OSCILLATION IN LINEAR DIFFERENTIAL EQUATIONS

by F. NEUMAN

1. Here a certain geometrical approach to global problems, in particular to oscillation, in the theory of linear differential equations of the n-th order will be described.

Studying zeros of solutions, GEORGE D. BIRKHOFF [1] in 1910 introduced a geometrical interpretation of solutions; he dealt, however, with curves in the projective plane corresponding to solutions of linear differential equations of the 3rd order only.

More closely to our approach there are investigations of H. GUGGENHEIMER [5]. He considered, however, only periodic solutions of linear differential equations of the n-th order.

Our approach is applicable both to general differential equations and those of arbitrary order. It makes possible to see the whole situation in behaviour of solutions for all differential equations of a given order and not only to consider separate simple examples (often with constant coefficients) as a motivation for a possible form of theorems. The proposed method is useful mainly for solving problems concerning existence or non-existence of differential equations with solutions having some prescribed properties. The investigation of oscillation problems can be done for all equivalent differential equations at the same time. There is also suggested a new canonical form to which every linear differential equation of the n-th order with only continuous coefficients can be reduced on its whole interval of definition, without any restriction on the interval of transformation or on smoothness of coefficients.

In the rank of the approach a geometrical interpretation of adjoint equations is also given.

The proposed method of investigation is not restricted to geometry since some concepts and theorems from the theory of functional equations (in [13]), topology and theory of dynamical systems ([10] and [18] used in [14]) play their important role.

The proposed approach is described in [12] (preliminary communication was published in [11]). Here we are going to give some basic ideas of the method.

2. Hence, consider

$$L_n(y) \equiv y^{(n)} + a_1(x)\, y^{(n-1)} + \dots + a_n(x)\, y = 0 \text{ on } I, \tag{1}$$

where I is an open interval (bounded or unbounded), $a_i \in C^\circ(I)$ for $i = 1, \dots, n$. $C^n(I)$ denotes the class of all continuous functions on I having here continuous derivatives up to and including the n-th order, $n \geqq 0$.

A differential equation is determined knowing its n linearly independent solutions $y_1, \dots, y_n$, simply written in vector form as $\boldsymbol{y}$ and called a fundamental solution. Such the equation will be denoted as $L_n[\boldsymbol{y}] = 0$. Those solutions are all of the class $C^n(I)$ that will be briefly expressed as $\boldsymbol{y} \in C^n(I)$. Wronski determinant of $\boldsymbol{y}(\in C^{n-1}(I))$ will be $W_n[\boldsymbol{y}]$.

Two differential equations $L_n^{(1)}(y) = 0$ on $I^{(1)}$ and

$L_n^{(2)}(z) = 0$ on $I^{(2)}$ are called (positively) *equivalent*

if there exist two functions $f: I^{(2)} \to \mathbf{R}$ and $\varphi : I^{(2)} \to I^{(1)}$ such that $f \in C^n(I^{(2)})$, $\varphi \in C^n(I^{(2)})$, $f \neq 0$ on $I^{(2)}$, $\mathrm{d}\varphi/\mathrm{d}t > 0$ on $I^{(2)}$ and

$$z(t) = f(t) \, . \, y(\varphi(t))$$

is a solution of $L^{(2)}(z) = 0$ on $I^{(2)}$ for every solution y of $L^{(1)}(y) = 0$ on $I^{(1)}$. Or, in other words,

$$\boldsymbol{z}(t) = f(t) \, . \, \boldsymbol{y}(\varphi(t)) \tag{2}$$

is a fundamental solution of $L_n^{(2)}(z) = 0$ if $\boldsymbol{y}$ is a fundamental solution of $L_n^{(1)}(y) = 0$.

From (2) it can be seen that two equivalent equations have the same behaviour of zeros of their solutions.

Hence, trying to find a suitable representation for the whole class of equivalent equations, we have two functions f and φ at our disposal.

Fortunately, it can be shown (see [16] or [12]) that for our f and φ

$$W_n[f\boldsymbol{y}] \neq 0 \text{ iff } W_n[\boldsymbol{y}] \neq 0 \text{ and } W_n[\boldsymbol{y}(\varphi)] \equiv W_n[\boldsymbol{z}] \neq 0 \text{ iff } W_n[\boldsymbol{y}] \neq 0.$$

Hence, instead of an equation $L^n[\boldsymbol{y}] = 0$ on I we may consider its equivalent equation, fundamental solution $\boldsymbol{z}$ of which is the central projection of $\boldsymbol{y}$ on the unit sphere $S_{n-1} \subset E_n$ (n-dimensional Euclidean vector space), i.e.,

$$\boldsymbol{z}(x) \overset{\text{df.}}{=} \boldsymbol{y}(x)/|\,\boldsymbol{y}(x)\,|,$$

where $f(x) = 1/|\,\boldsymbol{y}(x)\,|$, $\varphi = \mathrm{id}_I$, and $|\,\boldsymbol{y}(x)\,| = \sqrt{\sum_i y_i^2(x)}$.

Having $\boldsymbol{z}(x)$ on the unit sphere, we may introduce a new parameter s requiring the unit tangent vector at every point, i.e.,

$$|\,\mathrm{d}\boldsymbol{z}(\varphi(s))/\mathrm{d}s\,| = |\,\mathrm{d}\boldsymbol{u}(s)/\mathrm{d}s\,| = 1.$$

This is always possible since φ determined in this way satisfies our conditions $\varphi \in C^n$, $\mathrm{d}\varphi/\mathrm{d}s > 0$.

Hence we have constructed $L_n[\boldsymbol{u}] = 0$ on J equivalent to the previous $L_n[\boldsymbol{y}] = 0$ on I, where $|\,\boldsymbol{u}(s)\,| = 1$, $|\,\mathrm{d}\boldsymbol{u}(s)/\mathrm{d}s\,| = 1$ for $s \in J$.

Such a fundamental solution $\boldsymbol{u}(s)$ can be considered as a curve on S_{n-1} and the corresponding Frenet formulae have the form

$$
\begin{aligned}
\boldsymbol{u}'(s) \overset{\text{df.}}{=} \boldsymbol{u}_1'(s) &= \qquad\quad \boldsymbol{u}_2(s) \\
\boldsymbol{u}_2'(s) &= -\boldsymbol{u}_1(s) \qquad + \alpha_2(s)\,\boldsymbol{u}_3(s) \\
&\dots \\
\boldsymbol{u}_n'(s) &= \qquad\qquad\qquad -\alpha_{n-1}(s)\,\boldsymbol{u}_{n-1}(s),
\end{aligned}
\tag{3}
$$

where $\boldsymbol{u}_1 \equiv \boldsymbol{u}, \boldsymbol{u}_2, \dots, \boldsymbol{u}_n$ form an orthogonal system of unit vectors at each $s \in J$, $\alpha_i \in C^{n-i}(J)$ and could be chosen such that all $\alpha_i \geqq 0$. Due to $W_n[\boldsymbol{y}] \neq 0 \Rightarrow W_n[\boldsymbol{u}] \neq 0$ we have also $\alpha_i > 0$ for all i and all s.

3. Using the Frener formulae (3) we may naturally derive *canonical forms* for equivalent equations of arbitrary order. For $n = 2$:

$$
\begin{aligned}
\boldsymbol{u}_1'(s) &= \qquad \boldsymbol{u}_2(s) \\
\boldsymbol{u}_2'(s) &= -\boldsymbol{u}_1(s)
\end{aligned}
\qquad \text{on} \quad J
$$

that gives
$$\boldsymbol{u}_1''(s) = \boldsymbol{u}_2'(s) = -\boldsymbol{u}_1(s)$$
or
$$u'' + u = 0 \quad \text{on} \quad J \tag{4}$$

as a canonical form for all linear differential equations of the 2nd order. We see that the form depends only on the interval of definition J, whereas the equation is always the same. That gives the known result that every two linear differential equations of the 2nd order are locally equivalent. If we want, however, to transform them on their whole intervals of definitions, then the condition on the corresponding intervals J is only another form of O. Borůvka's result of the character of such equations [2].

For $n = 3$ we get

$$u''' - \frac{\alpha_2'(s)}{\alpha_2(s)} \,.\, u'' + \left(1 + \alpha_2^2(s)\right) u' - \frac{\alpha_2'(s)}{\alpha_2(s)} u = 0 \quad \text{on} \quad J$$

and similarly for higher orders.

For a given n the canonical form depends on an interval of definition J and on $n - 2$ positive functions $\alpha_i \in C^{n-i}(J)$, $i = 2, \dots, n - 1$. The form is (generally) not unique for a given class of equivalent equations since we might have started from another fundamental solution than our $\boldsymbol{y}$.

Let us emphasize, however, the fact that for the transformation to the canonical form there are neither restrictions on the interval of definition nor on smoothness of coefficients. That allows us to study global properties, especially oscillation, only considering the mentioned canonical forms.

Coming to *adjoint equation*, let us recall that adjoint equation ${}_aL_n(y) = 0$ on I to $L_n(y) = 0$ on I has a fundamental solution (see, e.g. [6])

$$(\boldsymbol{y} \times \boldsymbol{y}' \times \dots \times \boldsymbol{y}^{(n-2)}) / W_n[\boldsymbol{y}]$$

if $\boldsymbol{y}$ is a fundamental solution of $L_n(y) = 0$ on I. Here $\boldsymbol{y} \times \dots \times \boldsymbol{y}^{(n-2)}$ denotes the vector product of the $(n - 1)$ vectors $\boldsymbol{y}, \dots, \boldsymbol{y}^{(n-2)}$.

Having in mind the definition of equivalence, ${}_aL_n(y) = 0$ on I is then equivalent to $L_n[y \times \ldots \times y^{(n-2)}] = 0$ on I.

4. We have seen that every linear differential equation $L_n(y) = 0$ on I can be represented by its fundamental solution $\boldsymbol{y}$ on I considered as a curve in E_n, i.e., with $y_i(x)$ as its coordinates. This $\boldsymbol{y} \in C^n(I)$ and $W_n[\boldsymbol{y}] \neq 0$ on I. Any of equivalent equations with the $L_n[\boldsymbol{y}] = 0$ is of the form $L_n[\boldsymbol{z}] = 0$ on a suitable interval, where $\boldsymbol{z} = f \cdot \boldsymbol{y}(\varphi)$. Especially $\boldsymbol{z}$ could be a curve on the unit sphere S_{n-1} with or without a particular parametrization described above.

The curve formed by the vector product of $\boldsymbol{y}, \boldsymbol{y}', \ldots, \boldsymbol{y}^{(n-2)}$ corresponds to an equation that belongs to the class of equations equivalent to the adjoint equation ${}_aL_n(y) = 0$ on I. The projection of the curve on the unit sphere with a respective change of parameter can be also considered.

Having in mind this geometrical representation of solutions we come to an interpretation of zeros of these solutions.

For this purpose, hyperplanes in E_n will be only those passing the origin: for $\boldsymbol{c} \in E_n$, $\boldsymbol{c} = (c_1, \ldots, c_n) \neq \boldsymbol{0}$ let $H(\boldsymbol{c}) \overset{\text{df.}}{=} \{\boldsymbol{v} \in E_n ;\ \boldsymbol{v} \cdot \boldsymbol{c} = \sum_i v_i c_i = 0\}$.

Hyperplanes $H(\boldsymbol{c}_1), \ldots, H(\boldsymbol{c}_j)$ will be called independent if the rank of the matrix $(\boldsymbol{c}_1, \ldots, \boldsymbol{c}_j)$ is j.

It can be proved ([12]) that:

"*To every linear differential equation $L_n(y) = 0$ on I there correspond*

a curve $\boldsymbol{u}(s)$ for $s \in J$,

a mapping $\varphi : J \to I$, and

a linear correspondence h between all solutions of $L_n(y) = 0$ and all hyperplanes $H(\boldsymbol{c})$ in E_n such that to linearly independent solutions y_1 and y_2 there correspond independent hyperplanes $h(y_1)$ and $h(y_2)$, and a solution y has a k-multiple zero ($0 \leqq k \leqq n - 1$) at x_0 iff the corresponding hyperplane $h(y)$ and our curve $\boldsymbol{u}(s)$ have the contact of $(k - 1)$-order at s_0, $\varphi(s_0) = x_0$.

The curve $\boldsymbol{u}$ is any of fundamental solutions of any of equations equivalent to the given $L_n(y) = 0$ on I and hence it may be the same for the whole class of equivalent equations. It may or may not be chosen such that it lies on the unit sphere S_{n-1}."

Let us illustrate the theorem on examples of known results.

Example 1. Separation theorem for linear differential equations of the 2nd order.

Any such equation $y'' + a_1(x)\, y' + a_2(x)\, y = 0$ on I can be transformed to $u'' + u = 0$ on J. Take a fundamental solution of the last, e.g., $\boldsymbol{u}(s) = (\cos s, \sin s)$, $s \in J$. Now, hyperplanes here are straight lines passing the origin, and two linearly independent solutions y_1, y_2 are mapped into different lines $h(y_1)$, $h(y_2)$.

It is evident that the contacts of $h(y)$ with $\boldsymbol{u}$ could be only simple intersections and that between any two successive intersections of $h(y_1)$ with $\boldsymbol{u}$ ($s_1 < s_3$) there is just one (s_2) intersection of $h(y_2)$ with $\boldsymbol{u}$. This corresponds to the known fact that solutions of linear differential equations of the 2nd order have only simple zeros and zeros of linearly independent solutions separate each other.

Let us note that, e.g., our equation of the 2nd order is both-side oscillatoric iff $J = (-\infty, \infty)$, i.e., the unit circle is encircled by $\boldsymbol{u}$ infinitely many times both for $s \to -\infty$ and for $s \to +\infty$.

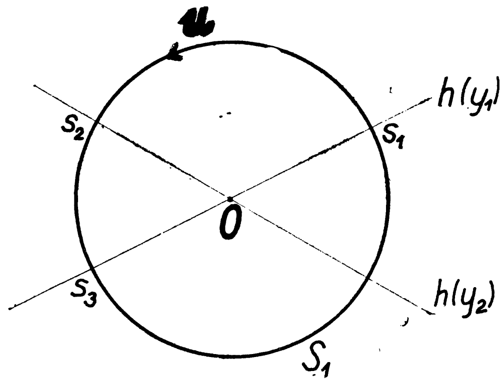

Fig. 1

Example 2. For linear differential equations of the 3rd order there were questions whether there exist

(i) an equation all solutions of which are oscillatoric (solved by G. SANSONE [15] in 1948)

(ii) an example of an equation such that all solutions of it at the same time as all solutions of its adjoint equation are oscillatoric (proposed by J. M. DOLAN [4] in 1970).

From the point of view of our approach it is not difficult to give the affirmative answers to both questions, since only a curve $\boldsymbol{u}(s)$ on the unit sphere $S_2 \subset E_3$ should be considered, $\boldsymbol{u} \in C^3$, $W_3[\boldsymbol{u}] \neq 0$, such that

(i) it is intersected infinitely many times by every plane going through the origin

(ii) the above requirement (i) is satisfied together with the property that the vector product $\boldsymbol{u} \times \boldsymbol{u}'$ satisfies also (i).

Example 3. Recently V. ŠEDA [17] and Z. MIKULÍK [9] dealt with differential equations with zero invariants (or locally equivalent to $y^{(n)} = 0$, or often called iterated equations) and the following result for these equations of the 3rd order was established: "There always exists a nonvanishing solution. If a solution has a zero, then

either all zeros of the solution are simple,
or all zeros of the solution are double."

The situation described by the theorem becomes very descriptive using our approach since it can be shown that a curve $\boldsymbol{u}$ corresponding to any iterated differential equation of the 3rd order can be written as $\boldsymbol{u}(s) = (\cos^2 s,\ 2\cos s \,.\, \sin s,\ \sin^2 s)$, which is (a part of) a circle in the plane $u_1 + u_3 = 1$ (not going through the origin). Hence, really,

there exists a plane (passing the origin) which does not intersect $\boldsymbol{u}$,

if a plane (passing the origin) intersects $\boldsymbol{u}$, then either all intersections are simple, or all contacts are of the 1st order always at the same point on the circle (with different parameters).

5. The proposed approach is not only suitable for illustrating theorems like in above examples, or e.g., theorems of M. Hanan [6] concerning zeros of solutions of linear differential equations of the 3rd order of the class C_{I} or C_{II} as it was shown in [12].

In [13] you can find the necessary and sufficient condition for the existence of only periodic solutions of linear differential equations of the n-th order. This condition is expressed in terms of distribution of zeros of solutions, and besides the described geometrical approach, some results (M. Kuczma [8] and B. Choczewski [3]) from theory of functional equations play their role in the proof of the condition.

Topology and some concepts from dynamical systems are used in [14] when answering the second problem proposed by J. M. Dolan [4] in the negative way. The problem was whether there exists an example of a linear differential equation of the 3rd order with the property that to every couple y_1, y_2 of linearly independent solutions of it there exist constants $c_{11}, c_{12}, c_{21}, c_{22}$ such that

$c_{11}y_1 + c_{12}y_2$ is an oscillatoric non-trivial solution and
$c_{21}y_1 + c_{22}y_2$ is a non-oscillatoric non-trivial solution.

The non-existence of such an equation is based on proving the non-existence of a curve on the unit sphere having the property that can be easily formulated using our geometrical interpretation. The compactness of S_2 plays here its important role.

The last mentioned problems seem to be interesting not only from the point of view of differential equations, but they lead to results in geometry and topology.

Then applying our geometrical approach to differential equations, we may either answer the proposed problem if the corresponding theorems in geometry, topology, theory of functional equations or theory of dynamical systems are known, or we get new interesting problems in those theories that lead to new investigations, maybe to new methods suggested by the previous problem in the theory of differential equations. Possibly we may succeed in solving the correspoding problems in both theories.

REFERENCES

[1] G. D. BIRKHOFF: *On the solutions of ordinary linear homogeneous differential equations of the third order*, Annals of Math. 12 (1910–11), p. 103—127.

[2] O. BORŮVKA: *Linear Differential Transformations of the Second Order*, The English Universities Press, London, 1971.

[3] B. CHOCZEWSKI: *On differentiable solutions of a functional equation*, Ann. Polon. Math. 13 (1963), p. 133–138.

[4] J. M. DOLAN: *On the relationship between the oscillatory behaviour of a linear third order differential equation and its adjoint*, J. Differential Equations 7 (1970), p. 367–388.

[5] H. GUGGENHEIMER: *Homogeneous linear differential equations with only periodic solutions*, Israel J. Math. 9 (1971),

[6] M. HANAN: *Oscillation criteria for third order linear differential equations*, Pacif. J. of Math. 11 (1961), p. 919–944.

[7] E. KAMKE: *Differentialgleichungen, Losungsmethoden und Losungen I* (Russian translation), Nauka, Moscow 1965.

[8] M. KUCZMA: *Functional Equations in a Single Variable*, PWN, Warszawa 1968.

[9] Z. MIKULÍK: *Conjugate and focus points of self adjoint linear differential equations of the third order*, Ph. D. thesis.

[10] V. V. NĚMYCKIJ, V. V. STĚPANOV: *Kačestvennaja teoria differencial'nykh uravněnij*, GITL, Moskva—Leningrad, 1949.

[11] F. NEUMAN: *Some results on geometrical approach to linear differential equations of the n-th order*, Comment. Math. Univ. Carol. 12 (1971), p. 307–315.

[12] F. NEUMAN: *Geometrical approach to linear differential equations of the n-th order, to* appear in Rend. di Matem. in 1972–73.

[13] F. NEUMAN: *On n-dimensional closed curves and periodic solutions of linear differential equations of the n-th order*, to appear in Demonstratio Mathematica (Poland).

[14] F. NEUMAN: *On two problems on oscillations of linear differential equations of the third order*, to appear in J. Differential Equations.

[15] G. SANSONE: *Studi sulle equazioni differenziali lineari omogenee di terzo ordine nel campo reale*, Revista Mat. Fis. Teor. (Tucuman), 6 (1948), p. 195–253.

[16] V. ŠEDA: *Über die Transformation der linearen Differentialgleichungen n-ter Ordnung I*, Čas. Pěst. Mat. 90 (1965), p. 385–412.

[17] V. ŠEDA: *On a class of linear differential equations of order n, $n \geqq 3$*, Čas. Pěst. Mat. 92 (1967) p. 247–261.

[18] G. R. SELL: *Nonautonomous differential equations and topological dynamics, $I + II$*, Trans. Math. Soc. 127 (1967), p. 241–283.

Author's address:

František Neuman
Department of Mathematics
J. E. Purkyně University of Brno
Janáčkovo náměstí 2a, Brno
Czechoslovakia