PARTIAL REGULARITY OF MINIMIZERS

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After the examples shown by E. De Giorgi, E. Giusti-M. Miranda, V.G. Mazja, J. Nečas, J. Souček, it is well known that the minimizers of variational integrals

$$(1) \qquad \mathfrak{F}[u;\Omega] = \int_\Omega F(x,u(x),\, Du(x))dx$$

in the vector valued case, even in simple situations, are in general *non* continuous. There is only hope to show *partial regularity* of minimizers, i.e. regularity except on a closed set hopefully small.

The study of the partial regularity of minimizers and of solutions of non linear elliptic systems starts with the works by Morrey and Giusti-Miranda in 1968, and it is the aim of this lecture to refer about some of the results obtained. I shall restrict myself to some results concerning the partial regularity of minimizers referring to [7] for a general account.

Let me start by stating the most general and recent result.

THEOREM 1. *Let* Ω *be a bounded open set in* $\mathbf{R}^n$ *and let* $F(x,u,p) : \Omega \times \mathbf{R}^N \times \mathbf{R}^{nN} \to \mathbf{R}$ *be a function such that*

i) $|p|^m \le F(x,u,p) \le c_0|p|^m, \quad m \ge 2$

ii) F *is of class* C^2 *with respect to* p *and*

$$|F_{pp}(x,u,p)| \le c_1(1+|p|^2)^{\frac{m-2}{2}}$$

iii) $(1+|p|^2)^{-\frac{m}{2}}F(x,u,p)$ *is Hölder-continuous in* (x,u) *uniformly with respect to* p

iv) F *is strictly quasi-convex* i.e. *for all* x_0, u_0, p_0 *and all* $\varphi \in C_0^\infty(\Omega,\mathbf{R}^N)$

Let $u \in H^{1,m}_{\mathrm{loc}}(\Omega,\mathbf{R}^N)$ *be a minimizer for*

$$\mathfrak{F}[u;\Omega] = \int_\Omega F(x,u,Du)dx$$

i.e. $\mathfrak{F}[u;\operatorname{supp}\varphi] \le \mathfrak{F}[u+\varphi;\operatorname{supp}\varphi]$. *Then there exists an open set* Ω_0 *such that* $u \in C^{1,\mu}(\Omega_0,\mathbf{R}^N)$, *moreover* $\operatorname{meas}(\Omega - \Omega_0) = 0$.

Theorem 1, proved in [12], is the result of a series of steps due to different authors.

Under the stronger condition of ellipticity

$$F_{p^i_\alpha p^j_\beta}\xi^\alpha_i\xi^\beta_j \geq \nu(1 + |p|^2)^{\frac{m-2}{2}} |\xi|^2 \qquad \forall\xi \in \mathbf{R}^{nN}; \quad \nu > 0$$

theorem 1 was proved for $m \geq 2$ by C.Morrey and E.Giusti, for $1 < m < 2$ by L.Pepe in 1968 in the case $F = F(p)$; in the case $m = 2$, $F = F(x,u,p)$ by Giaquinta - Giusti and Ivert in 1983, in the case $m \geq 2$, $F=F(x,u,p)$ by Giaquinta - Ivert in 1984. Fro these results I refer to [7] [9] [11]. Under the weaker assumption of quasi-convexity in (2) it was proved by L. Evans [5] in the case $F = F(p)$, $m \geq 2$.

The case $1 < m < 2$ is open, and essentially open are all the questions concerning the singular set; for instance

1. what about the structure of the singular set? what about the Hausdorff dimension of the singular set?
2. are there resonable structures under which minimizers are regular? (see the interesting paper [22])
3. what about the stability or instability properties of the singular set? or what about topological properties of the set of smooth minimizers?

We have results improving theorem 1 roughly only in case of quadratic functionals if we exclude the case in which F does not depend explicitly on u. So let us consider a quadratic functional

$$(3) \qquad A(u) = \int_\Omega A^{\alpha\beta}_{ij}(x,u)D_\alpha u^i D_\beta u^j dx$$

where the coefficients $A^{\alpha\beta}_{ij}$ are smooth (for example Hölder-continuous) and satisfy the ellipticity condition

$$(4) \qquad A^{\alpha\beta}_{ij}(x,u)\xi^\alpha_i\xi^\beta_j \geq |\xi|^2 \qquad \forall\xi \in \mathbf{R}^{nN}$$

Notice that the functional A is not differentiable. Concerning the strong condition of ellipticity (4), we remark that there is not much hope to weaken it. In fact in [14] it is shown that for weak solutions of the simple quasilinear system

$$\int_\Omega A^{\alpha\beta}_{ij}(x,u)D_\alpha u^i D_\beta \varphi^j dx = 0 \qquad \forall\varphi \in H^1_0(\Omega,\mathbf{R}^N)$$

with coefficients satisfying the strict Legendre-Hadamard condition

$$A^{\alpha\beta}_{ij}(x,u)\xi^\alpha\xi^\beta\eta_i\eta_j \geq |\xi|^2|\eta|^2 \qquad \forall\xi \in \mathbf{R}^n \quad \forall\eta \in \mathbf{R}^N$$

Caccioppoli's inequality may not be true; and Caccioppoli's inequality is indeed the starting point for the regularity theory.

THEOREM 2. (*Giaquinta - Giusti* [8]) - *Let u be a minimizer for* $A(u)$.*Then the Hausdorff dimension of the singular set* $\Omega - \Omega_0$ *is strictly less than* $n-2$. *In particular minimizers are smooth in dimension* $n = 2$.[1)]

Now the first natural question is whether the singularities are at most isolated in dimension $n = 3$, where first we can have singularities. The question is open in that generality, but it has a positive answer under the extra assumption that the coefficients split as

$$A_{ij}^{\alpha\beta}(x,u) = G^{\alpha\beta}(x)g_{ij}(u) \tag{5}$$

THEOREM 3. (*Giaquinta - Giusti* [10]) - *Let u be a bounded minimizer of*

$$\int_\Omega G^{\alpha\beta}(x)g_{ij}(u)D_\alpha u^i D_\beta u^j\,dx$$

where G and g are smooth symmetric definite positive matrices. Then in dimension $n = 3$ *the singularities of u are at most isolated and in general the singular set of u has Hausdorff dimension no larger than* $n - 3$.

THEOREM 4. (*Jost - Meier* [18]) - *Under the assumption of theorem 3 if u is a bounded minimizer with smooth boundary datum, then singularities may occur only far from the boundary.*

We recall that solutions of quasilinear elliptic systems may instead have singularities at the boundary [6].

The functional (3) (4) (5) that can be rewritten as

$$\mathcal{E}(u) = \int_\Omega G^{\alpha\beta}(x)g_{ij}(u)D_\alpha u^i D_\beta u^j\sqrt{G}\,dx \tag{6}$$

where

$$G(x) = \det(G_{\alpha\beta}(x)) \qquad (G_{\alpha\beta}(x)) = (G^{\alpha\beta}(x))^{-1}$$

represents in local coordinates the energy of a map between two Riemannian manifolds $u : M^n \to M^N$ with metric tensors respectively $G_{\alpha\beta}$. g_{ij}. Smooth stationary points are called *harmonic maps*. We refer to [2][3][17] for more information.

From the general point of view of differential geometry, theorems

1) Actually, under some more restrictive assumptions, in the general situation of theorem 1 minimizers are also smooth in dimension 2, see [7].

2 and 3 are limited.

In fact, while we can always localize in M^n, this is in general not possible in the target manifold M^N, except that we assume that it is covered by one chart (or, worse still, that u is continuous). In the general setting of a map from M^n into M^N, theorems 2 and 3 have been proved independently by Schoen - Uhlenbeck [19] [20].

At this point we may resonably ask whether the (bounded) minimizers of (6) may be really singular. In that respect the classical result by Eells - Sampson [4] can be read: *if the sectional curvature of M^N is non-positive then the minimizers (as well as the stationary points) of (6) are smooth.* Hildebrandt - Kaul - Widman [15] in the case of target manifold with positive sectional curvature proved: *if $u(M^n)$ is contained in a geodetic ball $B_r(q)$ which is disjoint from the cut locus of its center and has radius*

(7) $$R < \frac{\pi}{2\sqrt{k}}$$

where k is an upper bound for the sectional curvature, then the minimizers (and even the stationary points) are smooth.

In case of a map from the unit ball $B_1(0)$ of $\mathbf{R}^n$ into the standard sphere S^n of $\mathbf{R}^{n+1}$ condition (7) means that $u(B_1(0))$ is strictly contained in a hemisphere. Hildebrandt - Kaul - Widman showed that the *equator map* u^* defined by $u^*(x) = (\frac{x}{|x|}, 0)$ is a stationary point for $\mathcal{E}(u)$.

Then Jäger-Kaul [16] proved that u^* *is a minimizer for* $n > 6$, *while it is even unstable for* $n < 7$; more recently Baldes [1] showed that u^* is *stable* even for $n = 3$ if considered as a mapping from $B_1(0)$ into a suitable ellipsoid.

In general we have

THEOREM 5. *(Schoen-Uhlenbeck [21], Giaquinta-Souček [13]) - Every energy minimizing map u from a domain in some n-dimensional Riemannian manifold into the hemisphere S^N_+ is regular provided $n \leq 6$, and in general its singular set has Hausdorff dimension no larger than $n - 7$.*

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