; on the other hand, $\mathsf{N}(a^{b+1}) = \mathsf{N}(a^b \,.\, a)$ is, as already shown, equipollent with $\mathsf{N}(a^b) \times \mathsf{N}(a)$. From this it follows that $(**)$ holds for $b + 1$. This proves that $(**)$ holds for all natural numbers a, b. Now, let f be a one-to-one relation on A onto $\mathsf{N}(a)$, g a one-to-one relation on B onto $\mathsf{N}(b)$. Then the relation $\{\varphi \to f \circ \varphi \circ g^{-1} \mid \varphi \in A^B\}$ is clearly one-to-one, with domain A^B and range $\mathsf{N}(a)^{\mathsf{N}(b)}$. The rest of the proof is left to the reader.

F. MULTIPLETS

We shall now examine some additional questions concerning, in particular, "iterations" of relations, as well as sequences and multiplets.

First, let us introduce one auxiliary definition.

3 F.1. Definition. Let $\alpha = \{a_k \mid k \in \mathsf{N}_p\}$, $\beta = \{b_l \mid l \in \mathsf{N}_q\}$ be finite sequences. Then the family $\alpha \cup \{b_{l-p} \mid l \in \mathsf{N}_{p+q} - \mathsf{N}_p\}$, which is clearly a sequence equal to $\{c_h \mid h \in \mathsf{N}_{p+q}\}$, where $c_h = a_h$ for $h \in \mathsf{N}_p$, $c_h = b_{h-p}$ for $h \in \mathsf{N}_{p+q}$, $h \notin \mathsf{N}_p$, is said to be obtained from α by the *apposition* of β (*from the right*) or from β by the *apposition* of α (*from the left*) and it will be denoted, if no misunderstanding is likely to arise, by $\alpha \,.\, \beta$ or $\alpha\beta$.

For instance, $\{a, b, c\} \{d, e\} = \{a, b, c, d, e\}$ (cf. 3 D.3).

3 F.2. Definition. If ϱ is a relation, $k \in \mathsf{N}$, $k \geqq 1$, then the relation consisting of all $\langle x, y \rangle$ for which there exists a sequence $\{z_i \mid i \in \mathsf{N}_{k+1}\}$ such that $z_0 = x$, $z_k = y$ and $i \in \mathsf{N}_k \Rightarrow \langle z_i, z_{i+1} \rangle \in \varrho$, will be called the k-th *power* of the relation ϱ and will be denoted by the symbol ϱ^k when there is no danger of ambiguity.

Remark. The notation introduced above is inconsistent with 3 D.4, according to which ϱ^k is the class of all sequences $\{\xi_i \mid i \in \mathsf{N}_k\}$ such that $\xi_i \in \varrho$. Nevertheless, it will always be clear from the context in what sense ϱ^k is being used.

3 F.3. *If ϱ is a relation, $h \in \mathsf{N}$, $k \in \mathsf{N}$, $h \geqq 1$, $k \geqq 1$, then $\varrho^{h+k} = \varrho^k \circ \varrho^h$.*

Proof. If $\langle a, b \rangle \in \varrho^h$, $\langle b, c \rangle \in \varrho^k$, then there exist finite sequences $\xi = \{x_i \mid i \in \mathsf{N}_{h+1}\}$, $\eta = \{y_j \mid j \in \mathsf{N}_{k+1}\}$ such that $x_0 = a$, $x_h = b$, $y_0 = b$, $y_k = c$, and $i \in \mathsf{N}_h \Rightarrow \langle x_i, x_{i+1} \rangle \in \varrho$, $j \in \mathsf{N}_k \Rightarrow \langle y_j, y_{j+1} \rangle \in \varrho$. Put $\xi' = \{x_i \mid i \in \mathsf{N}_h\}$, $\zeta = \xi' \,.\, \eta$ (see 3 F.1), and $\zeta = \{z_i \mid i \in \mathsf{N}_{h+k+1}\}$. Then $z_0 = a$, $z_{h+k} = c$, and $i \in \mathsf{N}_{h+k+1} \Rightarrow \langle z_i, z_{i+1} \rangle \in \varrho$. The rest of the proof is left to the reader.

3 F.4. Theorem. *Let ϱ be a relation. Let σ consist of all $\langle x, y \rangle$ for which there exists a finite sequence $\{z_i \mid i \in \mathsf{N}_{p+1}\}$ such that $p \geqq 1$, $z_0 = x$, $z_p = y$ and $i \in \mathsf{N}_p \Rightarrow \langle z_i, z_{i+1} \rangle \in \varrho$. Then σ is the smallest transitive relation containing ϱ (see 1 C.4); in other words, σ is transitive, $\sigma \supset \varrho$, and if σ' is transitive, $\sigma' \supset \varrho$, then $\sigma' \supset \sigma$. If $\tau = \sigma \cup \mathsf{J}_{\mathbf{D}\varrho \cup \mathbf{E}\varrho}$, then τ is the smallest reflexive transitive relation containing ϱ.*

Proof. Clearly, $\sigma \supset \varrho^k$ for $k = 1, 2, 3, \ldots$. If $\langle a, b \rangle \in \sigma$, $\langle b, c \rangle \in \sigma$, then $\langle a, b \rangle \in \varrho^h$, $\langle b, c \rangle \in \varrho^k$ for some natural $h \geqq 1$, $k \geqq 1$, and therefore, by 3 F.3, $\langle a, c \rangle \in \varrho^{h+k} \subset \sigma$. Thus σ is transitive.

If $\sigma' \supset \varrho$ is transitive, let $\langle a, b \rangle \in \sigma$. Then there exists a finite sequence $\{x_i \mid i \in \mathsf{N}_{m+1}\}$ with $x_0 = a$, $x_m = b$ and $i \in \mathsf{N}_m \Rightarrow \langle x_i, x_{i+1} \rangle \in \varrho$. Since $\sigma' \supset \varrho$, we have $\langle x_i, x_{i+1} \rangle \in \sigma'$ for each $i \in \mathsf{N}_m$. Since σ' is transitive, this implies, by an easy inductive reasoning, that $\langle a, b \rangle \in \sigma'$. The assertion concerning τ is an immediate consequence.

Remark. The relation σ consists, evidently, of all $\langle x, y\rangle$ such that $\langle x, y\rangle \in \varrho^k$ for some natural $k \geqq 1$. Of course, it cannot be said that "$\sigma = \bigcup\{\varrho^k \mid k \in \mathsf{N}, k \geqq 1\}$" because the relations ϱ^k need not be sets.

3 F.5. Definition. If ϱ is a relation, A is a class, then A is said to be *saturated* (or *closed*) *under* ϱ if $\varrho[A] \subset A$.

Examples. (A) A set $M \subset \mathsf{N}$ is saturated under s if and only if either $M = \emptyset$ or $M = \mathsf{N} - \mathsf{N}_p$ for some $p \in \mathsf{N}$. — (B) A class $\mathscr{A}$ of sets is saturated under the relation $\supset$ if and only if it can be expressed in the form $\mathscr{A} = \bigcup\{\exp X \mid X \in \mathscr{B}\}$.

3 F.6. *If A is a class and ϱ is a relation, then there exists exactly one class B containing A, saturated under ϱ and such that if $B' \supset A$ is saturated under ϱ, then $B' \supset B$.*

Proof. Let σ be the smallest reflexive transitive relation containing the relation $\varrho \cup \mathsf{J}_A$ (see 1 C.4). Put $B = \sigma[A]$. Then clearly $A \subset B$, $\varrho[B] \subset \sigma[B] = (\sigma \circ \sigma)[A] = \sigma[A] = B$. Now let $A \subset B'$, $\varrho[B'] \subset B'$. If $x \in B$, then by 3 F.4 there exists a finite sequence $\{z_k\}_{k=0}^p$ such that $z_0 \in A$, $z_p = x$ and $\langle z_k, z_{k+1}\rangle \in \varrho \cup \mathsf{J}_A$ for $k \in \mathsf{N}_p$. Since $z_0 \in B'$, $\varrho[B'] \subset B'$, it is easy to prove, by induction, that $x \in B'$. — The proof of the uniqueness is left to the reader.

3 F.7. *Let ϱ, μ be relations and let $\mathbf{D}\mu$, $\mathbf{E}\mu$ consist of pairs. Then there exists exactly one relation σ such that (1) $\sigma \supset \varrho$, (2) $\mu[\sigma] \subset \sigma$, (3) if σ' is a relation, $\sigma' \supset \varrho$, $\mu[\sigma'] \subset \sigma'$, then $\sigma' \supset \sigma$.*

This follows easily from 3 F.6.

The meaning of this proposition may be indicated as follows: there is given a relation ϱ and a certain procedure for "transforming" pairs into pairs; it is asserted that there exists a (unique) smallest relation $\sigma \supset \varrho$ such that if $x\sigma y$ and $\langle x', y'\rangle$ is obtained by the prescribed procedure from $\langle x, y\rangle$, then $x'\sigma y'$.

We shall now use the above results for the proper introduction of the concept of an n-tuple of elements and other related notions.

3 F.8. Definition. Let $\mathscr{A}$ be a class. Denote by μ the relation consisting of all elements of the form $\langle x, \langle y, x\rangle\rangle$, where $x \in \mathscr{A}$, $y \in \mathscr{A}$. Let $n = k + 1$, $k \in \mathsf{N}$, $k \geqq 1$. Then every $z \in \mathbf{E}\mu^k$ will be called an *n-tuple of elements of $\mathscr{A}$*; also, for convenience, every $x \in \mathscr{A}$ will be called a *1-tuple of elements of $\mathscr{A}$*. An n-tuple of elements of $\mathscr{A}$ will also be called a *regular multiplet of elements of $\mathscr{A}$*.

If $\mathscr{A}$ is the universal class, we shall speak simply of an *n-tuple of elements* (or of a *regular multiplet of elements*). Instead of an n-tuple of elements from the class of all elements satisfying a prescribed condition $\mathbf{C}$ (respectively, possessing a certain property $\mathbf{P}$), we shall speak of an *n-tuple of elements satisfying* $\mathbf{C}$ (respectively, *possessing property* $\mathbf{P}$).

Intuitively, by means of the relation μ we pass (of course, by no means in a unique manner) from an element x to an element $\langle y, x\rangle$, from it to an element $\langle v, y, x\rangle =$

$= \langle v, \langle y, x \rangle \rangle$, then to $\langle t, v, y, x \rangle = \langle t, \langle v, y, x \rangle \rangle = \langle t, \langle v, \langle y, x \rangle \rangle \rangle$, etc.; this procedure is exactly expressed in our definition.

Remarks. 1) Observe that e.g. a 4-tuple is at the same time a triple and a pair. It is not even apparent that the existence of an element which would be an n-tuple for each $n \in \mathsf{N}$, $n \geqq 1$, would contradict our axioms. On the other hand, for any given $n \in \mathsf{N}$, an element can be expressed in at most one manner as an n-tuple of elements. — 2) An element of the form $\langle \langle a, b \rangle, (c) \rangle$ is, of course, a pair of elements, but not a triple since (c) is not a pair; on the other hand, $\langle a, \langle b, (c) \rangle \rangle$ is a triple. — 3) We have considered the successive forming of pairs from elements only. This has been necessary because otherwise the procedure indicated in 3 F.2 could not be used.

3 F.9. Definition. Let A be a class. If an element x belongs to every class X such that $X \supset A$, $X \times X \subset X$, then we shall say that x is a *multiplet of elements of* A.

Remarks. 1) Clearly, if x, y are multiplets of elements of A, then $\langle x, y \rangle$ also is a multiplet of elements of A. This fact also gives a motivation of the above definition, whose meaning is simply that multiplets of elements of A are precisely those elements which can be obtained from those belonging to A by forming of pairs, in other words precisely those elements which belong either to A or to $A \times A$ (i.e. are of the form $\langle x, y \rangle$, $x \in A$, $y \in A$) or to $A \times (A \times A)$ (i.e. are of the form $\langle x, \langle y, z \rangle \rangle$, $x \in A$, $y \in A$, $z \in A$) or to $(A \times A) \times A$ and so on. — 2) A regular multiplet of elements of A is a multiplet of elements of A. The proof is left to the reader. — 3) The definition of an n-multiplet of elements will be given in exercises, since this notion is not essential for further developments. — 4) For reasons indicated below (3 F.13) we defer to Notes (at the end of the book) the introduction of the general concept of a multiplet and an n-tuple, e.g. of non-comprisable classes. The term "multiplet", unspecified, will meanwhile be used informally to indicate objects of the form e.g. $\langle a, \langle b, c \rangle, d \rangle$ where a, b, c, d are quite arbitrary.

We shall now discuss the relationship between finite sequences and n-tuples of elements.

3 F.10. Theorem. *There exists exactly one single-valued relation σ, whose domain is the class of all non-void finite sequences and which has the following properties:*

(1) *for each element x the relation σ assigns to the one-element sequence $\{x\}$ the element x;*

(2) *if $\sigma\{a_{k+1}\}_{k=0}^{n} = y$, and a_0 is an element, then $\sigma\{a_k\}_{k=0}^{n+1} = \langle a_0, y \rangle$; in other words, using the symbol introduced in* 3 F.1, *$\sigma(\{a_0\} \,.\, \alpha) = \langle a_0, \sigma\alpha \rangle$ for any finite sequence α.*

If $n \geqq 1$ is a natural number, then the restriction of σ to the class of all sequences of length n is a relation bijective for this class and that of all n-tuples of elements.

Convention. The relation σ will be called the *canonical relation for the class of finite sequences and that of all regular multiplets of elements.*

Proof. The assertion is intuitively self-evident. To every one-element sequence $\{x\}$ the element x is assigned; to the sequence $\{x\} . \{y\}$ (see 3 F.1), we assign $\langle x, y\rangle$ and, in general, if a regular multiplet ξ is already assigned to a sequence x, then we assign, for any element y, the regular multiplet $\langle y, \xi\rangle$ to $\{y\} . \xi$. Now this procedure of simultaneous transition is covered by the pattern described formally in 3 F.7. – We proceed to the formal proof. Let ϱ denote the relation $\{\{x\} \to x\}$; let μ denote the relation consisting of pairs with the left member $\langle a, \alpha\rangle$ where a is a finite sequence, α a multiplet, and the right member $\langle\{x\}. a, \langle x, \alpha\rangle\rangle$, x being an arbitrary element. Let σ be the relation described in 3 F.7. It is easy to prove that σ possesses the properties required. We omit the rest of the proof.

3 F.11. We shall now discuss one more relation for elements whose discussion in this place has rather the character of an exercise. It will, however, play an important role in Section 7.

It is intuitively clear what we mean when we say that the element $\langle a, b\rangle$ is obtained from $\langle a, \langle b, c\rangle\rangle$ or from $\langle\langle a, c\rangle, b\rangle$ or from $\langle\langle a, b\rangle, c\rangle$, etc., by deleting the element c; let us now try, however, to give a precise general formulation of this relation.

It is clear that if an element α is obtained from an element β by deleting a in the sense indicated above, then the same holds, with an arbitrary element x, for $\langle\alpha, x\rangle$ and $\langle\beta, x\rangle$ as well as for $\langle x, \alpha\rangle$ and $\langle x, \beta\rangle$; on the other hand, it is intuitively clear that this procedure, i.e. transition from α and β to $\langle\alpha, x\rangle$ and $\langle\beta, x\rangle$ or to $\langle x, \alpha\rangle$ and $\langle x, \beta\rangle$, carried out step by step starting from pairs such as x and $\langle x, a\rangle$ or x and $\langle a, x\rangle$, gives all cases where some ξ is obtained from some η by deleting a. We are now ready to state the following assertion.

3 F.12. *Let a be an element. Then there exists exactly one relation ϑ_a such that* (1) *for any element x we have $x\vartheta_a\langle a, x\rangle$, $x\vartheta_a\langle x, a\rangle$*; (2) *if $\xi\vartheta_a\eta$, z is an element, then $\langle z, \xi\rangle\, \vartheta_a\langle z, \eta\rangle$ and $\langle\xi, z\rangle\, \vartheta_a\langle\eta, z\rangle$*; (3) *if, for a relation δ,* (1) *and* (2) *hold, with ϑ_a replaced by δ, then $\delta \supset \vartheta_a$.*

This assertion is obtained as a corollary to 3 F.7, if we put $\varrho = \{x \to \langle a, x\rangle\} \cup \cup \{x \to \langle x, a\rangle\}$, $\mu = \{\langle z, \xi\rangle \to \langle z, \eta\rangle\} \cup \{\langle\xi, z\rangle \to \langle\eta, z\rangle\}$.

Definition. If ξ and η are arbitrary elements and $\langle\xi, \eta\rangle \in \vartheta_a$, we shall say that ξ is *obtained from η by deleting a* and that η is *obtained by enriching ξ with a*.

3 F.13. We shall return to these questions later in Section 7 and in the Notes at the end of the book. One remark, however, is in order here. We have defined regular multiplets (n-tuples) and, as a more general notion, multiplets of elements. However, if we try to define, within the framework of the theory of classes and sets presented here, the concept, say, of a "regular multiplet of classes", i.e. of an n-tuple of classes, $n \in \mathrm{N}$ arbitrary, we encounter serious difficulties. Namely, for any actually given n we can define an n-tuple (e.g. of classes), but we cannot give a definition valid for all natural numbers n. Indeed, it is not possible to proceed indefinitely from a to $\langle b, a\rangle$, then to $\langle c, b, a\rangle$, etc., because such a procedure would involve essentially the use of "sequences of non-comprisable objects" which do not exist. A more profound discussion of these topics would lead to considerations belonging to mathematical logic. For our purposes it would be quite sufficient to choose once and for all a fixed natural number $k \geqq 1$ and to define, e.g., a regular multiplet of classes in an obvious, cumbersome but logically correct way (for instance, for $k = 3$: "A is regular multiplet of classes, if either A is a class, or $A = \langle X, Y \rangle$, X, Y being classes, or $A = \langle X, Y, Z \rangle$ where X, Y, Z are classes").

An analogous reasoning applies, of course, to concepts such as "X is obtained from Y by deleting A", etc.

G. COUNTABLE SETS

We shall now consider countable sets (i.e. countably infinite sets and sets with a finite number of elements). At this stage we cannot prove that every finite class has a finite number of elements, i.e. is equipollent with some N_p (the converse holds, of course, see 3 E.7); this will be shown in Section 4. Fortunately, these two concepts (finite classes and sets with a finite number of elements) are easily seen to coincide, without the Axiom of Choice, for the case when countable sets are considered. Therefore, we formulate propositions below for sets with a finite number of elements inviting the reader to keep in mind that, as a matter of fact, they hold (if the Axiom of Choice is assumed) for finite sets.

However, some important propositions on countable sets cannot be proved without the Axiom of Choice. They will be proved in Section 4 but stated here for the sake of completeness.

3 G.1. Definition. A class A is called *countable* if there exists a single-valued relation φ such that $A = \varphi[\mathsf{N}]$. A class A is called *countably infinite* if it is equipollent with N.

Remarks. 1) Clearly, every countably infinite class, as well as every class with a finite number of elements (see 3 E.7) is countable. — 2) By 1 E.4, every countable class is a set (since N is a set).

3 G.2. We recall that a class X is said to have p elements, p a natural number (see 3 E.6, 3 E.7), if it is equipollent with N_p, and that every set with a finite number of elements is finite (see 3 E.4).

We now give some lemmas.

3 G.3. *Let $M \subset \mathsf{N}$. Suppose that (1) $0 \in M$, (2) if $n \in M$, then either $n + 1 \in M$ or n is the greatest element in M (i.e. $n \in M$, and $m \leqq n$ whenever $m \in M$). Then either $M = \mathsf{N}$ or $M = \mathsf{N}_p$ for some $p \in \mathsf{N}$.*

Proof. If $n + 1 \in M$ whenever $n \in M$, then by the principle of mathematical induction, $M = \mathsf{N}$. If not, then some $n \in M$ is the greatest element in M. By 3 D.5 (remark on induction for the sets N_p), every $m \in \mathsf{N}_{n+1}$ belongs to M and thus $M = \mathsf{N}_{n+1}$.

3 G.4. *Let $M \subset \mathsf{N}$. Let φ be the single-valued relation which assigns to every $x \in M$ the cardinality* (see 3 E.7) *of the set $\mathbf{E}\{z \mid z \in M, z < x\}$. Then φ is one-to-one and $\varphi[M]$ is equal either to N or to some N_p. Moreover, for any $x \in M$, $y \in M$ we have $x < y$ if and only if $\varphi x < \varphi y$.*

Proof. For brevity, put $M_x = \mathbf{E}\{z \mid z \in M, z < x\}$. Clearly, $0 \in \varphi[M]$ since M_x is void if x is the smallest number in M. If $n \in \varphi[M]$, then for some $x \in M$, $n = \varphi x$, N_n is equipollent with M_x. If x is the greatest number in M, then clearly n is the greatest one in $\varphi[M]$. If not, choose the least $y \in M$ with $x < y$. Then $M_y = M_x \cup \cup (x)$, $\varphi y = n + 1, n + 1 \in \varphi[M]$. Now, by the preceding lemma (3 G.3), $\varphi[M] = \mathsf{N}$ or $\varphi[M] = \mathsf{N}_p$ for some $p \in \mathsf{N}$. Finally, if $x \in M$, $y \in M$, $x < y$, then clearly $M_x \subset M_y$, $x \in M_y$, $x \notin M_x$ which clearly implies, since M_x is equipollent with $\mathsf{N}_{\varphi x}$, M_y is equipollent with $\mathsf{N}_{\varphi y}$, that $\varphi x < \varphi y$.

3 G.5. *If $M \subset \mathsf{N}$ is finite non-void, then there exists a number which is the greatest one in M.*

Proof. Let φ be the relation considered in 3 G.4. Clearly $\varphi[M]$ is finite, hence equal to some N_{q+1}. It is easy to see that $\varphi^{-1}q$ is the greatest number in M.

3 G.6. Theorem. *If A is a countable set, then either A is countably infinite or has a finite number of elements.*

Proof. Since A is countable, there exists a single-valued relation f such that $A = f[\mathsf{N}]$. Let g be the single-valued relation assigning to every $x \in A$ the least number in the inverse fibre $f^{-1}[x]$. Then g is one-to-one on A into N. By 3 G.4, $g[A]$ is equipollent either with N or with some N_p; in the first case, A is countably infinite; in the second case it has p elements.

3 G.7. Theorem. *Let A be a class, $B \subset A$. If A is countable, then B is countable. If A has a finite number m of elements, then B has a finite number n of elements, $n \leqq m$.*

Proof. I. Since A is countable, there exists a single-valued relation φ such that $\varphi[\mathsf{N}] = A$. Let ψ denote the range-restriction of φ to B (i.e. $\psi = \varphi \cap (\mathsf{N} \times B)$); clearly, $\psi[\mathsf{N}] = B$. — II. Clearly it is sufficient to prove the second assertion for the case $A = \mathsf{N}_m$. Apply 3 G.4 (with M replaced by B). Clearly, $\varphi[B] = \mathsf{N}_n$ for some n, and $n \leqq m$ (if $m < n$, A would be equipollent with a proper subset of B, hence of A, which is impossible since A is finite).

3 G.8. Theorem. *Let f be a single-valued relation. If A is countable, then $f[A]$ is countable. If A has a finite number m of elements, then $f[A]$ has a finite number n of elements, $n \leqq m$.*

Proof. I. There exists a single-valued φ with $A = \varphi[\mathsf{N}]$. Putting $\psi = f \circ \varphi$ we have $f[A] = \psi[\mathsf{N}]$. — II. We prove the second assertion for $A = \mathsf{N}_m$. For every $x \in f[A]$ let gx be the least of all numbers z such that $fz = x$. Then g is a one-to-one relation on $f[A]$ into A from which, by 3 G.7, we obtain that $f[A]$ has a finite number n of elements, $n \leqq m$.

3 G.9. *The sets* N × N *and* N *are equipollent.*

Proof. Denote by f the relation assigning $2^m \cdot (2n + 1) - 1$ to $\langle m, n \rangle \in \mathsf{N} \times \mathsf{N}$. It is easy to see that f is one-to-one on $\mathsf{N} \times \mathsf{N}$ onto N.

3 G.10. Theorem. *Let A, B be countable. Then $A \times B$ is countable; if A, B are non-void and one of them is infinite, then $A \times B$ is countably infinite; if both A and B are finite, then $A \times B$ has ab elements where a, b denote the number of elements of A, B respectively.*

Proof. There exist single-valued relations φ, ψ such that $\varphi[\mathsf{N}] = A$, $\psi[\mathsf{N}] = B$. Now let Φ assign to every $m \in \mathsf{N}$ the pair $\langle \varphi(f^{-1}m), \psi(f^{-1}m) \rangle$ where f is the relation indicated in 3 G.9. It is easy to see that Φ is single-valued, $\Phi[\mathsf{N}] = A \times B$. – If, say, A is countably infinite, $B \neq \emptyset$, then choose $b \in B$. Clearly, the subset $A \times (b)$ of $A \times B$, hence also $A \times B$ itself is also countably infinite. – The last assertion is contained in 3 E.8 because A and B are countable and finite, and hence, by 3 G.6, have a finite number of elements.

3 G.11. *If ϱ is a relation, A is a class, and A as well as every $\varrho[(x)]$, $x \in A$, has a finite number of elements, then $\varrho[A]$ has a finite number of elements.*

This easily follows from 3 G.5 (which implies the existence of a number $p \in \mathsf{N}$ not less than the number of elements of any $\varrho[(x)]$, $x \in A$).

3 G.12. Theorem. *Every infinite class contains a countably infinite subset.*

Proof. Let A be an infinite class. Let φ be a one-to-one relation on A into A such that $\varphi[A] \neq A$. Choose $x \in A - \varphi[A]$. Let ψ consist of $\langle 0, x \rangle$ and all $\langle k, \varphi^k x \rangle$, $k \in \mathsf{N}$, $k \geqq 1$. Clearly, $\mathbf{D}\psi = \mathsf{N}$, ψ is single-valued. We are going to prove that ψ is one-to-one. Suppose, on the contrary, that there exists a number $k \in \mathsf{N}$ such that $\psi k = \psi m$ for some $m \in \mathsf{N}$, $m > k$; choose the least k with this property. Then, clearly, $\psi k = \psi m \neq x$, for $x \notin \varphi[A]$ whereas $\psi m = \varphi^k x \in \varphi[A]$. Thus, $k \neq 0$. Since φ is one-to-one, we get $\psi(k - 1) = \psi(m - 1)$ which is a contradiction. We have proved that ψ is one-to-one. Therefore, $\psi[\mathsf{N}] \subset A$ is a countably infinite set.

3 G.13. The proofs of the following propositions require the Axiom of Choice. We formulate them here, although they cannot be proved now, but we shall restate them with proofs in Section 4 (see 4 C.9, 4 C.10). – 1) A class is finite if and only if it has a finite number of elements. – 2) If A is a countable set, ϱ is a relation and every $\varrho[(x)]$, $x \in A$, is countable, then $\varrho[A]$ is countable. – 3) If $\{X_a \mid a \in A\}$ is a countable family of countable sets, then $\bigcup X_a$ is countable.

4. CHOICE

Roughly speaking, the Axiom of Choice asserts the following: if we have a class of non-void sets, then we can choose "simultaneously" an element in each set. In order to illustrate this situation, it is sometimes said that an election is held in all the sets in question at the same time, and every set elects its "chairman" from among its elements. This principle is of fundamental importance for modern mathematics based on set theory. Its significance in "classical" mathematics is not so essential; nevertheless, we can find it in a weaker form in very simple cases, for example in the following reasoning: "Let M be a non-empty set of positive real numbers, $\inf M = 0$. We are going to prove that there exist $x_n \in M$ such that $\lim x_n = 0$. For every $n = 1, 2, \ldots$ there exist $x \in M$ with $x < n^{-1}$; hence, we choose for each n one such number x_n, and we get the required sequence". The last step here is based on the Axiom of Choice.

We shall not now discuss in detail the sense and role of the Axiom of Choice and its relationship to the other axioms and theorems. We remark only that, roughly speaking, for a certain fairly reasonable axiomatic system of set theory containing the Axiom of Choice, it has been proved that this system is consistent provided it is consistent without the Axiom of Choice. Before formulating the Axiom of Choice (in a form stronger than usual) we prove some propositions useful for further developments but not based on the Axiom of Choice. We point out that there is a deeply rooted similarity between these propositions and assertions 3 B.4, 3 B.5 in Section 3.

A. MONOTONICALLY ADDITIVE CLASSES

First, we introduce notions which will be used at various places in the sequel. We recall that a class of sets $\mathscr{A}$ is called monotone (see 3 B.1) if for every $X \in \mathscr{A}$, $Y \in \mathscr{A}$ either $X \supset Y$ or $Y \subset X$.

4 A.1. Definition. Let $\mathscr{A}$ be a class of sets. $\mathscr{A}$ is called *additive* if $X \cup Y \in \mathscr{A}$ whenever $X \in \mathscr{A}$, $Y \in \mathscr{A}$, *completely additive* if $\bigcup \mathscr{X} \in \mathscr{A}$ for every non-empty collection $\mathscr{X} \subset \mathscr{A}$; $\mathscr{A}$ is said to be *monotonically additive* if $\bigcup \mathscr{X} \in \mathscr{A}$ for every non-empty monotone collection $\mathscr{X} \subset \mathscr{A}$.

We observe that the definition of complete additivity requires, of course, $\bigcup \mathscr{X} \in \mathscr{A}$ only if $\mathscr{X}$ is a collection, not a non-comprisable class, and similarly for monotone additivity.

Examples. (A) Every class of the form $\exp X$ is completely additive. – (B) If ϱ is a relation, $\mathscr{A}$ is a completely additive class of sets, and every $\varrho[X]$, $X \in \mathscr{A}$, is comprisable, then $\mathbf{E}\{\varrho[X] \mid X \in \mathscr{A}\}$ is completely additive. – (C) The class of one-element sets is monotonically additive but not completely additive. – (D) The class (the collection) of all connected subsets (see 20 B.1) of a given topological space is monotonically additive but not completely additive. – (E) The class of all sets with a finite number of elements is additive (see 3 E.8) but not monotonically additive and hence also not completely additive.

4 A.2. *The following classes are monotonically additive: the class of all comprisable single-valued relations, the class of all comprisable one-to-one relations, the class consisting of all monotone collections of sets.*

A proof of the last assertion only will be given. Let $\mathbf{A}$ consist of all monotone collections of sets. If $\mathbf{X} \subset \mathbf{A}$ is a monotone collection, consider $\bigcup \mathbf{X}$. If $A \in \bigcup \mathbf{X}$, $B \in \bigcup \mathbf{X}$, then $A \in \mathscr{X}_1$, $B \in \mathscr{X}_2$ for some $\mathscr{X}_1 \in \mathbf{X}$, $\mathscr{X}_2 \in \mathbf{X}$; since $\mathbf{X}$ is monotone, we have either $\mathscr{X}_1 \supset \mathscr{X}_2$ or $\mathscr{X}_2 \supset \mathscr{X}_1$. It is sufficient to consider one of these cases, say $\mathscr{X}_1 \supset \mathscr{X}_2$. Then we obtain $A \in \mathscr{X}_1$, $B \in \mathscr{X}_1$ and therefore, $\mathscr{X}_1$ being monotone, either $A \supset B$ or $B \subset A$.

4 A.3. *If $\mathbf{X}$ is a non-void class of additive (respectively, completely additive, monotonically additive) collections of sets, then $\bigcap \mathbf{X}$ is additive (respectively, completely additive, monotonically additive).*

We omit the easy proof.

4 A.4. *If $\mathscr{A}$ is a monotone class of sets, then the class $\mathscr{B}$ of all $\bigcup \mathscr{X}$, where $\mathscr{X} \subset \mathscr{A}$ is a collection, is monotone and completely additive.*

Proof. I. Let $\mathscr{X} \subset \mathscr{A}$, $\mathscr{Y} \subset \mathscr{A}$ be collections. If for every $X \in \mathscr{X}$ there is a $Y \in \mathscr{Y}$ with $Y \supset X$, then clearly $\bigcup \mathscr{Y} \supset \bigcup \mathscr{X}$; if not, then, for some $X_0 \in \mathscr{X}$, $Y \supset X_0$ holds for no $Y \in \mathscr{Y}$; therefore, $\mathscr{A}$ being monotone, $X_0 \supset Y$ for every $Y \in \mathscr{Y}$ and therefore $X_0 \supset \bigcup \mathscr{Y}$, $\bigcup \mathscr{X} \supset \bigcup \mathscr{Y}$. This proves the monotonicity of $\mathscr{B}$. – II. Let $\mathscr{Z} \subset \mathscr{B}$ be a collection. For every $Z \in \mathscr{Z}$, let $\mathscr{X}_Z$ be the collection of all sets $X \in \mathscr{A}$ such that $X \subset Z$; clearly, $\bigcup \mathscr{X}_Z = Z$. Now put $\mathscr{X} = \bigcup\{\mathscr{X}_Z \mid Z \in \mathscr{Z}\}$. Then $\mathscr{X}$ is a collection, $\mathscr{X} \subset \mathscr{A}$ and clearly $\bigcup \mathscr{X} = \bigcup \mathscr{Z}$.

For the sake of completeness we introduce analogous definitions relating to intersections.

4 A.5. Definition. Let $\mathscr{A}$ be a class of sets. $\mathscr{A}$ is said to be *multiplicative* if $X \cap Y \in \mathscr{A}$ for $X \in \mathscr{A}$, $Y \in \mathscr{A}$, *completely multiplicative* if $\bigcap \mathscr{X} \in \mathscr{A}$ for every non-void collection $\mathscr{X} \subset \mathscr{A}$; $\mathscr{A}$ is called *monotonically multiplicative* if $\bigcap \mathscr{X} \in \mathscr{A}$ for every non-void monotone collection $\mathscr{X} \subset \mathscr{A}$.

The same remarks as for the previous definition also hold for this definition.

4 A.6. We turn now to the theorems announced above, namely to propositions 4 A.7 and 4 A.8 which are quite analogous to the theorems 3 B.4 and 3 B.5. This analogy also concerns the proof and is not essentially diminished by the fact that the assumption $X \supset \varphi X$ in Section 3 is replaced here by $X \subset \varphi X$; however, the class $\mathscr{B}$ may be non-comprisable, a striking difference from the result in Section 3.

It is to be noted also that there is an analogy (though not so complete) between 4 D.1 and 3 C.2 as well as between theorem 11 A.9 (and its consequences) and the uniqueness theorem 3 C.5.

Roughly speaking (and anticipating some concepts to be introduced in the sequel), the main purpose of the theorems in the subsection 3 C lies in a "construction" of the set of natural numbers (or, at least, "segments" of it), whereas 4 A.7 and 4 A.8 besides preparing the way for the "maximality principle" (4 C.1), contain a "construction" of the class of all ordinal numbers (or, at least, "segments" of it). The theorems from 3 C justify the use of the principle of mathematical induction, theorems 4 A.7, 4 A.8 that of the principle of transfinite induction.

4 A.7. Theorem. *Let $\mathscr{A}$ be a monotonically additive class of sets. Let $A \in \mathscr{A}$. Let φ be a single-valued relation with domain $\mathscr{A}$ and let $\varphi X \in \mathscr{A}$, $X \subset \varphi X$ for each $X \in \mathscr{A}$. Then there exists precisely one class $\mathscr{B} \subset \mathscr{A}$ such that*

(a_0) *$A \in \mathscr{B}$,* (a_1) *$X \in \mathscr{B} \Rightarrow \varphi X \in \mathscr{B}$,* ($a_2$) *$\mathscr{B}$ is monotonically additive;*

(b) *if $\mathscr{C} \subset \mathscr{B}$ and the following holds:* (1) *$A \in \mathscr{C}$,* (2) *$X \in \mathscr{C}$ implies $\varphi X \in \mathscr{C}$,* (3) *$\mathscr{C}$ is monotonically additive, then $\mathscr{C} = \mathscr{B}$.*

The set A is the smallest set in $\mathscr{B}$.

Proof. Let us say that a class $\mathscr{X} \subset \mathscr{A}$ has property $\boldsymbol{S}$ if (1) $A \in \mathscr{X}$, (2) $X \in \mathscr{X} \Rightarrow \Rightarrow \varphi X \in \mathscr{X}$, (3) $\mathscr{X}$ is monotonically additive. Let $\mathscr{B}$ be the class of all $X \in \mathscr{A}$ which belong to every class with property $\boldsymbol{S}$ (such classes exist, for $\mathscr{A}$ has property $\boldsymbol{S}$). Evidently, $\mathscr{B}$ also has property $\boldsymbol{S}$. In fact, A belongs to every class with property $\boldsymbol{S}$, and hence $A \in \mathscr{B}$; if $X \in \mathscr{B}$, then $X \in \mathscr{X}$, hence $\varphi X \in \mathscr{X}$ for each class $\mathscr{X}$ with property $\boldsymbol{S}$, and therefore $\varphi X \in \mathscr{B}$; finally, if $\mathscr{Y} \subset \mathscr{B}$ is a monotone collection, then, for every class $\mathscr{X}$ with property $\boldsymbol{S}$, $\mathscr{Y} \subset \mathscr{X}$; hence $\bigcup \mathscr{Y} \in \mathscr{X}$, and this implies $\bigcup \mathscr{Y} \in \mathscr{B}$.

We shall show now that $\mathscr{B}$ has the properties required in the theorem. The properties (a_0), (a_1), (a_2) have already been proved. If $\mathscr{C} \subset \mathscr{B}$ and $\mathscr{C}$ fulfils conditions (1), (2), (3) in (b), then $\mathscr{C}$ has property $\boldsymbol{S}$, which implies $\mathscr{B} \subset \mathscr{C}$.

We have proved that there exists a class $\mathscr{B}$ with the required properties. The proof that $\mathscr{B}$ is unique (for given $\mathscr{A}$, φ and A) is similar to the proof of theorem 3 B.4: if $\mathscr{B}'$ is a class with the properties in the theorem, then also the class $\mathscr{B} \cap \mathscr{B}'$ has properties (a_0) – (a_2). By (b), we get $\mathscr{B} \cap \mathscr{B}' = \mathscr{B}$ and at the same time $\mathscr{B} \cap \mathscr{B}' = = \mathscr{B}'$. Hence, $\mathscr{B} = \mathscr{B}'$.

Finally $X \supset A$ for every $X \in \mathscr{B}$; for if $\mathscr{C}$ denotes the class (evidently, monotonically additive) of sets $X \in \mathscr{B}$ such that $X \supset A$, then by (b) we get $\mathscr{C} = \mathscr{B}$.

4 A.8. Theorem. *Let $\mathscr{B}$ be a non-empty class of sets. Let $A \in \mathscr{B}$. Let φ be a single-valued relation with domain $\mathscr{B}$ such that $X \in \mathscr{B} \Rightarrow X \subset \varphi X$. Let assumptions* (a_0), (a_1), (a_2), (b) *in* 4 A.7 *be fulfilled. Then*

(m) *if $X \in \mathscr{B}$, $Y \in \mathscr{B}$, then either $X = Y$ or $X \supset \varphi Y$ or $Y \supset \varphi X$ (hence, $\mathscr{B}$ is monotone)*;

(w) *if a class $\mathscr{X} \subset \mathscr{B}$ is non-void, then it contains a smallest set*;

(s) *if $X \in \mathscr{B}$, then $X \subset \varphi X$, $X \neq \varphi X$ unless X is the largest set in $\mathscr{B}$; if $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \subset Y$, $X \neq Y$, then $\varphi X \subset \varphi Y$, $\varphi X \neq \varphi Y$ unless $\varphi X = Y$ and Y is the largest set in $\mathscr{B}$; finally, $\mathscr{B}$ is comprisable (i.e. is a set) if and only if there is a largest set in $\mathscr{B}$.*

Proof. I. We shall prove assertion (m). If $X \in \mathscr{B}$, $Y \in \mathscr{B}$, then $X \varrho Y$ denotes that either $X \supset Y$ or $\varphi X \subset Y$. We shall show that the following holds:

$(*)$ if $X \varrho Y$, $Y \varrho X$, then $X \varrho (\varphi Y)$.

Let $X \varrho Y$, $Y \varrho X$. If $\varphi X \subset \varphi Y$, then evidently $X \varrho (\varphi Y)$. If $\varphi X \subset \varphi Y$ does not hold, then neither $X = Y$ nor (since $Y \subset \varphi Y$) $\varphi X \subset Y$. As $X \varrho Y$, we have simultaneously $X \supset Y$, $X \neq Y$, hence $Y \supset X$ does not hold. We have shown that $Y \varrho X$ implies $X \supset \varphi Y$, and this means that $X \varrho (\varphi Y)$.

Let us denote by $\mathscr{B}'$ the class of all $Y \in \mathscr{B}$ such that $X \varrho Y$ for each $X \in \mathscr{B}$, and by $\mathscr{B}''$ the class of all $Z \in \mathscr{B}$ such that $Y \varrho Z$ for every $Y \in \mathscr{B}'$. We shall show that $\mathscr{B}' = \mathscr{B}'' = \mathscr{B}$. Evidently, if $X \in \mathscr{B}$, and $\mathscr{Y} \subset \mathscr{B}$ is a nonempty monotone collection and $X \varrho Y$ for every $Y \in \mathscr{Y}$, then $X \varrho (\bigcup \mathscr{Y})$. This implies that $\mathscr{B}'$ and $\mathscr{B}''$ are monotonically additive, hence they have property (a_2). By theorem 4 A.7, $X \varrho A$ for every $X \in \mathscr{B}$. Hence $A \in \mathscr{B}'$, $A \in \mathscr{B}''$, and $\mathscr{B}'$ and $\mathscr{B}''$ fulfil (a_0). If $Y \in \mathscr{B}'$, $Z \in \mathscr{B}''$, then $Z \varrho Y$, $Y \varrho Z$ by the definition of classes $\mathscr{B}'$, $\mathscr{B}''$. By $(*)$ we have $Y \varrho (\varphi Z)$; hence $Z \in \mathscr{B}'' \Rightarrow \varphi Z \in \mathscr{B}''$. If we put $\mathscr{B}''$ instead of $\mathscr{C}$ in (b), we have $\mathscr{B}'' = \mathscr{B}$. Therefore, if $X \in \mathscr{B}$, then also $X \in \mathscr{B}''$, and if $Y \in \mathscr{B}'$, then again by the definition of the classes $\mathscr{B}'$, $\mathscr{B}''$ we have $X \varrho Y$, $Y \varrho X$. Therefore by $(*)$, $X \varrho (\varphi Y)$ also holds; hence $Y \in \mathscr{B}' \Rightarrow \varphi Y \in \mathscr{B}'$. By (b), we get $\mathscr{B}' = \mathscr{B}$.

Thus, for arbitrary $X \in \mathscr{B}$, $Y \in \mathscr{B}$ we have $X \varrho Y$, that is either $X \supset Y$ or $\varphi X \subset Y$ (in view of $X \subset \varphi X$, this implies that $\mathscr{B}$ is monotone). It is easy to see, in a manner similar to the proof of theorem 3 B.5, that this condition is equivalent to (m).

II. We shall prove that (w) holds. Let $\mathscr{X} \subset \mathscr{B}$, $\mathscr{X} \neq \emptyset$. Let $\mathscr{M}$ be the class of all $X \in \mathscr{B}$ such that $X \subset Y$ for every $Y \in \mathscr{X}$. Let $X_0 = \bigcup \mathscr{M}$; then, clearly, $X_0 \in \mathscr{B}$. If $Y \in \mathscr{X}$, then $X \subset Y$ for every $X \in \mathscr{M}$, hence $X_0 \subset Y$; we get $X_0 \in \mathscr{M}$. It suffices to show that $X_0 \in \mathscr{X}$. We shall assume that $X_0 \subset Y$, $X_0 \neq Y$ for each $Y \in \mathscr{X}$ and we shall derive a contradiction. Then (since $\varphi Y \supset Y$) $X_0 \supset \varphi Y$ cannot hold for any $Y \in \mathscr{X}$, and hence, by property (m), $\varphi X_0 \subset Y$ for each $Y \in \mathscr{X}$, also $\varphi X_0 \in \mathscr{M}$. It is easy to see that the class $\mathscr{M}$ is monotonically additive. By 4 A.7, $A \in \mathscr{M}$. Let $X \in \mathscr{M}$. We are going to show that $\varphi X \in \mathscr{M}$. Certainly $X \subset X_0$. As we have proved that $\varphi X_0 \in \mathscr{M}$, $\varphi X \in \mathscr{M}$ is evident for $X = X_0$. If $X \subset X_0$, $X \neq X_0$, then (because $\varphi X_0 \supset X_0$) $X \supset \varphi X_0$ cannot hold. Hence, by (m), $X_0 \supset \varphi X$ and we get again $\varphi X \in \mathscr{M}$. Then, by (b), $\mathscr{M} = \mathscr{B}$, which contradicts the assumption $X_0 \notin \mathscr{X}$.

III. Finally we prove (s). Clearly, if $X \in \mathscr{B}$ is the largest set in $\mathscr{B}$, then $\varphi X = X$. Conversely, suppose that $X_0 \in \mathscr{B}$, $\varphi X_0 = X_0$. Put $\mathscr{C} = \mathbf{E}\{X \mid X \in \mathscr{B},\ X \subset X_0\}$. Evidently, $A \in \mathscr{C}$. Let $Y \in \mathscr{C}$. If $Y = X_0$, then $\varphi Y = \varphi X_0 = X_0 \subset X_0$, and thus $\varphi Y \in \mathscr{C}$. If $Y \neq X_0$, then $Y \subset \varphi X_0$, $Y \neq \varphi X_0$, hence $Y \supset \varphi X_0$ cannot hold. Therefore, by (m), $X_0 \supset \varphi Y$ and we get again $\varphi Y \in \mathscr{C}$. Finally, it is evident that $\mathscr{C}$ is monotonically additive. Hence $\mathscr{C} = \mathscr{B}$ by (b), and X_0 is the largest set in $\mathscr{B}$. Omitting the proof of the second assertion in (s), which may be left to the reader, we pass to the third statement. If $\mathscr{B}$ is a set, then also $\bigcup \mathscr{B}$ is a set by theorem 2.13. Since $\mathscr{B}$ is monotonically additive, $\bigcup \mathscr{B} \in \mathscr{B}$. Evidently, $\bigcup \mathscr{B}$ is the largest set in $\mathscr{B}$. Conversely, if $\mathscr{B}$ contains the largest set B, then $\mathscr{B} \subset \exp B$, and $\mathscr{B}$ is a set (by Axiom 1 E.1, (c) and Theorem 1 E.5).

The following proposition will be useful in the proof of theorem 4 C.1.

4 A.9. Lemma. *Let $\mathscr{B}$ be a non-void class of sets; let A be a set. Let φ be a single-valued relation defined on a class $\mathscr{A} \supset \mathscr{B}$ such that for any $X \in \mathscr{A}$, $\varphi X \in \mathscr{A}$, $\varphi X \supset X$. Let $\mathscr{B}$ possess properties* (a_0), (a_1), (a_2), (b) *from* 4 A.7. *Let $\mathscr{A}'$, $\mathscr{A}''$ be classes such that* (1) $\mathscr{A}' \subset \mathscr{A}'' \subset \mathscr{A}$, (2) $A \in \mathscr{A}'$, (3) *if $\mathscr{X} \subset \mathscr{A}'$ is a monotone collection, then $\bigcup \mathscr{X} \in \mathscr{A}''$,* (4) *if $X \in \mathscr{A}''$, then $\varphi X \in \mathscr{A}'$. Then $\mathscr{B} \subset \mathscr{A}''$ and for each $X \in \mathscr{B}$ there exists a $Y \in \mathscr{B} \cap \mathscr{A}'$ such that $X \subset Y$ (in particular, if $\mathscr{B}$ possesses a largest set M, then $M \in \mathscr{A}'$).*

Proof. To prove that $\mathscr{B} \subset \mathscr{A}''$ we assume, on the contrary, that $\mathscr{B} - \mathscr{A}'' \neq \emptyset$. Then, by 4 A.8 (w), there exists in $\mathscr{B} - \mathscr{A}''$ a smallest set X_0. Let $\mathscr{C}$ consist of all $X \in \mathscr{B}$ such that $X \subset X_0$, $X \neq X_0$. Put $X_1 = \bigcup\{\varphi X \mid X \in \mathscr{C}\}$. Evidently, $\mathscr{C} \subset \mathscr{A}''$, hence $\varphi X \in \mathscr{A}'$ for each $X \in \mathscr{C}$, and by assumption (3), $X_1 \in \mathscr{A}''$. Since $X \supset X_0$ for no $X \in \mathscr{C}$, we have, in view of part I of the proof of 4 A.8, $\varphi X \subset X_0$ for every $X \in \mathscr{C}$, hence also $X_1 \subset X_0$. Since $\mathscr{B}$ is monotone and monotonically additive, we have $X_1 \in \mathscr{B}$. We now show that both assumptions $X_1 = X_0$ and $X_1 \neq X_0$ lead to a contradiction; this will prove $\mathscr{B} \subset \mathscr{A}''$. Since $X_1 \in \mathscr{A}''$ and $X_0 \notin \mathscr{A}''$, $X_1 = X_0$ cannot hold. Let $X_1 \neq X_0$, hence $X_1 \in \mathscr{C}$. Since $X_0 \in \mathscr{B}$, $X_1 \in \mathscr{B}$, and $X_1 \supset X_0$ does not hold, $\varphi X_1 \subset X_0$ (by part I of the proof of 4 A.8); since $X_1 \in \mathscr{C}$, we have $\varphi X_1 \subset X_1$ and therefore $\varphi X_1 = X_1$. It follows, by part III of the proof of 4 A.8, that X_1 is the largest set in $\mathscr{B}$. This is a contradiction, as, on the other hand, $X_1 \subset X_0$, $X_1 \neq X_0$. Therefore $\mathscr{B} \subset \mathscr{A}''$.

We now prove the second assertion of the lemma. Let $X \in \mathscr{B}$. Then $X \in \mathscr{A}''$, hence $\varphi X \in \mathscr{A}'$, $\varphi X \in \mathscr{B} \cap \mathscr{A}'$. Since $\varphi X \supset X$, it is sufficient to put $Y = \varphi X$.

B. AXIOM OF CHOICE

The Axiom of Choice can be formulated in different ways. Often we can find for example the following formulations: "If $\mathscr{M}$ is a disjoint family of non-void sets, then there exists a set A which has precisely one common element with each $M \in \mathscr{M}$"

or "if $\{X_a \mid a \in A\}$ is a family of non-void sets, then there exists a family $\{x_a \mid a \in A\}$ such that $x_a \in X_a$ for each $a \in A$"; probably this last formulation most clearly expresses the idea of "simultaneous selection" which is important for the Axiom of Choice.

There is a close connection between the Axiom of Choice and another general principle, which is sometimes called "the principle of definition by means of abstraction". This idea, which has a very wide methodological significance, can be formulated, as far as mathematical concepts are concerned, in the following form: if ϱ is an equivalence on a class Z, then we can assign to every $z \in Z$ an element $\psi(z)$ in such a way that $\psi(z) = \psi(z')$ if and only if $z\varrho z'$. This principle is used for example for the definition of cardinal numbers: we associate (see 9.A) with every set X an element $\operatorname{card} X$, called the cardinality of X, such that $\operatorname{card} X = \operatorname{card} Y$ if and only if X and Y are equipollent. Of course, the principle of definition by abstraction in the above formulation does not imply the Axiom of Choice; e.g. if Z is a set, then we can take $\psi(z) = \mathbf{E}\{x \mid x\varrho z\}$ for $z \in Z$, and this has nothing in common with the choice of elements.

On the other hand, if we assume the Axiom of Choice in a suitable form, then in the situation described above we may choose elements φz in such a way that always $z\varrho(\psi z)$, i.e. $\psi z \in \varrho[(z)]$. To this end consider, provided $\varrho[(z)]$ are sets, the relation $\mu = \{\varrho[(z)] \to z \mid z \in Z\}$, which is not single-valued in general; choose a single-valued $\varphi \subset \mu$ and put, for any $z \in Z$, $\psi z = \varphi(\varrho[(z)])$. Let us note that the classes $\varrho[(z)]$ are sets whenever Z is a set.

The facts roughly indicated above as well as the necessity of choosing, from time to time, elements from non-comprisable classes, leads to a rather strong form of the Axiom of Choice (which is, in fact, rather similar to the form of the postulate of choice contained implicitly in N. Bourbaki's formulations in his "Éléments de mathématique").

4 B.1. Axiom of Choice. *Let A be a class; let ϱ be an equivalence on A. Then there exists a class $B \subset A$ with the following property: for every $x \in A$ there exists precisely one $y \in B$ such that $x\varrho y$.*

Remark. Evidently, the property indicated above may be expressed as follows: there exists a single-valued relation $\varphi \subset \varrho$ such that $\mathbf{D}\varphi = \mathbf{D}\varrho$, $x\varrho x' \Rightarrow \varphi x = \varphi x'$.

Now, some immediate consequences of the Axiom of Choice are given.

4 B.2. Theorem. *Let ϱ be a relation. Then there exists a single-valued relation $\varphi \subset \varrho$ such that $\mathbf{D}\varphi = \mathbf{D}\varrho$.*

Proof. Let us denote by $\boldsymbol{R}$ the relation defined as follows: $\langle u, v\rangle \in \boldsymbol{R}$ if and only if $u \in \varrho$, $v \in \varrho$ and there exist x, y, z such that $u = \langle x, y\rangle$, $v = \langle x, z\rangle$, that is the first members of u and of v are equal. It is easy to see that $\boldsymbol{R}$ is a transitive, symmetric and reflexive relation, $\mathbf{D}\boldsymbol{R} = \varrho$, $\mathbf{E}\boldsymbol{R} = \varrho$, hence $\boldsymbol{R}$ is an equivalence on the class ϱ. By the Axiom of Choice, there exists a class $\varphi \subset \varrho$ such that

(*) for each $u \in \varrho$ there exists precisely one $v \in \varphi$ such that $u\boldsymbol{R}v$.

We shall prove that $\mathbf{D}\varphi = \mathbf{D}\varrho$ and that φ is a single-valued relation. Evidently, $\mathbf{D}\varphi \subset \mathbf{D}\varrho$. Let $x \in \mathbf{D}\varrho$; there exists a y such that $u = \langle x, y\rangle \in \varrho$. By $(*)$, there exists precisely one $v = \langle s, t\rangle \in \varphi$ such that $u\boldsymbol{R}v$; we have $s = x$, hence $\langle x, t\rangle = v \in \varphi$ and therefore $x \in \mathbf{D}\varphi$. Thus we have proved $\mathbf{D}\varrho \subset \mathbf{D}\varphi$; hence, $\mathbf{D}\varphi = \mathbf{D}\varrho$. Let $x \in \mathbf{D}\varphi$; let $\langle x, y\rangle \in \varphi$, $\langle x, y'\rangle \in \varphi$; we are going to prove that $y = y'$. By the definition of the relation $\boldsymbol{R}$, we have $\langle x, y\rangle \boldsymbol{R}\langle x, y'\rangle$; this implies, by $(*)$, $\langle x, y\rangle = \langle x, y'\rangle$, hence $y = y'$.

Corollary. *If ϱ is a single-valued relation, then $\mathbf{E}\varrho$ is equipollent with a subclass of $\mathbf{D}\varrho$.*

The proof is left to the reader.

4 B.3. Theorem. *Let $\{X_a \mid a \in A\}$ be an indexed class of non-empty sets. Then there exists an indexed class $\{x_a \mid a \in A\}$ such that $x_a \in X_a$ for each $a \in A$.*

Proof. Let ϱ be the class of all pairs $\langle a, y\rangle$ such that $y \in X_a$. Then ϱ is a relation, $\mathbf{D}\varrho = A$. By the preceding theorem, there exists a single-valued relation $\varphi \subset \varrho$ such that $\mathbf{D}\varphi = \mathbf{D}\varrho$, hence $\mathbf{D}\varphi = A$. For $\varphi \subset \varrho$, we have $\langle a, \varphi a\rangle \in \varrho$ for each $a \in A$, hence $\varphi a \in X_a$. Put $x_a = \varphi a$; then $\{x_a \mid a \in A\} = \varphi$ has the required properties.

4 B.4. Theorem. *Let $\mathscr{M}$ be a disjoint class (respectively, a disjoint collection) of non-empty sets. Then there exists a class (respectively, a collection) F such that, for every $M \in \mathscr{M}$, $M \cap F$ is a singleton.*

Proof. We consider the indexed class $\{M \mid M \in \mathscr{M}\}$. By 4 B.3, there exists an indexed class $\{x_M \mid M \in \mathscr{M}\}$ such that $x_M \in M$ for each $M \in \mathscr{M}$. Put $F = \mathbf{E}\{x_M \mid M \in \mathscr{M}\}$. Since $\mathscr{M}$ is disjoint, $F \cap M = (x_M)$ for each $M \in \mathscr{M}$.

C. MAXIMALITY PRINCIPLE AND RELATED PROPOSITIONS

In addition to the above trivial consequences of the Axiom of Choice we are going to give several essentially more profound ones.

4 C.1. Theorem ("maximality principle"). *Let $\mathscr{M}$ be a collection of sets with the following property: for every monotone collection $\mathscr{X} \subset \mathscr{M}$ there exists a $Y \in \mathscr{M}$ such that $X \subset Y$ for every $X \in \mathscr{X}$. Then there exists, for every $A \in \mathscr{M}$, a maximal set $M \in \mathscr{M}$ containing A.*

Proof. Let $S = \bigcup \mathscr{M}$, $\mathscr{A} = \exp S$. Put $\mathscr{A}' = \mathscr{M}$. Let $\mathscr{A}''$ be the collection of all $\bigcup\mathscr{X}$, where $\mathscr{X} \subset \mathscr{M}$ is a non-empty monotone collection. It is easy to see that $\mathscr{A}' \subset \mathscr{A}'' \subset \mathscr{A}$ and that $\mathscr{A}$ is a monotonically additive collection. Next, for every $X \in \mathscr{A}''$, there exists a $Y \in \mathscr{A}'$ such that $Y \supset X$; this is shown as follows: since $X \in \mathscr{A}''$, we have $X = \bigcup\mathscr{X}$, where $\mathscr{X} \subset \mathscr{M}$ is a non-void monotone collection; by the assumption in the theorem, there exists a set $Y \in \mathscr{M} = \mathscr{A}'$ such that $Z \subset Y$ for every $Z \in \mathscr{X}$, hence, $X \subset Y$. Now put $X\varrho Y$ if and only if $X \in \mathscr{A}$, $Y \in \mathscr{A}$ and either (1) $X \in \mathscr{A}''$, $Y \in \mathscr{A}'$, $X \subset Y$, $X \neq Y$, or (2) $X \in \mathscr{A}''$, $X = Y$, there is no $Z \in \mathscr{A}$

with $Z \supset X$, $Z \neq X$, or (3) $X \notin \mathscr{A}''$, $Y = S$; then it is easy to see that ϱ is a relation with domain $\mathscr{A}$. By 4 B.2 there exists a single-valued relation $\varphi \subset \varrho$ with $\mathbf{D}\varphi = \mathbf{D}\varrho = \mathscr{A}$. For every $X \in \mathscr{A}$ we have $\varphi X \supset X$; for if $X \in \mathscr{A}''$, then, by the definition of ϱ, $X \subset \varphi X$; if $X \notin \mathscr{A}''$, then $X \subset \varphi X = S$.

Now we have a monotonically additive class $\mathscr{A}$, a single-valued relation φ such that $\mathbf{D}\varphi = \mathscr{A}$, $\varphi X \supset X$ and $\varphi X \in \mathscr{A}$ for every $X \in \mathscr{A}$, a set $A \in \mathscr{A}'$, and a class $\mathscr{A}''$ such that the conditions (1), (2), (3), (4) in 4 A.9 are fulfilled. Moreover, the assumptions of the Theorem 4 A.7 are fulfilled. Hence, there exists a $\mathscr{B} \subset \mathscr{A}$ with the properties from 4 A.7. As $\mathscr{A}$ is a set, $\mathscr{B}$ is a set; hence, by 4 A.8, (s), there exists a largest set M in $\mathscr{B}$ and $\varphi M = M$. By 4 A.9, $M \in \mathscr{A}' = \mathscr{M}$. Since $A \in \mathscr{B}$, we have $A \subset M$. As $\varphi M = M$, the definition of ϱ implies that there exists no $Y \in \mathscr{M}$ such that $Y \supset M$, $Y \neq M$.

4 C.2. Theorem. *Let $\mathscr{M}$ be a collection of sets with the following property: for every monotone collection $\mathscr{X} \subset \mathscr{M}$ there exists a $Y \in \mathscr{M}$ such that $Y \subset X$ for every $X \in \mathscr{X}$. Then there exists, for every $A \in \mathscr{M}$, a minimal set $M \in \mathscr{M}$ contained in A.*

This follows at once from the preceding theorem applied to the collection of all sets $S - X$ where $S = \bigcup \mathscr{M}$, $X \in \mathscr{M}$.

4 C.3. Theorem. *A non-void monotonically additive (multiplicative) collection of sets contains a maximal (minimal) set.*

Proof. Let $\mathscr{A}$ be monotonically additive. Then, for any monotone $\mathscr{X} \subset \mathscr{A}$, we have $\bigcup \mathscr{X} \in \mathscr{A}$ and $X \subset \bigcup \mathscr{X}$ for every $X \in \mathscr{X}$. Thus the assumptions from 4 C.1 are fulfilled. The proof for a monotonically multiplicative collection is analogous.

4 C.4. In the following theorem (4 C.5) the concept of a finite set plays an essential role. We have not yet established the well-known connection between finite sets and the "segments" N_p; this will be done somewhat later in this section. We could, of course, state and discuss the theorem in question at a later place, but we prefer to give it now, preceded by a lemma, the proof of which will also serve as a useful exercise.

Lemma. *Every finite non-void class of sets contains a minimal set.*

Proof. Suppose that $\mathscr{A}$ is a finite non-void class of sets containing no minimal set. Let ϱ consist of all pairs $\langle X, Y \rangle$ where $X \in \mathscr{A}$, $Y \in \mathscr{A}$, $X \supset Y$, $X \neq Y$. By the supposition, $\mathbf{D}\varrho = \mathscr{A}$. By 4 B.2, there exists a single-valued relation $\varphi \subset \varrho$ with $\mathbf{D}\varphi = \mathbf{D}\varrho = \mathscr{A}$. Choose a set $A \in \mathscr{A}$. By 3 B.4, there exists a collection $\mathscr{B} \subset \mathscr{A}$ with properties (a_0), (a_1), (b) indicated in 3 B.4. Clearly, there is no smallest set in $\mathscr{B}$ (since if $X \in \mathscr{B}$, then $\varphi X \in \mathscr{B}$, $\varphi X \subset X$, $\varphi X \neq X$). Hence, by 3 B.5, assertion (s), $X \in \mathscr{B}$, $Y \in \mathscr{B}$, $X \neq Y$ implies $\varphi X \neq \varphi Y$. Thus $\varphi_{\mathscr{B}}$ is one-to-one. Clearly, $A \neq \varphi X$ for every $X \in \mathscr{B}$ (otherwise we would have $A \subset X$, $A \neq X$ which contradicts the fact that A is the largest set in $\mathscr{B}$; cf. 3 B.4). We have shown that φ is one-to-one on $\mathscr{B}$ into $\mathscr{B}$ and $\varphi[\mathscr{B}] \neq \mathscr{B}$; hence, $\mathscr{B}$ is infinite. This is a contradiction for $\mathscr{A} \supset \mathscr{B}$ is finite.

Corollary. *Every finite non-void collection of sets contains a maximal set. In a finite non-void monotone collection of sets, there is a largest and a smallest set.*

Remark. Of course, all these assertions are trivial as soon as we show that every finite class is equipollent with some N_p; this will be done in 4 C.9.

4 C.5. Theorem. *Let $\mathscr{A}$ be a non-void class of sets such that $X \in \mathscr{A}$ if and only if every finite subset of X belongs to $\mathscr{A}$ (such a class of sets is sometimes said to be of "finite character"). Then $\mathscr{A}$ is monotonically additive. If $\mathscr{A}$ is comprisable, then it contains a maximal set.*

Proof. Let $\mathscr{X} \subset \mathscr{A}$ be a non-void monotone collection; put $S = \bigcup \mathscr{X}$. If $\mathscr{X} = (\emptyset)$, then clearly $S = \emptyset$, $S \in \mathscr{A}$. Hence we may suppose that $\mathscr{X}$ contains a non-empty set. Let $M \subset S$ be finite. Put $\varrho = \{x \to X \mid x \in M,\ X \in \mathscr{X},\ x \in X\}$; clearly, $\mathbf{D}\varrho = M$. By 4 B.2, there exists a single-valued $\varphi \subset \varrho$ with $\mathbf{D}\varphi = M$. By the corollary to 4 B.2, $\mathscr{M} = \mathbf{E}\varphi$ is finite. Now, $\mathscr{M} \subset \mathscr{X}$ is a collection and therefore, by the corollary to 4 C.4, there exists in $\mathscr{M}$ a largest set, say X_0. For every $x \in M$ we have $x \in \varphi x$, $\varphi x \in \mathscr{M}$, hence $\varphi x \subset X_0$. Thus $x \in M \Rightarrow x \in X_0$, and $M \subset X_0$.

We have shown that every finite $M \subset S$ is contained in some $X \in \mathscr{X} \subset \mathscr{A}$. Since $\mathscr{A}$ is of "finite character", this proves that every finite $M \subset S$ belongs to $\mathscr{A}$. Hence S belongs to $\mathscr{A}$ as well. This proves that $\mathscr{A}$ is monotonically additive.

4 C.6. Theorem. *Let $\mathscr{F}$ be a non-empty monotonically additive class of single-valued relations. Let A be a set and let $\mathbf{D}f \subset A$ for any $f \in \mathscr{F}$. Suppose that for every $f \in \mathscr{F}$ such that $\mathbf{D}f \neq A$ there exists a relation $g \in \mathscr{F}$ such that $g \supset f$, $g \neq f$. Then there exists a relation $h \in \mathscr{F}$ such that $\mathbf{D}h = A$.*

Proof. Let ϱ consist of all pairs $\langle f, g\rangle$ where $f \in \mathscr{F}$, $g \in \mathscr{F}$, $g \supset f$, and $g \neq f$ provided $\mathbf{D}f \neq A$. By the assumptions stated in the theorem, $\mathbf{D}\varrho = \mathscr{F}$. Hence, by 4 B.2, there exists a single-valued relation $\varphi \subset \varrho$ with $\mathbf{D}\varphi = \mathbf{D}\varrho = \mathscr{F}$. By 4 A.7 and 4 A.8, there exists a monotone class $\mathscr{B} \subset \mathscr{F}$ such that $f \in \mathscr{B} \Rightarrow \varphi f \in \mathscr{B}$ and $\mathscr{B}$ is comprisable if and only if there is a largest set in $\mathscr{B}$. Now, clearly $\{f \to \mathbf{D}f \mid f \in \mathscr{B}\}$ is a one-to-one relation on $\mathscr{B}$ into $\exp A$ (if $f_1 \in \mathscr{B}$, $f_2 \in \mathscr{B}$, $\mathbf{D}f_1 = \mathbf{D}f_2$, then, by the monotonicity of $\mathscr{B}$, either $f_1 \supset f_2$ or $f_2 \subset f_1$ which, f_1, f_2 being single-valued, implies $f_1 = f_2$). This implies that $\mathscr{B}$ is comprisable. Hence, there exists a largest set, say h, in $\mathscr{B}$. Clearly, $\mathbf{D}h = A$ since otherwise we would have $\varphi h \supset h$, $\varphi h \neq h$, $\varphi h \in \mathscr{B}$.

Remark. Under a stronger assumption, namely of $\mathscr{F}$ being a set, the theorem would follow at once from 4 C.1.

4 C.7. Theorem. *Let $\mathscr{A}$ be a class of finite sequences, let $m \in \mathsf{N}$ and let the following conditions be satisfied:* (a) *$\mathscr{A}$ contains a sequence of length m;* (b) *if $n \in \mathsf{N}$, $\{x_k \mid k \in \mathsf{N}_n\} \in \mathscr{A}$, then there exists an element y such that $\{x_k \mid k \in \mathsf{N}_{n+1}\} \in \mathscr{A}$ where $x_n = y$.*

Then there exists an infinite sequence $\{a_k \mid k \in \mathsf{N}\}$ such that for every $n \in \mathsf{N}$, $n \geqq m$, $\{a_k \mid k \in \mathsf{N}_n\}$ belongs to $\mathscr{A}$.

Proof. Denote by $\mathscr{F}$ the class of all single-valued relations φ such that (1) either $\mathbf{D}\varphi = \mathsf{N}_n$, $n \geqq m$, or $\mathbf{D}\varphi = \mathsf{N}$, (2) for every $p \in \mathsf{N}$, $p \geqq m$, the restriction of φ to $(\mathbf{D}\varphi) \cap \mathsf{N}_p$ belongs to $\mathscr{A}$. It is easy to see that $\mathscr{F}$ satisfies the assumptions from 4 C.6 (with $A = \mathsf{N}$). Thus there exists a $\varphi \in \mathscr{F}$ such that $\mathbf{D}\varphi = \mathsf{N}$. This completes the proof.

Remark. The above theorem is a form of the so-called principle of construction by induction (on natural numbers) or of recurrent construction. Observe that, in this theorem, the element x_n is, in general, not determined uniquely by the sequence $\{x_k \mid k \in \mathsf{N}_n\}$, in contradistinction to Theorem 3 D.6.

We now give some basic facts on finite and infinite sets the proof of which requires the Axiom of Choice. Thus the gap in thec onsiderations in Section 3 concerning countable sets will be filled (the propositions in question have already been mentioned in 3 G.13).

4 C.8. Theorem. *If a class A is equipollent with no N_p, $p \in \mathsf{N}$, then a subset of A is equipollent with N.*

Proof. Apply theorem 4 C.7 putting $m = 0$ and denoting by $\mathscr{A}$ the set of all one-to-one finite sequences ranging in A. Clearly, the condition (b) from 4 C.7 is satisfied since, for any one-to-one finite sequence φ of length n, the class $A - \varphi[\mathsf{N}_n]$ is non-void. Hence, there exists an infinite sequence $\{a_k \mid k \in \mathsf{N}\}$ such that every $\{a_k \mid k \in \in \mathsf{N}_n\}$ belongs to $\mathscr{A}$. Thus $\{a_k \mid k \in \mathsf{N}\}$ is one-to-one, which completes the proof.

4 C.9. Theorem. *Every finite class is a set equipollent to some N_p, $p \in \mathsf{N}$; conversely, every class equipollent to some N_p is a finite set. Every infinite class contains a countably infinite set.*

Proof. Let A be a finite class; if A were equipollent with no N_p, there would exist $M \subset A$ equipollent with N which is impossible (see 3 A.9); thus A is equipollent to some N_p, hence A is a set. The converse follows from 3 E.4. The last assertion has already been proved (see 3 G.12); an alternative proof is as follows: If A is infinite, it is equipollent with no N_p; therefore it contains a set which is equipollent with N, hence countably infinite.

4 C.10. Theorem. *Let A be a countable set, ϱ a relation such that $\varrho[(x)]$ is countable for every $x \in A$. Then $\varrho[A]$ is countable.*

Proof. For any $x \in A$, there exists a single-valued relation (depending on x) σ such that $\mathbf{D}\sigma \subset \mathsf{N}$, $\mathbf{E}\sigma = \varrho[(x)]$. Now let r denote the relation consisting of all $\langle x, \psi \rangle$ where $x \in A$, ψ is a single-valued relation with $\mathbf{D}\psi \subset \mathsf{N}$, $\mathbf{E}\psi = \varrho[(x)]$. By 4 B.2, there exists a single-valued relation $f \subset r$ with $\mathbf{D}f = \mathbf{D}r = A$.

Let φ be a single-valued relation, $\mathbf{D}\varphi \subset \mathsf{N}$, $\mathbf{E}\varphi = A$. We define a relation Φ as follows: $\langle\langle m, n \rangle, y \rangle \in \Phi$ if and only if $\langle n, y \rangle \in f(\varphi m)$.

Now if $z \in \varrho[A]$, then $z \in \varrho[(x)]$ for some $x \in A$; clearly, fx is a single-valued relation for N ranging on $\varrho[(x)]$. Thus, for some m, n, $\langle n, z \rangle \in fx$, $\langle n, z \rangle \in f(\varphi m)$. Clearly, Φ is single-valued (since every $f(\varphi m)$ is a single-valued relation). Therefore we have a single-valued Φ with $\mathbf{D}\Phi \subset \mathsf{N} \times \mathsf{N}$, $\mathbf{E}\Phi = \varrho[A]$. This completes the proof, because, by 3 G.9, there is a single-valued relation on N onto $\mathsf{N} \times \mathsf{N}$.

Corollary. *Let $\{X_a \mid a \in A\}$ be a countable family of countable sets (this means that A and every X_a is countable). Then $\bigcup X_a$ is countable.*

Proof. It is clear that $\varrho = \{a \to x \mid x \in X_a, a \in A\}$ is a relation satisfying the assumption from the theorem. Evidently $\varrho[A] = \bigcup X_a$.

D. MINIMALLY NON-COMPRISABLE CLASSES

We conclude this section with some propositions concerning a certain special type of non-comprisable classes.

Up to now, non-comprisable classes usually appeared in this book mostly either as a "pathological" case beyond the scope of a proper investigation or as a kind of an unlimited depository of elements. In the present short subsection we give some positive facts concerning certain "relatively not too large" non-comprisable classes.

4 D.1. Theorem. *There exists a non-comprisable class $\mathscr{B}$ of sets such that, for a suitable set A and a single-valued relation φ on $\mathscr{B}$,*

(d_0) $A \in \mathscr{B}$, (d_1) $X \in \mathscr{B} \Rightarrow \varphi X \in \mathscr{B}$, ($d_1'$) $X \in \mathscr{B} \Rightarrow X \subset \varphi X$, $X \neq \varphi X$;

(d_2) *$\mathscr{B}$ is monotonically additive;*

(d_3) *if $\mathscr{C} \subset \mathscr{B}$ and the following holds: 1) $A \in \mathscr{C}$, (2) if $X \in \mathscr{C}$, then $\varphi X \in \mathscr{C}$, (3) $\mathscr{C}$ is monotonically additive, then $\mathscr{C} = \mathscr{B}$.*

Every such a class $\mathscr{B}$ is monotone, and every non-void subclass $\mathscr{X}$ of $\mathscr{B}$ contains a smallest set.

Proof. Let $\mathscr{A}$ be the class of all sets. Let ϱ consist of all pairs $\langle X, Y\rangle$ of sets such that $X \subset Y$, $X \neq Y$; it is clear that $\mathbf{D}\varrho = \mathscr{A}$.

By 4 B.2, there exists a single-valued $\varphi \subset \varrho$ with $\mathbf{D}\varphi = \mathscr{A}$. Let A be an arbitrary set. Then the existence of $\mathscr{B}$ with properties indicated above (except, perhaps, non-comprisability and (d_1')) follows from 4 A.7, 4 A.8. If $\mathscr{B}$ were comprisable, then by 4. A.8 (s), $\mathscr{B}$ would contain a largest set, say, M; this is a contradiction, because $M \in \mathscr{B}$, hence $\varphi M \in \mathscr{B}$, and $\varphi M \supset M$, $\varphi M \neq M$. Thus $\mathscr{B}$ is non-comprisable. Finally since $\mathscr{B}$ contains no largest set, (d_1') holds.

Remark. The class $\mathscr{B}$ is well-ordered (see Section 11), and every interval $]\!] \leftarrow, X]\!]$ of it is a set. In Section 11, we shall prove that an order on a non-comprisable class is uniquely determined, up to an isomorphism, by these properties.

We now append some auxiliary propositions.

4 D.2. *A class $\mathscr{A}$ of sets is comprisable if and only if $\bigcup\mathscr{A}$ is comprisable.*

Proof. "If" follows from $\mathscr{A} \subset \exp(\bigcup\mathscr{A})$, "only if" is asserted in 2.13.

4 D.3. *Let $\mathscr{A}$ be a non-comprisable monotone class of sets. If $\mathscr{B} \subset \mathscr{A}$, then either $\bigcup\mathscr{B}$ is a set and $\bigcup\mathscr{B} \subset A$ for some $A \in \mathscr{A}$ or $\bigcup\mathscr{B} = \bigcup\mathscr{A}$. If $X \subset \bigcup\mathscr{A}$ is a set, then $X \subset A$ for some $A \in \mathscr{A}$.*

Proof. Let $\mathscr{B} \subset \mathscr{A}$. Then either (1) there is a set $A \in \mathscr{A}$ such that $B \subset A$ for every $B \in \mathscr{B}$ or (2) for any $A \in \mathscr{A}$, there is a set $B \in \mathscr{B}$ with $A \subset B$. In the first case, $\bigcup\mathscr{B} \subset A$, $\mathscr{B} \subset \exp A$, in the second case, $\bigcup\mathscr{B} = \bigcup\mathscr{A}$. – Let $X \subset \bigcup\mathscr{A}$ be a set. Let ϱ consist of all $\langle x, Y\rangle$ where $x \in X$, $Y \in \mathscr{A}$, $x \in Y$. Clearly, $\mathbf{D}\varrho = X$, hence there exists a single-valued $\varphi \subset \varrho$ with $\mathbf{D}\varphi = X$. Obviously, $\mathbf{E}\varphi \subset \mathscr{A}$ is a set and therefore there exists a set $A \in \mathscr{A}$ such that $\varphi x \subset A$ for every $x \in X$, hence $X \subset A$.

4 D.4. Definition. A class A is called *minimally non-comprisable* if it is non-comprisable and there exists a monotone class $\mathscr{A}$ of sets such that $A = \bigcup\mathscr{A}$.

Remark. We have proved (4 D.1) that such classes exist. In the axiomatic system presented here, there is no possibility of proving that there exists a non-comprisable class which is not minimally non-comprisable; on the other hand, no means are known for proving that every non-comprisable class is minimally non-comprisable.

As for the term "minimally non-comprisable", its motivation is apparent from theorem 4 D.7.

4 D.5. *If A is a set, and B is an arbitrary non-comprisable class, then A is equipollent with a subset of B.*

Proof. Denote by F the class of all one-to-one relations for A into B. Clearly the suppositions of 4 C.6 are satisfied (observe that, for any $f \in F$, $f[A]$ is a set, hence $f[A] \neq B$). Therefore, there exists a $h \in F$ with $\mathbf{D}h = A$.

We now intend to show that the above assertion also holds if A is assumed to be a minimally non-comprisable class. This will be proved as a special case of a rather general theorem resembling (but by no means including) 4 C.6 and expressing (implicitly) a kind of an "over-transfinite induction" (for the concept of transfinite induction, etc., see Section 11).

4 D.6. Theorem. *Let F be a non-void monotonically additive class of single-valued relations. Let $\mathscr{M}$ be a monotone class of sets; put $M = \bigcup\mathscr{M}$. Let $\mathbf{D}f \subset M$ for every $f \in F$. Suppose that for any $f \in F$ such that $\mathbf{D}f \neq M$ there exists a $g \in F$ such that $g \supset f$, $g \neq f$ and, for some $X \in \mathscr{M}$, $\mathbf{D}g \supset X \supset \mathbf{D}f$. – Then there exists a single-valued relation Φ and a monotone class $\mathscr{M}^*$ of sets such that* (1) *$\mathbf{D}\Phi = M$,* (2) *for any $X \in \mathscr{M}^*$, $\Phi_X \in F$,* (3) *$\bigcup\mathscr{M}^* = M$.*

Proof. If $\mathscr{M}$, hence M is a set, we may apply 4 C.6 immediately and put $\mathscr{M}^* = (M)$. Therefore we shall suppose that $\mathscr{M}$ is non-comprisable; then $\mathbf{D}f \neq M$ for every $f \in F$. Let ϱ consist of all $\langle f, g\rangle$ where $f \in F$, $g \in F$, $f \subset g$, $f \neq g$ and, for some $X \in \mathscr{M}$, $\mathbf{D}f \subset X \subset \mathbf{D}g$ holds. By the assumptions made, $\mathbf{D}\varrho = F$. Now let $\varphi \subset \varrho$ be single-valued (see 4 B.2), $\mathbf{D}\varphi = \mathbf{D}\varrho = F$. Choose arbitrarily a relation $h \in F$. By 4 A.7 (with $\mathscr{A}$, A replaced by F, h) and 4 A.8 there is a class $B \subset F$ with the properties described in 4 A.7, 4 A.8; in particular, B is monotone, and it is easy to see that B is non-comprisable. Now let $\mathscr{M}^*$ denote the class of all $\mathbf{D}g$, $g \in B$, and let $\mathscr{X}$ denote the class of all $X \in \mathscr{M}$ such that $X \subset \mathbf{D}g$ for some $g \in B$. We are going to show that $\bigcup\mathscr{X} = M$. Suppose, on the contrary, that $\bigcup\mathscr{X} \neq M$; then by 4 D.3, there exists a $Y \in \mathscr{M}$ such that $\bigcup\mathscr{X} \subset Y$, i.e., $X \subset Y$ for every $X \in \mathscr{X}$. If $f \in B$, then $g = \varphi f \in B$, $\langle f, g\rangle \in \varrho$ and therefore there exists a $X \in \mathscr{M}$ with $\mathbf{D}f \subset X \subset \mathbf{D}g$. Hence $f \in B$ implies $\mathbf{D}f \subset Y$. From this we obtain at once that $\mathscr{M}^*$ is comprisable. However, this assertion clearly contradicts the fact that B is non-comprisable. This contradiction proves that the assumption $\bigcup\mathscr{X} \neq M$ is false. Therefore $\bigcup\mathscr{X} = M$, hence $\bigcup\mathscr{M}^* = M$. Now it is easy to see that the relation $\Phi = \bigcup B$ has the properties required.

4 D.7. Theorem. *If M is a set or a minimally non-comprisable class, and B is an arbitrary non-comprisable class, then M is equipollent with a subclass of B.*

Proof. The case in which M is a set has been considered in 4 D.5. Suppose that M is minimally non-comprisable, $M = \bigcup\mathscr{M}$ where $\mathscr{M}$ is a monotone (non-comprisable) class. Denote by $F$the class of all one-to-one relations f such that $\mathbf{D}f \subset M$, $\mathbf{E}f \subset B$. Clearly, F is monotonically additive. If $f \in F$, then, by 4 D.3, there exists a set $X \in \mathscr{M}$ such that $\mathbf{D}f \subset X$, $\mathbf{D}f \neq X$; by 4 D.5, there exists a one-to-one relation g' on $X - \mathbf{D}f$ into $B - \mathbf{E}f$. Putting $g = f \cup g'$, we have $g \in F$, $\mathbf{D}g = X$. This proves that all assumptions from 4 D.6 are satisfied. Therefore, there exists a single-valued Φ and a monotone class $\mathscr{M}^*$ with properties (1)–(3) indicated in 4 D.6. By property (1), the domain of Φ is equal to M. Since $\bigcup\mathscr{M}^* = M$, property (2) implies that Φ is a one-to-one relation with $\mathbf{E}\Phi \subset B$. This proves the theorem.

5. PRODUCTS

In this section, the cartesian product and some related notions will be considered. The cartesian product is, essentially, a generalization of the product $X \times Y$ of two classes X, Y already introduced in Section 1; roughly speaking, if certain sets X_a are given, their cartesian product consists of elements determined by their coordinates, the "a-th coordinate" being taken from X_a. Clearly, X^Y introduced in 1 E.8 is a special case of the cartesian product with all X_a equal to a given X, and a ranging over Y.

Besides this concept, we shall also consider the sum of sets, a concept which is dual, in a certain sense, to that of the product.

A. CARTESIAN PRODUCT

5 A.1. Definition. If A, B are classes, then the class $A \times B$, i.e. the class of all $\langle x, y\rangle$, where $x \in A$, $y \in B$ is called the *pair-product* or simply *product* of classes A and B. (This is a restatement of definition 1 C.9.)

By this definition, the meaning of $A \times (B \times C), (A \times B) \times C, A \times ((B \times C) \times D)$, etc., is already clear.

Convention. We shall write $A \times B \times C$ instead of $A \times (B \times C)$, $A \times B \times \times C \times D$ instead of $A \times (B \times (C \times D))$, etc.

Observe that this convention is in accordance with the notation $\langle a, b, c\rangle = = \langle a, \langle b, c\rangle\rangle$ introduced in 1 B.3.

Remark. Clearly, $(A \times B) \times C \neq A \times (B \times C)$, $A \times B \neq B \times A$ in general; in special cases, e.g. if some of the "factors" are void, equality may hold.

5 A.2. *If A, B, C are classes, then $A \times (B \cup C) = (A \times B) \cup (A \times C)$, $(B \cup C) \times A = (B \times A) \cup (C \times A)$, $A \times (B \cap C) = (A \times B) \cap (A \times C)$, $(B \cap C) \times \times A = (B \times A) \cap (C \times A)$, and similarly for the difference and the symmetric difference. If A, B, C, D are classes, A, B are non-void, then $A \times B \subset C \times D$ if and only if $A \subset C$, $B \subset D$.*

5 A.3. *If Y is a class, $\{Y_b\}$ is an indexed class of sets, then $A \times \bigcup\{Y_b\} = = \bigcup\{A \times Y_b\}$, $A \times \bigcap\{Y_b\} = \bigcap\{A \times Y_b\}$. If $\{X_a\}$, $\{Y_b\}$ are indexed classes of* *sets, then* $\bigcup\{X_a\} \times \bigcup\{Y_b\} = \bigcup_{a,b}\{X_a \times Y_b\}$, *and similarly for the intersection (provided* $\{X_a\}$, $\{Y_b\}$ *are non-void).*

These and similar results will be used in the sequel without reference.

Having introduced the product of two classes we can proceed, step by step, to the product of any actually given finite number of classes; however, this procedure cannot yield the product of infinitely many "factors". Therefore, another approach is adopted.

5 A.4. Definition. Let $\mathscr{X} = \{X_a \mid a \in A\}$ be a family of sets. The class of all families $x = \{x_a \mid a \in A\}$ such that $x_a \in X_a$ for every $a \in A$, will be called the *cartesian product* (sometimes only *product*) of the family $\mathscr{X}$ (or also, somewhat informally, the cartesian product of sets X_a, a running through A), and denoted by $\Pi\mathscr{X}$ or $\Pi\{X_a \mid a \in A\}$ or, briefly, $\prod_{a\in A} X_a$, or if the set of indexes A is clear from the context, then by $\prod_a X_a$ (or sometimes ΠX_a, provided there is no danger of ambiguity).

Definition. If $\mathscr{S}$ is a collection of sets, then the *cartesian product* (or simply *product*) of $\mathscr{S}$, denoted by $\Pi\mathscr{S}$, is, by definition, the product $\Pi\{X \mid X \in \mathscr{S}\}$.

In the sequel, we shall usually consider products of families of sets. Of course, all results can be easily carried over to products of collections of sets (the difference being purely formal).

Remarks. 1) Clearly, if X, A are sets, $X^A = \Pi\{a \to X \mid a \in A\}$ or, in other words, $X^A = \Pi\{X_a \mid a \in A\}$ where $X_a = X$ for every $a \in A$. – 2) It is evident that $\Pi\emptyset = (\emptyset) = X^\emptyset$. – 3) It is possible to introduce even the product of a "set of classes". Namely if A is a set and ϱ is a relation with domain A, consider the class $P(\varrho)$ consisting of all families $\{x_a \mid a \in A\}$ such that $a\varrho x_a$ for every $a \in A$. Now, if $\{X_a \mid a \in A\}$ is a family of sets and $\varrho = \{a \to x \mid x \in X_a, a \in A\}$, then clearly $P(\varrho) = \Pi\{X_a \mid a \in A\}$. – However, we shall not consider $P(\varrho)$ further.

5 A.5. Definition. If a is an element, then the relation $\{x \to y \mid x$ is a family, $\langle a, y\rangle \in x\}$ will be denoted by pr_a (and occasionally called the *projection relation associated with* a or simply the a-th *projection relation*).

Observe that pr_a is single-valued, $\mathbf{D}\mathrm{pr}_a$ is the class of all families whose domain (set of indexes) contains a, and $\mathbf{E}\mathrm{pr}_a$ is the universal class. Since pr_a is single-valued, the symbol $\mathrm{pr}_a x$ has a meaning for every $x \in \mathbf{D}\mathrm{pr}_a$.

Convention. If $x \in \mathbf{D}\mathrm{pr}_a$ we shall call $\mathrm{pr}_a x$ the a-th *coordinate* of x.

We shall now establish the fact that the pair-product (of two sets) and the cartesian product of a two-element family are "equivalent", in a sense to be specified. – The following assertion is clear:

5 A.6. *Let* A, B *be sets. Let* $T = (\alpha, \beta)$ *be a two-element set. Then the relation assigning to every* $\langle x, y\rangle \in A \times B$ *the family* $(\langle\alpha, x\rangle, \langle\beta, y\rangle)$ *is bijective for the pair-product* $A \times B$ *and the cartesian product* $\Pi(\langle\alpha, A\rangle, \langle\beta, B\rangle)$.

Convention. The above relation will be called (if the sets A, B and the elements α, β are given) the *canonical relation for* $A \times B$ *and the corresponding cartesian product.*

Observe that there is a similar relation for, say, $A \times B \times C$ and $\Pi\{X_\xi \mid \xi \in \in (\alpha, \beta, \gamma)\}$, where $X_\alpha = A$, $X_\beta = B$, $X_\gamma = C$, and so on. On the other hand, $A \times B$ and the above cartesian product are entirely different in the formal sense: elements of $A \times B$ are pairs, i.e. not sets, whereas elements of a cartesian product are always families, hence sets.

5 A.7. Convention. If no misunderstanding is likely to arise, we shall also use the symbol $A \times B$ for the product $\prod_{\xi = \alpha, \beta} X_\xi$ where $X_\alpha = A$, $X_\beta = B$, provided that either α, β are evident from the context or their choice is irrelevant for the reasoning in question (see also 5 A.11). Similarly for $A \times B \times C$, $A \times B \times C \times D$, and so on.

5 A.8. Convention. If $T = (a, b)$, $a \neq b$, A is a set, then the relation $\{\varphi \to \to \varphi^{-1}[a] \mid \varphi \in T^A\}$, which is bijective for T^A and $\exp A$ (see 1 E.9), will be called the *canonical relation for* T^A *and* $\exp A$.

Now we proceed to consider some properties of the cartesian product.

5 A.9. Theorem. *The product of a family of sets is a set. It is void if and only if some member of the family is void. (In symbols* $\Pi\{X_a \mid a \in A\} = \emptyset$ *if and only if* $X_a = \emptyset$ *for some* $a \in A$.)

Proof. Consider a family of sets $\{X_a \mid a \in A\}$. Put $X = \bigcup X_a$. Then X is a set, hence X^A is a set. Clearly, $\Pi X_a \subset X^A$. As for the second assertion, "if" is clear, "only if" is asserted in 4 B.3.

Remark. As just shown, the above theorem follows at once from the Axiom of Choice. On the other hand, this axiom in its strong form (see 4 B.1) given in this book can hardly be deduced from the above theorem; nevertheless, its weaker form obtained by replacing "class" by "set" in 4 B.1 is easily deduced from the theorem in question.

5 A.10. *If* X_a *are non-void, then* $\Pi\{X_a \mid a \in A\} \subset \Pi\{Y_a \mid a \in A\}$ *if and only if* $X_a \subset Y_a$ *for every* $a \in A$*; in particular,* $\Pi X_a = \Pi Y_a$ *if and only if* $X_a = Y_a$ *for every* $a \in A$. – This is clear, since $\mathrm{pr}_a[\Pi X_a] = X_a$.

5 A.11. *Let a relation* φ *be bijective for sets* A *and* B*. Let a family of sets* $\{X_b \mid b \in \in B\}$ *be given. Then the relation assigning to every* $x = \{x_b\}$ *from the product* $\Pi\{X_b \mid b \in B\}$ *the element* $\{x_{\varphi a}\}$ *from the product* $\Pi\{X_{\varphi a} \mid a \in A\}$ *is bijective for these products.*

The proof is easy and is left to the reader.

Convention. The above relation will be called the *canonical relation (under* φ*) for the products* in question.

5 A.12. An important remark of a general kind is in place here. The above proposition expresses the "commutativity" of the cartesian product, more precisely the fact that a cartesian product of a family "does not depend", essentially, either on the set of indexes or on the manner in which sets are assigned to indexes. The sets $\Pi X_b, \Pi X_{\varphi a}$ are distinct, in general, but there is a naturally defined (canonical) bijective relation for them.

Such a situation occurs in many other instances (cf. 5 B.2, 5 B.5, 5 B.6, etc.). In such cases, statements and properties relating to one of the sets (or other objects) in question are, as a rule, transformed almost automatically, by means of the appropriate canonical relation, to statements and properties relating to the other one. Thus, with due care, we can use in our reasoning any of such objects or even pass freely from one of them to another. We shall often proceed in this way, sometimes even without special comment.

5 A.13. *Let $\{B_a \mid a \in A\}$ be a disjoint family of sets, let $B = \bigcup B_a$. Let $\{X_b \mid b \in B\}$ be a family of sets. Then the relation assigning to every $\{x_b \mid b \in B\}$ the element $\{\{x_b \mid b \in B_a\} \mid a \in A\}$ is a bijective relation for $\Pi\{X_b \mid b \in B\}$ and $\Pi\{\Pi\{X_b \mid b \in B_a\}\}$, that is for $\prod\limits_{b\in B} X_b$ and $\prod\limits_{a\in A} \prod\limits_{b\in B_a} X_b$.*

The proof is immediate and is left to the reader.

Convention. The above relation will be called *canonical for $\prod\limits_{b\in B} X_b$ and $\prod\limits_{a\in A} \prod\limits_{b\in B_a} X_b$.*

Remark. The above propositions express a kind of "associativity" of the cartesian product. An associativity in the strict formal sense (cf. the corresponding propositions for $\bigcup$ and $\bigcap$ in Section 2) does not hold of course (e.g. the set of indexes of any family belonging to $\prod\limits_{b\in B} X_b$ is precisely B, that of any family belonging to $\prod\limits_{a\in A} \prod\limits_{b\in B_a} X_b$ is precisely A).

We are now going to introduce the notation $A \times [\mathscr{B}]$, $[\mathscr{A}] \times [\mathscr{B}]$ and so on in a sense analoguous to that of $A \cup [\mathscr{B}]$, $[\mathscr{A}] \cup [\mathscr{B}]$ etc. (see Section 2). The meaning of $A \times [\mathscr{B}]$ is clear if we conceive $A \times X$ as the value (at X) of the relation $\{X \to A \times \times X\}$, A fixed; this relation may be denoted by $A \times$, and then $A \times [\mathscr{B}]$ is well defined (see 1 B.7). Nevertheless we shall define $A \times [\mathscr{B}]$, and other similar symbols, explicitly.

5 A.14. Definition. If A is a set, $\mathscr{B}$ a class of sets, then $A \times [\mathscr{B}]$ is the class of all $A \times B$, $B \in \mathscr{B}$, and $[\mathscr{B}] \times A$ is the class of all $B \times A$, $B \in \mathscr{B}$. If $\mathscr{A}$, $\mathscr{B}$ are classes of sets, then $[\mathscr{A}] \times [\mathscr{B}]$ is the set of all $A \times B$, $A \in \mathscr{A}$, $B \in \mathscr{B}$.

Example. If $\mathscr{A} = \mathscr{B}$ is the class of all singletons, then $[\mathscr{A}] \times [\mathscr{B}]$ is the class of all singletons of the form $(\langle x, y\rangle)$, whereas $\mathscr{A} \times \mathscr{B}$ is the class of all pairs of the form $\langle (x), (y)\rangle$.

5 A.15. Definition. If $\mathbf{A}$ is a set of families, then $\Pi[\mathbf{A}]$ is the set of all ΠA, $A \in \mathbf{A}$, $\exp[\mathbf{A}]$ is the set of all $\exp A$, $A \in \mathbf{A}$, and $\Sigma[\mathbf{A}]$ is the set of all ΣA, $A \in \mathbf{A}$ (see 5 B.1).

We shall use these symbols (especially those in 5 A.15) only rarely and do not, therefore, consider their use or properties in any detail.

B. SUM

We are now going to introduce the concept of the sum (in the sense of set theory) of a family of sets. Relatively less important than that of the cartesian product, it is nevertheless very useful (and in many respects far simpler than that of the product).

5 B.1. Definition. If $\mathscr{X} = \{X_a \mid a \in A\}$ is a family of sets, then the set of all $\langle a, x\rangle$, $x \in X_a$, will be called the (set-theoretic) *sum* of the family $\mathscr{X}$ and denoted $\Sigma\mathscr{X}$ or $\Sigma\{X_a \mid a \in A\}$ or simply $\sum_{a\in A} X_a$ (sometimes even ΣX_a if the set of indexes is clear from the context).

Remarks. 1) Observe that, if $\mathscr{X}$ is a family of sets, $\Sigma\mathscr{X} = \in^{-1} \circ \mathscr{X}$. – 2) If $\{X_a\}$ is disjoint, then ΣX_a is, essentially, the same as $\bigcup X_a$. In general, ΣX_a can be described as obtained by replacing every X_a with a replica of it, in such a way as to get disjoint sets, and then forming the union. – 3) The sum can be defined also for an indexed class of sets (not necessarily comprisable); we shall, however, not investigate these questions.

Convention. If $\mathscr{S}$ is a collection of sets, then $\Sigma\mathscr{S}$ will mean $\Sigma \mathrm{J}_{\mathscr{S}}$ (cf. 1 B.10).

5 B.2. *If $\{X_a \mid a \in A\}$ is a family of sets, then $\{\langle a, x\rangle \to x \mid a \in A,\ x \in X_a\}$ is a single-valued relation on $\Sigma\{X_a \mid a \in A\}$ onto $\bigcup\{X_a \mid a \in A\}$. It will be called the* natural mapping-relation *for ΣX_a and $\bigcup\{X_a\}$. This relation is one-to-one (hence bijective for ΣX_a and $\bigcup X_a$) if and only if $\{X_a\}$ is disjoint. In this case it will be called* canonical.

Proof. Perhaps only the last assertion requires a proof. If $\{X_a\}$ is disjoint, then there exist no distinct $a \in A$, $a' \in A$, and x such that $x \in X_a$, $x \in X_{a'}$; thus if $\langle a, x\rangle \in \in \Sigma X_a$, $\langle a', y\rangle \in \Sigma X_a$, and $a \neq a'$, then $x \neq y$. – If the canonical relation is one-to-one, then x and x' are distinct whenever $\langle a, x\rangle$, $\langle a', x'\rangle$ are distinct; hence if a, a' are distinct, no x exists with $x \in X_a$, $x \in X_{a'}$.

5 B.3. Definition. Let a be an element. The relation assigning to every x the element $\langle a, x\rangle$ will be denoted by inj_a and occasionally called the *injection relation associated with a* or simply the a-th *injection relation.*

Observe that $\Sigma\{X_a \mid a \in A\} = \bigcup \mathrm{inj}_a [X_a]$.

5 B.4. *If $\{X_a \mid a \in A\}$ is a family of sets, then ΣX_a is a set. It is void if and only if, for every $a \in A$, X_a is void (hence, in particular, if $A = \emptyset$).* – This is clear.

Remark. Clearly, $\Sigma\{X_a \mid a \in A\} \subset \Sigma\{Y_a \mid a \in A\}$ if and only if $X_a \subset Y_a$ for every $a \in A$.

5 B.5. *Let φ be a bijective relation for sets A and B. Let a family of sets $\{X_b \mid b \in B\}$ be given. Then the relation assigning to every $\langle a, x\rangle \in \Sigma\{X_{\varphi a} \mid a \in A\}$ the element $\langle \varphi a, x\rangle$ from $\Sigma\{X_b \mid b \in B\}$ is a bijective relation (called* canonical*) for $\sum_{a\in A} X_{\varphi a}$ and $\sum_{b \in B} X_b$.* – This is clear .

5 B.6. *Let $\{B_a \mid a \in A\}$ be a disjoint family of sets, let $B = \bigcup\{B_a\}$. Let $\{X_b \mid b \in B\}$ be a family of sets. Then the relation assigning to every $\langle b, x\rangle \in \Sigma\{X_b \mid b \in B\}$ the element $\langle a, \langle b, x\rangle\rangle$ where $b \in B_a$ (which determines the element a uniquely) is a bijective relation for $\sum_{b\in B} X_b$ and $\sum_{a\in A}\sum_{b\in B_a} X_b$ and is called* canonical *(for $\sum_{b\in B} X_b$ and $\sum_{a\in A}\sum_{b\in B_a} X_b$).*

The easy proof is left to the reader.

5 B.7. *Let $\{X_a \mid a \in A\}$ be a family of sets; let Y be a set. Then the relation assigning to every $\langle\langle a, x\rangle, y\rangle \in (\Sigma X_a) \times Y$ the element $\langle a, \langle x, y\rangle\rangle \in \Sigma(X_a \times Y)$ is a bijective relation for $(\Sigma X_a) \times Y$ and $\Sigma(X_a \times Y)$ and is called* canonical *(for $(\Sigma X_a) \times Y$ and $\Sigma(X_a \times Y)$).*

The proof is left to the reader.

5 B.8. Remark. We have established, in 5 A.6, 5 A.8, 5 A.11, 5 A.13, 5 B.2, 5 B.5, 5 B.6, certain canonical relations (cf. in this connection remark 5 A.12). These relations are bijective, hence the sets in question are equipollent. Explicitly, we have (abbreviated notation is used): (a) $\exp A$ and T^A, T being a two-element set, are equipollent; (b) $\prod_{b\in B} X_b$ and $\prod_{a\in A} X_{\varphi a}$ are equipollent, provided φ is bijective for A and B; (c) $\prod_{b\in B} X_b$ and $\prod_{a\in A}\prod_{b\in B_a} X_b$ are equipollent, $\{B_a\}$ being a disjoint cover of B, (d) $\bigcup X_a$ and ΣX_a are equipollent, if $\{X_a\}$ is disjoint; (e) $\sum_{b\in B} X_b$ and $\sum_{a\in A} X_{\varphi a}$ are equipollent, φ being bijective for A and B; (f) $\sum_{b\in B} X_b$ and $\sum_{a\in A}\sum_{b\in B_a} X_b$ are equipollent, $\{B_a\}$ being a disjoint cover of B.

C. RELATIONAL PRODUCT

If ϱ, σ are relations, then their pair-product in the sense of 5 A.1 consists of all pairs $\langle\langle a, b\rangle, \langle c, d\rangle\rangle$ where $\langle a, b\rangle \in \varrho$, $\langle c, d\rangle \in \sigma$. If $\{\varrho_a \mid a \in A\}$ is a family of relations, then the cartesian product consists of all $\{\langle x_a, y_a\rangle \mid a \in A\}$ where $\langle x_a, y_a\rangle \in \varrho_a$; hence, this product is not a relation. Such a situation, of course, is not satisfactory; intuitively, the product of two relations φ, ψ (for the sake of simplicity, single-valued ones) should be a relation assigning $\langle \varphi x, \psi y\rangle$ to every $\langle x, y\rangle \in \mathbf{D}\varphi \times \mathbf{D}\psi$, and the product of $\{\varrho_a\}$ should (provided ϱ_a are single-valued) assign $\{\varrho_a x_a\}$ to $\{x_a\} \in \Pi\{\mathbf{D}\varrho_a\}$. We are now going to introduce the appropriate definitions.

5 C.1. Definition. If ϱ, σ are relations, then the relation $\{\langle x, y\rangle \to \langle u, v\rangle \mid \langle x, u\rangle \in \varrho, \langle y, v\rangle \in \sigma\}$ is called the *relational pair-product* of ϱ and σ and denoted by $\varrho \times_{\mathrm{rel}} \sigma$; as a rule, we shall write simply $\varrho \times \sigma$ instead of $\varrho \times_{\mathrm{rel}} \sigma$ and call it, in short, the *product* of ϱ and σ (thus, if ϱ and σ are relations, $\varrho \times \sigma$ shall always mean the relational product unless it is clear from the context or stated explicitly that the product in the sense of 5 A.1 is considered).