Ivo Vrkoč

An optimal control problem for Ito stochastic equations

In: Miloš Ráb and Jaromír Vosmanský (eds.): Proceedings of Equadiff III, 3rd Czechoslovak Conference on Differential Equations and Their Applications. Brno, Czechoslovakia, August 28 - September 1, 1972. Univ. J. E. Purkyně - Přírodovědecká fakulta, Brno, 1973. Folia Facultatis Scientiarum Naturalium Universitatis Purkynianae Brunensis. Seria Monographia, Tomus I. pp. 205--207.

Persistent URL: http://dml.cz/dmlcz/700059

AN OPTIMAL CONTROL PROBLEM FOR ITÔ STOCHASTIC EQUATIONS

by IVO VRKOČ

Let an Itô equation

$$\mathrm{d}x = a(t, x)\,\mathrm{d}t + B(t, x)\,\mathrm{d}w(t) \tag{1}$$

be given in a region $Q = (0, L) \times (x_1, x_2)$ where L is a positive number, (x_1, x_2) is an open interval and $w(t)$ is a Wiener process ($Ew(t) = 0$, $Ew^2(t) = t$, E is the mathematical expectation). Assume that

i) *the functions* $a(t, x)$, $B(t, x)$ *are defined in the closure* $\bar{Q}$,

ii) $a(t, x)$, $B(t, x)$ *are Lipschitz continuous in* x *and Hölder continuous in* t,

iii) $B(t, x) \neq 0$ *in* $\bar{Q}$,

iv) *an initial value* x_0 *is a random variable which is stochastically independent of increments of* $w(t)$ *and is situated in* (x_1, x_2) (*i.e.* $P(x_0 \in (x_1, x_2)) = 1$).

Denote $S = \{[t, x_i];\ 0 \leqq t \leqq L,\ i = 1, 2\}$. *The solution* $x(t, x_0)$ *of* (1) *with adhesive barrier* S is defined as usual i.e. $x(t) = x^*(t)$ for $t \leqq \tau$, $x(t) = x^*(\tau)$ for $t > \tau$ where $x^*(t)$ is the solution of (1) (provided that $a(t, x)$, $B(t, x)$ are extended on the whole plane) with initial value x_0 and τ is the moment of the first exit of $x^*(t)$ from (x_1, x_2). Under the assumptions i) to iv) the solution with adhesive barrier exists and is unique (in the sense of equivalent processes—see [1]).

Denote $P(B, a, x_0, Q)$ *the probability that* $x(t, x_0)$ *leaves the interval* (x_1, x_2) *at least once on the time-interval* $(0, L)$ i.e.

$$P(B, a, x_0, Q) = P\{\exists\{\tau : x(\tau, x_0) \notin (x_1, x_2), \quad \tau \leqq L\}\}.$$

Definition 1. *A function* $B(t, x)$ *is called maximal with respect to a function* $a(t, x)$ *and to a region* Q ($Q = (0, L) \times (x_1, x_2)$) if the functions $a(t, x)$, $B(t, x)$ fulfil i), ii), iii) and if

$$P(B, a, x_0, Q) = \max_{B'} P(B', a, x_0, Q)$$

for all initial values x_0 fulfilling iv) where $B'(t, x)$ can be every function fulfilling i), ii), iii) and $|B'(t, x)| \leqq |B(t, x)|$.

Let $u(t, x)$ be a bounded solution of the parabolic equation

$$\frac{\partial u}{\partial t} = a(L - t, x)\frac{\partial u}{\partial x} + \frac{1}{2} B^2(L - t, x)\frac{\partial^2 u}{\partial x^2} \tag{2}$$

fulfilling the initial condition

$$u(0, x) = 0 \quad \text{for} \quad x \in (x_1, x_2) \tag{3}$$

and the boundary condition

$$u(t, x_i) = 1 \quad \text{for} \quad t > 0,\ i = 1, 2. \tag{4}$$

Under the conditions i), ii), iii) the solution $u(t, x)$ exists and is unique in the class of bounded functions (see [2], [3]).

The maximal functions $B(t, x)$ can be characterized by means of properties of the solutions $u(t, x)$.

Theorem 1. *Let $a(t, x)$, $B(t, x)$ fulfil conditions* i) *to* iii). *The function $B(t, x)$ is maximal with respect to the function $a(t, x)$ and to the region Q if and only if the bounded solution $u(t, x)$ of* (2) *fulfilling* (3) *and* (4) *is convex as a function of x in Q.*

Since it is very difficult to verify the condition given by this Theorem more explicit conditions are formulated in the following theorems. The simplest case is when $a(t, x)$ is a linear function in x.

Theorem 2. *Let functions $\alpha(t)$, $\beta(t)$ be defined and Hölder continuous on $\langle 0, L \rangle$. Let $B(t, x)$ fulfil conditions* i) *to* iii). *If $a(t, x) = \alpha(t) + x\beta(t)$ fulfils $a(t, x_1) \geqq 0$ and $a(t, x_2) \leqq 0$, then $B(t, x)$ is maximal with respect to the function $a(t, x)$ and to the region Q.*

Remark 1. The conditions $a(t, x_1) \geqq 0$, $a(t, x_2) \leqq 0$ are also necessary conditions.

In the following it will be assumed that coefficients a, B do not depend on t. We shall say that *the coefficients of* (1) *are symmetric if $x_1 = -x_2$, $a(x) = -a(-x)$ and $B(x) = B(-x)$.*

Theorem 3. *Let coefficients of* (1) *fulfil* i) *to* iii). *If the coefficients are symmetric and fulfil $a(t, x) \leqq 0$ for $x \geqq 0$, then $B(x)$ is maximal with respect to the function $a(x)$ and Q.*

Secondly, the case of nonnegative $a(x)$ will be treated.

Theorem 4. *Let functions $a(x)$, $B(x)$ fulfil conditions* i) *to* iii). *If $0 < K_1 \leqq B^2(x) \leqq K_2$ (K_1, K_2 are constants) and*

$$0 \leqq a(x) \leqq \frac{x_2 - x}{2} \frac{K_2}{(x_2 - x_1)^2} \left[\arcsin \sqrt{\frac{K_1}{K_2}}\right]^2 \quad in\ (x_1, x_2), \tag{5}$$

then the function $B(x)$ is maximal with respect to the function $a(x)$ and to the region Q.

Now we shall deal with the case when $a(x)$ can change the sign.

Theorem 5. *Let functions $a(x)$, $B(x)$ fulfil conditions* i) *to* iii). *If $0 < K_1 \leqq B^2(x) \leqq K_2$, $a(x_2) \leqq 0 \leqq a(x_1)$ and*

$$a(x) - a(x_2) \leqq \frac{x_2 - x}{2} \frac{K_2}{(x_2 - x_1)^2} \left[\arcsin\left(e^{2(\delta-\gamma)} \sqrt{\frac{K_1}{K_2}}\right)\right]^2,$$

$$a(x) - a(x_1) \geqq -\frac{x - x_1}{2} \frac{K_2}{(x_2 - x_1)^2} \left[\arcsin\left(e^{2(\delta-\gamma)} \sqrt{\frac{K_1}{K_2}}\right)\right]^2$$

where

$$\delta \leqq \min_x \int_{x_2}^{x} \frac{a(\xi)}{B^2(\xi)}\,d\xi, \qquad \gamma \geqq \max_x \int_{x_2}^{x} \frac{a(\xi)}{B^2(\xi)}\,d\xi,$$

then $B(x)$ is maximal with respect to the function $a(x)$ and to the region Q.

Remark 2. The conditions of the two last theorems are independent of L. It implies that if B is maximal with respect to $a(x)$ and to a region $Q^* = (0, L^*) \times (x_1, x_2)$, then it is maximal with respect to $a(x)$ and to a region $Q = (0, L) \times (x_1, x_2)$ for every $L > 0$.

The next theorem states that there actually exists a couple of functions $a(x)$, $B(x)$ such that $B(x)$ is not maximal with respect to $a(x)$ and especially for $B(x) \equiv 1$ there exists a function $a(x)$ (fulfilling i) to iii)) such that $B(x) \equiv 1$ is not maximal with respect to the function $a(x)$. This theorem deals also with the necessity of condition (5). Due to the statement of the following theorem condition (5) cannot be omitted.

Definition 2. *A function $\varphi(x, K_1, K_2)$ has property (M)* if it is defined and continuous and positive in the domain $0 \leqq x \leqq 1$, $0 < K_1 \leqq K_2$ and if a function $B(x)$ is maximal with respect to a function $a(x)$ and to $Q = (0,1) \times (0,1)$ under conditions that $K_1 \leqq B^2(x) \leqq K_2$, $0 \leqq a(x) \leqq \varphi(x, K_1, K_2)$.

For example the function in the right-hand side of (5) has property (M).

Theorem 6. *Let a function $\varphi(x, K_1, K_2)$ have property (M), then*

$$\varphi(x, 1, 1) \leqq \frac{1}{2} \min_t \left[\frac{\partial^2 v}{\partial x^2}(t, x) \Big/ \frac{\partial v}{\partial x}(t, x) \right] \quad \text{for} \quad x > \frac{1}{2}, \qquad t \in (0, 1),$$

where $v(t, x)$ is the bounded solution of $\partial v / \partial t = \frac{1}{2}\, \partial^2 v / \partial x^2$ fulfilling $v(0, x) = 0$ for $0 < x < 1$ and $v(t, 0) = v(t, 1) = 1$ for $t > 0$.

REFERENCES

[1] I. Vrkoč: *Some maximum principles for stochastic equations.* Czech. Math. J. V. 19 (94), 1969, 569–604.

[2] I. Vrkoč: *Some explicit conditions for maximal local diffusions in one-dimensional case.* Czech. Math. J. V. 21 (96), 1971, 236–256.

[3] G. Schleinkofer: *Die erste Randwertaufgabe und das Cauchy-Problem für parabolische Differentialgleichungen mit unstetigen Anfangswerten.* Mathematische Zeitschrift 1969, B 111, 87–97.

Author's address:
Ivo Vrkoč
Mathematical Institute, Czechoslovak Academy of Sciences
Žitná 25, Praha 1
Czechoslovakia