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PERRON INTEGRAL, PERRON PRODUCT INTEGRAL AND ORDINARY LINEAR DIFFERENTIAL EQUATIONS

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1. Perron integral and Perron product integral

A finite set $\Delta = \{x_0, t_1, x_1, \ldots, t_k, x_k\}$ is called a partition of an interval $[a,b]$ if

$$a = x_0 < x_1 < \ldots < x_k = b\ , \quad x_{j-1} \leqq t_j \leqq x_j$$

for $j = 1,2,\ldots,k$. Let $\delta : [a,b] \to (0,\infty)$ (no continuity or measurability properties required). A partition Δ is said to be δ-fine if $[x_{j-1}, x_j] \subset (t_j - \delta(t_j),\ t_j + \delta(t_j))$.

Let $f : [a,b] \to \mathbb{R}$, put $S(f,\Delta) = \sum_{j=1}^{k} f(t_j)(x_j - x_{j-1})$. It is well known (cf. [1], [2]) that the following two conditions are equivalent:

$$\begin{aligned} &f \text{ is Perron integrable (P-integrable) over } [a,b]\ , \\ &q = (P)\int_a^b f(t)\, dt\ ; \end{aligned} \tag{1.1}$$

$$\begin{aligned} &\text{for every } \varepsilon > 0 \text{ there exists such a } \delta : [a,b] \to (0,\infty) \text{ that} \\ &|q - S(f,\Delta)| \leqq \varepsilon \text{ for every } \delta\text{-fine partition } \Delta \text{ of } [a,b]\ . \end{aligned} \tag{1.2}$$

Condition (1.2) makes good sense since

$$\begin{aligned} &\text{for every } \delta : [a,b] \to (0,\infty) \text{ there exists a } \delta\text{-fine} \\ &\text{partition } \Delta \text{ of } [a,b]\ . \end{aligned} \tag{1.3}$$

1.1. REMARK. The proof of (1.3) is easy: If (1.3) were false for a δ on $[a,b]$, it would be false either for δ on $[a, (a+b)/2]$ or for δ on $[(a+b)/2, b]$ and this procedure, if continued, leads to a contradiction.

Denote by M the ring of real or complex $n \times n$ matrices. For

$A : [a,b] \to M$ and a partition Δ of $[a,b]$ put

$$P(A,\Delta) = (I + A(t_k)(x_k - x_{k-1}))\dots(I + A(t_1)(x_1 - x_0)) ,$$

$$\tilde{P}(A,\Delta) = \exp(A(t_k)(x_k - x_{k-1}))\dots \exp(A(t_1)(x_1 - x_0)) .$$

The following result is well known (cf. [4], [5]): If A is continuous and if U is the matrix solution of

$$\dot{x} = A(t)x , \tag{1.4}$$

$U(a) = I$, then both $P(A,\Delta)$, $\tilde{P}(A,\Delta)$ converge to $U(b)$ in the following sense:

For every $\varepsilon > 0$ there exists such an $\eta > 0$ that $||U(b) - P(A,\Delta)|| \leqq \varepsilon$, $||U(b) - \tilde{P}(A,\Delta)|| \leqq \varepsilon$ for every partition Δ of $[a,b]$ satisfying $x_j - x_{j-1} < \eta$, $j = 1,\dots,k$. (1.5)

In [5] the Lebesgue product integral was introduced in a way analogous to the usual introduction of the Bochner integral and it was proved that $U(b)$ is equal to the Lebesgue product integral of $\exp(A(t)\,dt)$ over $[a,b]$ provided A is Lebesgue integrable in the usual sense. In the next definition, the limiting process from (1.2) is applied to the product $P(A,\Delta)$ - of course without any continuity or measurability condition on A.

1.2. DEFINITION. Let $Q \in M$ be regular. A is said to be *Perron product-integrable over* $[a,b]$ ($\mathcal{P}$-integrable), Q is called the *Perron product integral* ($\mathcal{P}$-integral) of A and denoted by $\mathcal{P}\int_a^b (I + A(t)\,dt)$, if for every $\varepsilon > 0$ there exists such a $\delta : [a,b] \to (0,\infty)$ that $||Q - P(A,\Delta)|| \leqq \varepsilon$ for every δ-fine partition Δ of $[a,b]$.

1.3. REMARK. The same concept of the $\mathcal{P}$-integral is obtained if $P(A,\Delta)$ is replaced by $\tilde{P}(A,\Delta)$ in Definition 1.2.

The integral $\mathcal{P}\int_a^b (I + A(t)\,dt)$ has properties analogous to those of the integral $(P)\int_a^b f(t)\,dt$. The properties of the latter are listed in Section 2, the analogous properties of the former in Section 3. In Section 4 some relations to ACG_*-functions and to the equation (1.4) are mentioned.

2. Properties of the Perron integral

If f is P-integrable over $[a,b]$ then $F(t) = (P)\int_a^t f(s)\,ds$ exists for $t \in (a,b]$. We put $F(a) = 0$. (2.1)

If f is P-integrable, then F is continuous and $\dot{F}(t) = f(t)$ a.e. Moreover, f is measurable. (2.2)

Let f be P-integrable over $[a,b]$. Then the following assertion holds: if $C \subset [a,b]$ is of measure zero and $\varepsilon > 0$, then there exists such a $\delta : C \to (0,\infty)$ that $\sum_{j=1}^{r} |F(\eta_j) - F(\xi_j)| < \varepsilon$ provided $\tau_j \in C$, $\xi_j \leqq \tau_j \leqq \eta_j \leqq \xi_{j+1}$ and $[\xi_j,\eta_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j))$ for $j = 1,\ldots,r$. (2.3)

Let $F : [a,b] \to \mathbb{R}$ have derivative a.e. and satisfy the assertion from (2.3). Put $f(t) = \dot{F}(t)$ if $\dot{F}(t)$ exists, $f(t)$ arbitrary otherwise. (2.4)

Then $(P)\int_a^b f(t)\,dt$ exists and equals $F(b) - F(a)$.

If $(P)\int_a^t f(s)\,ds$ exists for $t < b$ and if $\lim_{t\to b-} (P)\int_a^t f(s)\,ds = q \in \mathbb{R}$, then $(P)\int_a^b f(s)\,ds$ exists and equals q. (2.5)

3. Properties of the Perron product integral

If A is $\mathcal{P}$-integrable over $[a,b]$ then $U(t) = \mathcal{P}\int_a^t (I + A(s)\,ds)$ exists for $t \in (a,b]$. We put $U(a) = I$. (3.1)

If A is $\mathcal{P}$-integrable, then $U(t)$ is regular at every t, U is continuous and $\dot{U}(t)U^{-1}(t) = A(t)$ a.e. Moreover, A is measurable. (3.2)

Let A be $\mathcal{P}$-integrable. Then the assertion (2.3) holds with F replaced by U. (3.3)

Let $U : [a,b] \to M$ be continuous, regular at every t and differentiable a.e., and let it satisfy the modified assertion of (2.3) (cf. (3.3)). Put $A(t) = \dot{U}(t)U^{-1}(t)$ if $\dot{U}(t)$ exists, $A(t)$ arbitrary otherwise. Then $P\int_a^b (I + A(t)\,dt)$ exists and equals $U(b)U^{-1}(a)$. (3.4)

If $P\int_a^t (I + A(s)\,ds)$ exists for $t < b$ and if $\lim_{t\to b-} P\int_a^t (I + A(s)\,ds) = Q \in M$ is regular, then $P\int_a^b (I + A(s)\,ds)$ exists and equals Q. (3.5)

4. ACG_*-solutions of linear ordinary differential equations

The concept of an ACG_*-function (cf. [6]) extends without complications to functions with values in finitedimensional linear spaces. A function $u : [a,b] \to \mathbb{R}^n$ $(\mathbb{C}^n)$ is called an ACG_*-solution of (1.4) if u is an ACG_*-function and satisfies (1.4) a.e. In an analogous manner the concept of a matrix ACG_*-solution of (1.4) is to be understood.

It is well known that F from (2.1) is an ACG_*-function and that every ACG_*-function is the primitive of its derivative (in the sense of (2.1), (2.2)). It follows from (2.3) and (2.4) that F is an ACG_*-function iff it satisfies (2.4). It follows from (3.1) - (3.4) that U from (3.1) is an ACG_*-function and that every ACG_*-function $U : [a,b] \to M$ can be written in the form

$$U(t)U^{-1}(a) = P\int_a^t (I + A(s)\,ds)$$

provided $U(t)$ is regular for every t. At the same time we have

$$U(t) - U(a) = (P)\int_a^t \dot{U}(s)\,ds = (P)\int_a^t A(s)\,U(s)\,ds$$

for $t \in [a,b]$, that is, U is an ACG_*-matrix solution of (1.4).

Thus we have obtained a class of LDE's the solutions of which are ACG_*-functions; these LDE's have the usual existence and uniqueness properties.

Denote by $H([a,b])$ the set of such $A : [a,b] \to M$ that $P\int_a^b (I + A(t)\,dt)$ exists. Let us find some effective conditions for $A \in H([a,b])$.

Assume that $c > 0$, $B : [-c,c] \to M$ is continuous, $I + B(t)$ is regular for $t \in [-c,c]$ and B is locally absolutely continuous on $[-c,c] \setminus \{0\}$. We have

$$\text{if } A(t) = \dot{B}(t)[I + B(t)]^{-1} \text{ a.e. then } A \in H([-c,c]) , \tag{4.1}$$

$$\text{if } k \in \{0,1,\dots\} , \ A(t) = \dot{B}(t)[I - B(t) + \dots + (-1)^k B^k(t)] \text{ a.e.}, \ \int_{-c}^{c} ||\dot{B}(t)B^{k+1}(t)||\,dt < \infty , \text{ then } A \in H([-c,c]) . \tag{4.2}$$

In the case (4.1), $I + B(t)$ is a fundamental matrix of (1.4), in the case (4.2) the substitution $x = [I + B(t)]y$ leads to the result.

Let $\alpha > 0$, $\beta > 0$, $T, S \in M$. If $\alpha < 1 + \beta$, then there exists such a continuous $B : \mathbb{R} \to M$ that $B(0) = 0$, $\dot{B}(t) = |t|^{-\alpha}[T \cos |t|^{-\beta} + S \sin |t|^{-\beta}]$ for $t \neq 0$. Let $c > 0$ be so small that $I + B(t)$ is regular for $t \in [-c,c]$. Then (4.1) may be applied. If $\alpha < 1 + \beta/2$ then (4.2) may be applied with $k = 0$. If $1 + \beta/2 \leq \alpha < 1 + 2\beta/3$ then (4.2) may be applied with $k = 1$; moreover, if $TS - ST \neq 0$ then $\int_{-1}^{t} A(s)\,ds$ is unbounded for $t \to 0-$ so that $(P)\int_{-1}^{1} A(t)\,dt$ does not exist.

5. The Saks-Henstock Lemma

In the proof of the properties (2.2) and (2.3) of the Perron integral the key part is played by the following

5.1. LEMMA (Saks, Henstock). Assume that f is P-integrable over $[a,b]$, $F(t) = (P)\int_a^t f(s)\,ds$. Let $\varepsilon > 0$ and let the gauge δ correspond to ε according to (1.2). Let

$$\xi_j, \tau_j, \eta_j \in [a,b] , \quad \xi_j \leq \tau_j \leq \eta_j \leq \xi_{j+1}$$
$$[\xi_j,\eta_j] \subset (\tau_j - \delta(\tau_j), \tau_j + \delta(\tau_j)) , \quad j = 1,2,\dots,r . \tag{5.1}$$

Then

$$\sum_{j=1}^{r} |f(\tau_j)(\eta_j - \xi_j) - F(\eta_j) + F(\xi_j)| \leqq 2\varepsilon \ .$$

For the Perron product integral, an analogous role in the proof of the properties (3.2) and (3.3) is played by

5.2. LEMMA. There exist $\varepsilon_0 > 0$ and $K > 0$ depending on n only so that the following holds:

Assume that A is P-integrable over $[a,b]$,

$$U(t) = P\int_a^t (I + A(s))ds \ , \ U(b) = Q \ .$$

Let $0 < \varepsilon < \varepsilon_0/||Q^{-1}||$ and let the gauge δ correspond to ε according to Definition 1.2. Let (5.1) hold. Then

$$\sum_{j=1}^{r} ||I + A(\tau_j)(\eta_j - \xi_j) - U(\eta_j)U^{-1}(\xi_j)|| \leqq K\varepsilon \ .$$

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