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The Use of Semiregular Finite Elements

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Abstract. This text is extended Equadiff 9 plenary lecture. Sections 1–4 contain a survey of published results which concern triangular and quadrilateral finite elements. Sections 1 and 2 are devoted to interpolation problems. These two sections contain also results of other authors. The analysis of both the effect of numerical integration and approximation of a boundary is restricted to triangular elements with linear polynomials and to quadrilateral elements with four-node isoparametric functions. The corresponding results in the case of smooth solutions are introduced in Section 3, where the rate of convergence $O(h)$ is proved; the case of nonsmooth solutions is studied in Section 5. This section is restricted to triangular elements. In Sections 3 and 5 the domain considered has a form of a narrow ring with a great diameter. In this case the elements cannot be arbitrarily narrow. In Section 4 a composite domain indicated in Fig. 5 is approximated by triangular elements and applications of the finite element method in magnetostatical problems are introduced. In this case the triangular elements can be arbitrarily narrow. Section 6 is an Appendix where a special form of a discrete Friedrichs' inequality, suitable for semiregular elements, is proved. Sections 5 and 6, which complete the survey introduced in Sections 1–4, have not yet been published and were written specially for Equadiff 9.
The notation of derivatives and Sobolev spaces is identical with the notation used in [9].
As to the notion of semiregular elements, semiregular triangles can have one angle arbitrarily small. Triangles with two arbitrarily small angles are irregular. A semiregular quadrilateral K can be arbitrarily narrow and it satisfies the condition

$$|\cos\vartheta_i| \le \sigma < 1 \quad (i = 1, \dots, 4),$$

where $\vartheta_1, \dots, \vartheta_4$ are the angles of K.

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1 Triangular and quadrilateral elements of the Lagrange type

First interpolation estimates which can be used in the finite element theory were derived by Synge in the year 1957 (see [14, pp. 209–213]). His a little improved result can be formulated in the following theorem:

Theorem 1.1. *Let u be a function continuous on a closed triangle $\overline{T}$ with bounded second partial derivatives in its interior T,*

$$\left|\frac{\partial^2 u}{\partial x_i \partial x_j}\right| \leq M_2,$$

and let $p(x_1, x_2)$ be a linear polynomial satisfying

$$p(P_i) = u(P_i) \quad (i = 1, 2, 3)$$

with P_1, P_2, P_3 the vertices of $\overline{T}$. Then it holds on $\overline{T}$

$$\left|\frac{\partial u}{\partial x_i} - \frac{\partial p}{\partial x_i}\right| \leq \frac{2M_2 h}{\cos(\gamma/2)} \quad (i = 1, 2) \tag{1.1}$$

$$|u - p| \leq \frac{2M_2 h^2}{\cos(\gamma/2)} \tag{1.2}$$

Result (1.2) was obtained by means of (1.1). Another independent consideration (where we first estimate the difference $g = u - p$ on P_2P_3 and then on P_1P' with $P' \in P_2P_3$ an arbitrary point) gives us

$$|u - p| \leq \frac{1}{2} M_2 h^2. \tag{1.3}$$

This result implies a question whether estimate (1.1) cannot be improved, as far as the geometry is concerned. An example showing that the answer is negative was presented in [15]. Here is its simplified version: Let us consider a set of triangles with vertices

$$P_1(-h/2, 0), P_2(h/2, 0), P_3(0, y_0),$$

where h is fixed and y_0 $(0 < y_0 < \sqrt{3}h/2)$ is variable, and a function $u(x_1, x_2) = x_1^2$. Its first degree interpolant has the form

$$p(x_1, x_2) = \frac{h^2}{4}\left(1 - \frac{x_2}{y_0}\right).$$

Hence

$$\left|\frac{\partial u}{\partial x_2} - \frac{\partial p}{\partial x_2}\right| = \left|\frac{\partial p}{\partial x_2}\right| = \frac{h^2}{4y_0} = \frac{h}{2}\cot\alpha = \frac{h}{2}\tan(\gamma/2), \tag{1.4}$$

where α and γ are the minimum and maximum angles of $\overline{T}$, respectively. If $y_0 \to 0$ then $\alpha \to 0$, $\gamma \to \pi$ and

$$\left| \frac{\partial u}{\partial x_2} - \frac{\partial p}{\partial x_2} \right| \to \infty .$$

Zlámal knew both estimate (1.1) and result (1.4) when he started to work on his paper "On the finite element method" (see [24]). Nevertheless, instead of the maximum angle condition

$$\gamma_T \le \gamma_0 < \pi \quad \forall T \in \mathcal{T}_h, \quad \forall h \in (0, h_0) \tag{1.5}$$

where $\mathcal{T}_h$ denotes a triangulation of a given (polygonal) domain, he introduced the minimum angle condition

$$\vartheta_T \ge \vartheta_0 > 0 \quad \forall T \in \mathcal{T}_h, \quad \forall h \in (0, h_0) \tag{1.6}$$

where ϑ_T is the minimum angle of T. Reading Zlámal's papers one sees that the finite element theory is relatively easy under condition (1.6). Also other mathematicians started to use condition (1.6) and when it was used in Ciarlet's 1978-book [3] it has become a standard finite element condition.

However, there are situations where the minimum angle condition (1.6) is too restrictive because it forbids to use triangles with one small angle. Such triangles are permitted according to the maximum angle condition. Thus it is quite natural to try to generalize the standard finite element theory to the case of condition (1.5).

We start with the interpolation theorems and first we remind Jamet's result [5].

For a better understanding we introduce from [5] only a special situation which is for applications quite sufficient. Let $\mathcal{L}(X, Y)$ denote the set of all linear bounded operators from a normed space X into a normed space Y. Let

$$\Pi \in \mathcal{L}(W^{k,p}(T), W^{1,p}(T)),$$

where k is a positive integer and $p \in [1, \infty]$, be an operator satisfying the following hypotheses:

(H.1) We have

$$\Pi u = u \quad \forall u \in P_{k,2}\,,$$

where $P_{k,n}$ denotes the set of all polynomials in n variables of degree not greater than k.

(H.2) There exists a unit vector ξ such that

$$\frac{\partial u}{\partial \xi}(P) = 0 \quad \forall P \in T \quad \Rightarrow \quad \frac{\partial (\Pi u)}{\partial \xi}(P) = 0 \quad \forall P \in T.$$

(We restrict ourselves to this special type of (H.2) because we are interested only in estimates of type (1.7).)

Theorem 1.2. *Let $\overline{T}$ be a closed triangle with the interior T and vertices P_1, P_2, P_3 and let α_T, β_T and γ_T be the angles at P_1, P_2 and P_3, respectively. Let the vertices be denoted in such a way that $\alpha_T \leq \beta_T \leq \gamma_T$. Let s_1 and s_2 be the unit vectors parallel to the sides P_3P_2 and P_3P_1, respectively. Let $\Pi \in \mathcal{L}(W^{k,p}(T), W^{1,p}(T))$ be an operator satisfying hypotheses (H.1) and (H.2) for $\xi = s_1$ and $\xi = s_2$. Let $u \in W^{k+1,p}(T)$. Then we have for $m = 0$ and $m = 1$*

$$|u - \Pi u|_{m,p,T} \leq C \, \frac{h_T^{k+1-m}}{(\cos(\gamma_T/2))^m} |u|_{k+1,p,T} \,, \tag{1.7}$$

where $h_T = \operatorname{dist}(P_1, P_2)$ and C is a constant not depending on u and T.

Proof. The assertion is a special case of [5, Theorem 2.2]. □

In [5] Theorem 1.2 is applied on compatible triangular finite elements of the Lagrange type for arbitrary k. (For $k = 1$, $p = \infty$ estimates (1.7) are identical with Synge's result.) This means that the operator Π is defined by the relations

$$(\Pi u)(P_i) = u(P_i) \quad (i = 1, \dots, N, \;\; N := (n+1)(n+2)/2),$$

where $P_1, \dots, P_N$ are the nodal points which are situated on $\overline{T}$ as the first N integers in the Pascal triangle (see Fig. 1 where the black circles denote prescribed function values).

However, in the case $k = 1$ estimates (1.7) hold only for $p \in (2, \infty]$. The important case $p = 2$ is treated in [2] for $k \geq 1$. A further generalization in the case $k = 1$ is given in [6]. The interpolation result proved in [6] can be formulated as follows.

Theorem 1.3. *Let $\overline{T}$ be the same triangle as in Theorem 1.2 and let $p \in (1, \infty)$. Let $u \in W^{2,p}(T)$ and let $I_h u$ be the linear function satisfying $(I_h u)(P_i) = u(P_i)$ $(i = 1, 2, 3)$. Then we have*

$$|u - I_h u|_{m,p,T} \leq C \, \frac{h_T^{2-m}}{(\sin \gamma_T)^m} |u|_{2,p,T} \quad (m = 0, 1), \tag{1.8}$$

where C is a constant independent of u and T.

Theorem 1.3 will be useful in our further considerations.

Now we introduce interpolation results in the case of semiregular (i.e., narrow) convex four-node quadrilateral isoparametric finite elements. In [1] such elements are called anisotropic. However, in [1] the error of the interpolation is estimated on rectangular elements; quadrilaterals are not considered.

The symbol $\overline{K}_0$ will denote the closed square in the (ξ, η)-plane with vertices $\widehat{M}_1(1, 0)$, $\widehat{M}_2(1, 1)$, $\widehat{M}_3(0, 1)$, $\widehat{M}_4(0, 0)$. The functions $\widehat{\varphi}^{(i)} : (\xi, \eta) \to R^1$ with

$$\begin{aligned} &\widehat{\varphi}^{(1)}(\xi, \eta) = \xi(1 - \eta), \;\; \widehat{\varphi}^{(2)}(\xi, \eta) = \xi\eta, \\ &\widehat{\varphi}^{(3)}(\xi, \eta) = (1 - \xi)\eta, \;\; \widehat{\varphi}^{(4)}(\xi, \eta) = (1 - \xi)(1 - \eta) \end{aligned} \tag{1.9}$$

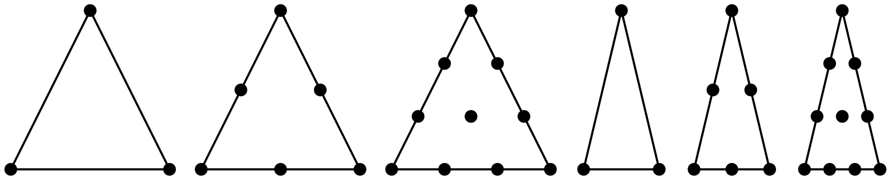

Fig. 1. Triangular finite elements of the Lagrange type.

are called bilinear basis functions; they have the property

$$\widehat{\varphi}^{(i)}(\widehat{M}_j) = \delta_{ij}.$$

Let $\overline{K}$ be a closed convex quadrilateral in the (x, y)-plane. Let two sides of $\overline{K}$ be much greater than the remaining two ones. Let us consider first the case that these two longer sides are parallel. (Such quadrilaterals are important, for example, in modelling a gap between rotor and stator in an electrical machine.) Let α_K be the smallest angle of $\overline{K}$ and let us denote by M_1 the vertex of $\overline{K}$ at the angle α_K. (If $\overline{K}$ has two or four angles which can be denoted by α_K then, of course, we have two or four choices.) One short side and one long side of $\overline{K}$ meet at M_1. The second end-point of the long one will be denoted by M_2 and the second end-point of the short one by M_4. The numbering of the vertices of $\overline{K}$ is thus either anticlockwise, or clockwise.

In applications the local numbering of the vertices of $\overline{K}$ obeys a different rule which is usually anticlockwise; let $N_1, \ldots, N_4$ denote the vertices of $\overline{K}$ according to this different rule, let (for simplicity) the numbering of $M_1, \ldots, M_4$ be also anticlockwise and let

$$M_1 = N_{j+1},\ M_2 = N_{j+2},\ M_3 = N_{j+3},\ M_4 = N_j,$$

where $N_{j+i} \equiv N_{j+i-4}$ if $j + i \geq 5$. As N_i corresponds by definition to $\widehat{M}_i$ the isoparametric transformation of $\overline{K}_0$ onto $\overline{K}$ has the form

$$\begin{aligned} x = x_K(\xi, \eta) &:= \sum_{i=1}^{4} x_i \widehat{\varphi}^{(j+i)}(\xi, \eta), \\ y = y_K(\xi, \eta) &:= \sum_{i=1}^{4} y_i \widehat{\varphi}^{(j+i)}(\xi, \eta), \end{aligned} \tag{1.10}$$

where x_i, y_i are the coordinates of M_i $(i = 1, \ldots, 4)$ and where the indices $j + i$ $(0 \leq j \leq 3$ fixed, $i = 1, \ldots, 4)$ are considered modulo 4. (In the case when the numbering of $M_1, \ldots, M_4$ is clockwise the corresponding isoparametric transformation of $\overline{K}_0$ onto $\overline{K}$ has again the form of (1.10).) As $\overline{K}$ is convex, transformation (1.10) maps $\overline{K}_0$ one-to-one onto $\overline{K}$.

Let

$$\xi = \xi_K(x,y),\ \eta = \eta_K(x,y) \tag{1.11}$$

denote the inverse transformation to transformation (1.10). We set

$$\varphi^{(i)}(x,y) := \widehat{\varphi}^{(i)}(\xi_K(x,y), y_K(x,y)) \quad (i = 1,\dots,4). \tag{1.12}$$

If $u \in C(\overline{K})$, then we define the isoparametric interpolation of u on $\overline{K}$ by

$$(Qu)(x,y) = \sum_{i=1}^{4} u(M_i)\varphi^{(j+i)}(x,y). \tag{1.13}$$

Theorem 1.4. *Let $\overline{K}$ be a narrow quadrilateral with parallel long sides which satisfy the assumption*

$$\operatorname{dist}(M_1, M_4) \le \frac{1}{12}\operatorname{dist}(M_1, M_2). \tag{1.14}$$

Let $u \in H^2(K)$. Then we have

$$\|u - Qu\|_{0,K} \le \left(C_1 + \frac{C_2\varepsilon_K}{h_K \sin\beta_K}\right) h_K^2 |u|_{2,K}, \tag{1.15}$$

$$|u - Qu|_{1,K} \le \left(C_3 + \frac{C_4}{\sin\alpha_K}\right)\frac{h_K}{\sin\beta_K}|u|_{2,K}, \tag{1.16}$$

where Qu is defined in (1.13), ε_K=dist$(M_1, M_4) < h_K$=dist(M_1, M_2), $\alpha_K \le \beta_K$, α_K and β_K being the angles at M_1 and M_2, respectively, and the constants C_1, C_2, C_3, C_4 satisfy

$$C_1 = 55.019093,\ C_2 = 21.658241,\ C_3 = 12.801823,\ C_4 = 19.47235264.$$

For the proof see [22].

Remark 1.5. Using the more standard approach with the bilinear isoparametric mapping of K_0 onto K we obtain (by means of the sharp form of the Bramble-Hilbert lemma) the estimate $\|u - Qu\|_{0,K} \le Ch_K^2|u|_{2,K}$ which does not depend on the geometry of K. However, this approach completely fails in estimating $|u - Qu|_{1,K}$ where we loose all powers of h_K.

Remark 1.6. It can be shown by an example that the dependence of the estimate of $|u - Qu|_{1,K}$ on $\sin^{-1}\alpha_K$ is essential (see [23]). The dependence on $\sin^{-1}\beta_K$ in both (1.15) and (1.16) is a cosmetic defect which is a consequence of the approach used in [22].

Remark 1.7. If we change assumption (1.14) to

$$\operatorname{dist}(M_1, M_4) \le \frac{1}{2n}\operatorname{dist}(M_1, M_2), \quad n \ge 6,$$

then the numerical constants in Theorem 1.4 will be smaller. (In more detail see [22].)

Theorem 1.4 can be generalized to the case that the long sides are not parallel. We again assume that $\overline{K}$ is a convex quadrilateral. Moreover, we assume that the long sides do not have any common vertex.

Our considerations are based on the following simple fact: Let $\overline{K}$ be an arbitrary convex quadrilateral. Then there exists a parallelogram $\overline{D}$ which has three vertices common with $\overline{K}$ and is such that $\overline{K} \subset \overline{D}$.

Let us denote these three vertices by M_1, M_2, M_3 in such a way that M_1M_2 and M_2M_3 are sides of K with the property

$$\operatorname{dist}(M_2, M_3) < \operatorname{dist}(M_1, M_2). \tag{1.17}$$

We shall denote

$$h_K := \operatorname{dist}(M_1, M_2), \quad a_K := \operatorname{dist}(M_2, M_3). \tag{1.18}$$

Of course it may happen that h_K is not the length of the greatest side of $\overline{K}$ and that the numbering of M_1, M_2, M_3, M_4 is not anticlockwise.

We shall assume that

$$a_K \le \frac{1}{2n} h_K, \quad \varepsilon_K \le \frac{1}{2n} h_K, \tag{1.19}$$

$$\frac{1}{2} \le \frac{\operatorname{dist}(M_4, p)}{\operatorname{dist}(M_3, p)} \le 1, \tag{1.20}$$

where $n \ge 6$ is a given integer, $\varepsilon_K := \operatorname{dist}(M_1, M_4)$ and p denotes the straight-line passing through M_1 and M_2.

In applications we usually have

$$\frac{\pi}{4} \le \alpha_K \le \frac{3\pi}{4}, \quad \frac{\pi}{4} \le \beta_K \le \frac{3\pi}{4}.$$

The interpolation theorem has in this more general case the following form (see [22]).

Theorem 1.8. *Let $\overline{K}$ be a quadrilateral satisfying assumptions (1.17)–(1.20) and let $u \in H^2(K)$. Then we have*

$$\|u - Qu\|_{0,K} \le \left(\widehat{C}_1(n) + \frac{\widehat{C}_2(n)\sqrt{\varepsilon_K a_K}}{h_K \sqrt{\sin \beta_K \sin \alpha_K}} \right) h_K^2 |u|_{2,K}, \tag{1.21}$$

$$|u - Qu|_{1,K} \le \left(\widehat{C}_3(n) + \frac{\widehat{C}_4(n)\sqrt{\varepsilon_K}}{\sqrt{a_K \sin \beta_K \sin \alpha_K}} \right) \frac{h_K}{\sin \beta_K} |u|_{2,K}, \tag{1.22}$$

where Q is an interpolation operator of type (1.13), $a_K = \operatorname{dist}(M_2, M_3)$ and $\varepsilon_K = \operatorname{dist}(M_1, M_4)$ satisfy (1.19), α_K and β_K are the angles at M_1 and M_2, respectively, and the positive constants $\widehat{C}_1(n)$, $\widehat{C}_2(n)$, $\widehat{C}_3(n)$ and $\widehat{C}_4(n)$ are decreasing when n is increasing, n being the integer which appears in (1.19).

2 Triangular elements of the Hermite type

Let us define $\Pi u \in P_{3,2}$, where $u \in C^1(\overline{T})$ and $\overline{T}$ is the same as in Theorem 1.2, by the relations

$$\begin{aligned} (D^\alpha \Pi u)(P_i) &= D^\alpha u(P_i) \quad |\alpha| \le 1 \quad (i=1,2,3), \\ \frac{\partial(\Pi u)}{\partial s_2}(Q_1) &= \frac{\partial u}{\partial s_2}(Q_1), \end{aligned} \tag{2.1}$$

where Q_1 is the mid-point of the side P_2P_3.

Theorem 2.1. *The polynomial Πu is uniquely determined by relations (2.1). We have*

$$\Pi \in \mathcal{L}(W^{3,p}(T), W^{1,p}(T)),\ p \in [1,\infty]$$

and the operator Π satisfies hypotheses (H.1) and (H.2) for $\xi = s_1$ and $\xi = s_2$. Hence estimates (1.7) hold for $k=3$, $p \in [1,\infty]$ and $m = 0,1$:

$$|u - \Pi u|_{m,p,T} \le C\, \frac{h_T^{4-m}}{(\cos(\gamma_T/2))^m} |u|_{4,p,T}.$$

Proof. The unique determination will be proved in Remark 2.10. The property $\Pi \in \mathcal{L}(W^{3,p}(T), W^{1,p}(T))$ follows for $p > 1$ from the Sobolev imbedding theorem and for $p=1$ from the fact that $W^{2,1}(T) \subset C(\overline{T})$. Hypothesis (H.1) is obvious and hypothesis (H.2) is proved in [19]. □

Remark 2.2. The tenth parameter $(\partial(\Pi u)/\partial s_2)(Q_1)$ has no influence on the global smoothness of a global finite element function defined in a given triangulation; thus it can be different in two adjacent triangles with a common shortest side.

Now we introduce a triangular finite element of the Hermite type which does not satisfy Jamet's hypothesis (H.2); nevertheless, it satisfies estimates not depending on the minimum angle of T.

Theorem 2.3. *Let $\overline{T}$ be the same triangle as in Theorem 1.2 and let $a = \operatorname{dist}(P_2,P_3)$, $b = \operatorname{dist}(P_1,P_3)$, $c \equiv h_T = \operatorname{dist}(P_1,P_2)$. Let $\varphi \in C^1(\overline{T})$ and let*

$$|D^\alpha \varphi(P)| \le M_4 \quad \forall |\alpha| = 4, \quad \forall P \in T\,, \tag{2.2}$$

$$D^\alpha \varphi(P_j) = 0 \quad \forall |\alpha| \le 1\ (j=1,2,3), \qquad \frac{\partial \varphi}{\partial n_a}(Q_1) = 0 \tag{2.3}$$

where Q_1 is the mid-point of the side P_2P_3 and n_a the unit normal to P_2P_3. Then we have for all $P \in \overline{T}$

$$|\varphi(P)| \le \frac{1}{96}\left(1 + 4\left(\frac{a}{c}\right)^3\right) M_4\, c^4\,, \tag{2.4}$$

$$\left|\frac{\partial \varphi}{\partial x_j}(P)\right| \le \frac{4}{15}\left(1 + 5\left(\frac{a}{c}\right)^2\right) \frac{1}{\sin \beta_T} M_4\, c^3 \quad (j=1,2). \tag{2.5}$$

Proof. Theorem 2.3 is proved in [19]. Nevertheless, we reproduce this proof because it is surprisingly short. We restrict our considerations to the case

$$|D^i\varphi(P)| \le M_4 \quad \forall |i| = 4\,, \quad \forall P \in \overline{T}\,. \tag{2.6}$$

In the case (2.2) we can use the trick with an inscribed triangle $\overline{T}' \subset T$ in the same way as in [24]. The proof is based on the following four lemmas.

Lemma 2.4. *Let s_1, s_2 be two noncollinear directions making an angle ω. Let $\frac{\partial\varphi}{\partial s_j}(P) = k_j$ $(j = 1, 2)$, P being a point of the (x_1, x_2)-plane. Then*

$$\left|\frac{\partial\varphi}{\partial x_j}(P)\right| \le \frac{|k_1| + |k_2|}{|\sin\omega|} \quad (j = 1, 2)\,.$$

Further, let s_1 and s_2 be two directions orthogonal to one another. If $|\frac{\partial\psi}{\partial s_i}(P)| \le k_i$ $(i = 1, 2)$ then we have for an arbitrary direction s

$$\left|\frac{\partial\psi}{\partial s}(P)\right| \le |k_1| + |k_2|.$$

Lemma 2.5. *Let $g(0) = \eta_1$, $g(l) = \eta_2$, $g'(0) = k_1$, $g'(l) = k_2$ and $|g^{(4)}(s)| \le K_4$ in $(0, l)$. Then for $s \in [0, l]$*

$$|g(s)| \le \max|\eta_j| + \frac{4\,l}{27}(|k_1| + |k_2|) + \frac{K_4}{16 \cdot 24} l^4, \tag{2.7}$$

$$|g'(s)| \le \frac{3}{2\,l}(|\eta_1| + |\eta_2|) + \max|k_j| + \frac{K_4}{24} l^3 \tag{2.8}$$

Further, if $g(0) = g(l) = g'(0) = g'(l) = 0$ then

$$|g''(s)| \le \frac{1}{2} K_4 l^2. \tag{2.9}$$

Lemma 2.6. *Let $g(0) = \eta_1$, $g(l/2) = \eta_2$, $g(l) = \eta_3$ and $|g^{(3)}(s)| \le K_3$ in $(0, l)$. Then for $s \in [0, l]$*

$$|g(s)| \le \frac{5}{4} \max|\eta_j| + \frac{\sqrt{3}}{6^3} K_3 l^3\,, \tag{2.10}$$

$$|g'(s)| \le \frac{8}{l} \max|\eta_j| + \frac{1}{4} K_3 l^2. \tag{2.11}$$

Lemma 2.7. *Let $g(0) = \eta_1$, $g(l) = \eta_2$, $g'(l) = k_1$ and $|g^{(3)}(s)| \le K_3$ in $(0, l)$. Then for $s \in [0, l]$*

$$|g(s)| \le \max|\eta_i| + \frac{l}{4}|k_1| + \frac{2}{81} K_3 l^3. \tag{2.12}$$

Lemmas 2.4–2.7 are taken from [24] with a modification in (2.7) and improvements in (2.8) and (2.12).

We have by Lemma 2.5 (with $g = \varphi|_{P_2P_3}$) and assumptions (2.3) and (2.6)

$$\left|\left(\varphi\,\Big|_{P_2P_3}\right)\right| \le \frac{1}{16\cdot 24}\cdot 4\,M_4a^4 = \frac{1}{96}\,M_4a^4\,, \tag{2.13}$$

$$\left|\left(\frac{\partial\varphi}{\partial a}\,\Big|_{P_2P_3}\right)\right| \le \frac{1}{24}\cdot 4\,M_4a^3 = \frac{1}{6}\,M_4a^3\,, \tag{2.14}$$

where $\partial/\partial a$ denotes the derivative in the direction of P_2P_3. Similarly, Lemma 2.6 with $g = \partial\varphi/\partial n_a|_{P_2P_3}$ yields

$$\left|\left(\frac{\partial\varphi}{\partial n_a}\,\Big|_{P_2P_3}\right)\right| \le \frac{4\sqrt{3}}{6^3}\,M_4a^3\,. \tag{2.15}$$

Using estimates (2.14), (2.15) and Lemma 2.4 we find for an arbitrary direction s

$$\left|\left(\frac{\partial\varphi}{\partial s}\,\Big|_{P_2P_3}\right)\right| \le \frac{43}{6^3}\,M_4a^3\,. \tag{2.16}$$

Let $P \in \overline{T}$, $P \ne P_1$ and let B be the point of the segment P_2P_3 which lies on the straight line determined by P_1 and P. Setting $l = \operatorname{dist}(B, P_1)$ and considering the function $g = \varphi\,\big|_{P_1B}$ we obtain by means of Lemma 2.5 and (2.3), (2.6), (2.13), (2.16)

$$|\varphi(P)| \le \frac{1}{96}\,M_4a^4 + \frac{4\,l}{27}\,\frac{43}{6^3}\,M_4a^3 + \frac{1}{16\cdot 24}\cdot 4M_4l^4, \tag{2.17}$$

$$\left|\frac{\partial\varphi}{\partial s}(P)\right| \le \frac{3}{2\cdot 96}M_4\frac{a^4}{l} + \frac{43}{6^3}M_4a^3 + \frac{1}{6}M_4l^3. \tag{2.18}$$

Estimate (2.17) implies (2.4). Estimate (2.18) will be used in deriving (2.5).

Relation (2.9) from Lemma 2.5 with $g = \varphi|_{P_2P_3}$ and relation (2.11) from Lemma 2.6 with $g = \partial\varphi/\partial n_a|_{P_2P_3}$ together with assumption (2.3) yield

$$\left|\frac{\partial^2\varphi}{\partial a^2}(B)\right| \le 2M_4a^2, \qquad \left|\frac{\partial^2\varphi}{\partial a\partial n_a}(B)\right| \le M_4a^2.$$

Hence, according to the second part of Lemma 2.4 where we set $\psi = \partial\varphi/\partial a$,

$$\left|\frac{\partial^2\varphi}{\partial a\partial s}(B)\right| \le 3M_4a^2. \tag{2.19}$$

Using Lemma 2.7 with $g = \partial\varphi/\partial a|_{P_1B}$ and taking into account relations (2.3), (2.14), (2.19) we find

$$\left|\frac{\partial\varphi}{\partial a}(P)\right| \le \frac{1}{6}M_4a^3 + \frac{3}{4}M_4a^2l + \frac{8}{81}M_4l^3. \tag{2.20}$$

Inequalities (2.18) and (2.20) together with Lemma 2.4 imply (2.5). □

Now we introduce some consequences of Theorem 2.3.

Theorem 2.8. *A polynomial $p \in P_{3,2}$ is uniquely determined by its ten values*

$$D^\alpha p(P_j) \quad |\alpha| \le 1, \quad (j = 1, 2, 3); \quad \frac{\partial p}{\partial n_a}(Q_1)\,, \tag{2.21}$$

where the meaning of the symbols P_i, Q_1 and n_a is the same as in Theorem 2.3.

Proof. It is sufficient to prove the uniqueness. Let us assume that the values (2.21) are equal to zero. Setting $\varphi(x_1, x_2) = p(x_1, x_2)$ in Theorem 2.3 we have $M_4 = 0$ and estimate (2.4) implies $p(x_1, x_2) \equiv 0$. □

Theorem 2.9. *Let $u \in C^1(\overline{T})$ and let*

$$|D^\alpha u(P)| \le M_4 \quad \forall |\alpha| = 4, \quad \forall P \in T.$$

Let $p \in P_{3,2}$ satisfies the relations

$$\begin{gathered} D^\alpha p(P_j) = D^\alpha u(P_j), \quad |\alpha| \le 1 \quad (j = 1, 2, 3), \\ \frac{\partial p}{\partial n_a}(Q_1) = \frac{\partial u}{\partial n_a}(Q_1)\,. \end{gathered} \tag{2.22}$$

Then the function

$$\varphi(x_1, x_2) \equiv u(x_1, x_2) - p(x_1, x_2) \tag{2.23}$$

satisfies relations (2.4) and (2.5).

Proof. It follows from the assumptions of Theorem 2.9 that function (2.23) satisfies all conditions of Theorem 2.3. □

Remark 2.10. We return to the first part of the proof or Theorem 2.1: If the right-hand sides of (2.1) are equal to zero, then also $(\partial \Pi u / \partial n_a)(Q_1) = 0$ and $(\Pi u)(x, y) \equiv 0$, according to Theorem 2.8. □

It follows from Theorem 2.9 that triangular finite elements with polynomials $p \in P_{3,2}$ uniquely determined by parameters (2.21) can be used in triangulations satisfying the maximum angle condition: Estimate (2.5) requires the next-to-smallest angles of all triangles to be bounded away from zero. This requirement (we call it *the second angle condition*) is equivalent with the maximum angle condition.

Some triangular finite elements of the Hermite type are sketched in Fig. 2. The black circle denotes the function value, the arrows and double arrows denote the first and second normal derivatives, respectively, and the circled integers k denote the values $D^\alpha p(P_i)$, $|\alpha| \le k$, where P_i is the centre of the circle.

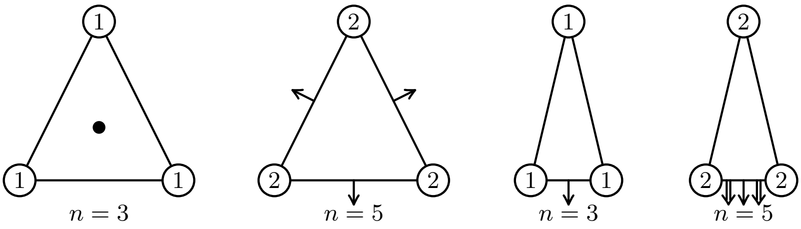

Fig. 2. Triangular finite elements of the Hermite type.

Remark 2.11. The method of the proof of Theorem 2.3 does not work successfully in the case of the classical Hermite triangular finite element of third degree where the last condition (2.3) is substituted by $\varphi(P_0)=0$, P_0 being the center of gravity of $\overline{T}$, because we obtain only

$$|(\partial^2\varphi/\partial a\partial n_a|_{P_2P_3})| \le KM_4 l^3/a \quad (l=\operatorname{dist}(P_1Q_1))$$

and $l/a \to \infty$ with $a\to 0$.

The hypothesis (H.2) is not also satisfied. This can be proved by the following example: Let $u(x,y)=y^4$ and let the triangle $\overline{T}$ have the vertices $P_1(0,0)$, $P_2(1,0)$, $P_3(0,1)$. Then the polynomial of third degree satisfying the first nine conditions (2.22) and condition $p(P_0)=u(P_0)$, where P_0 is the center of gravity of $\overline{T}$, has the form

$$p(x,y)=\frac{4}{3}\left(xy-\frac{3}{4}y^2-x^2y-xy^2+\frac{3}{2}y^3\right).$$

We see that $\partial u/\partial x \equiv 0$ while $\partial p/\partial x \ne 0$ in T. Thus hypothesis (H.2) is not satisfied and we cannot apply Jamet's theory on this finite element.

Remark 2.12. In [2, p. 222] the parameters

$$D^\alpha p(P_j) \quad |\alpha|\le 1 \quad (j=1,2,3); \qquad \iint_T \frac{\partial^2 p}{\partial x\partial y}dxdy \tag{2.24}$$

were considered in connection with the maximum angle condition for a cubic triangular finite element on a right triangle with the sides P_1P_2 and P_2P_3 lying on the axes x and y, respectively. However, parameters (2.24) do not determine in a general case a polynomial $p\in P_{3,2}$ uniquely. To prove it let us consider a triangle with vertices $P_i(x_i,y_i)$ $(i=1,2,3)$ and let T_0 be the triangle lying in the ξ,η-plane with vertices $P_1^*(0,0)$, $P_2^*(1,0)$, $P_3^*(0,1)$. The transformation

$$x=x(\xi,\eta)\equiv x_1+\overline{x}_2\xi+\overline{x}_3\eta, \quad y=y(\xi,\eta)\equiv y_1+\overline{y}_2\xi+\overline{x}_3\eta, \tag{2.25}$$

where

$$\overline{x}_j=x_j-x_1, \quad \overline{y}_j=y_j-y_1 \quad (j=2,3), \tag{2.26}$$

maps the triangle $\overline{T}_0$ one-to-one onto $\overline{T}$. Let us set

$$p^*(\xi,\eta) = p(x(\xi,\eta), y(\xi,\eta)). \tag{2.27}$$

If all ten parameters (2.24) are equal to zero then

$$D^\alpha p^*(P_j^*) = 0 \quad |\alpha| \le 1 \quad (j = 1,2,3), \tag{2.28}$$

$$\iint_{T_0} \left\{ -\overline{x}_3\overline{y}_3 \frac{\partial^2 p^*}{\partial \xi^2} + (\overline{x}_2\overline{y}_3 + \overline{x}_3\overline{y}_2)\frac{\partial^2 p^*}{\partial\xi\partial\eta} - \overline{x}_2\overline{y}_2\frac{\partial^2 p^*}{\partial \eta^2} \right\} d\xi d\eta = 0. \tag{2.29}$$

Relations (2.28) imply

$$p^*(\xi,\eta) = K\xi\eta(1-\xi-\eta). \tag{2.30}$$

Inserting (2.30) into (2.29) we obtain

$$K\{2(\overline{x}_2\overline{y}_2 + \overline{x}_3\overline{y}_3) - (\overline{x}_2\overline{y}_3 + \overline{x}_3\overline{y}_2)\} = 0. \tag{2.31}$$

If the difference standing in braces is different from zero then (2.31) implies $K = 0$ and parameters (2.24) determine uniquely $p \in P_{3,2}$. However, if

$$2(\overline{x}_2\overline{y}_2 + \overline{x}_3\overline{y}_3) = \overline{x}_2\overline{y}_3 + \overline{x}_3\overline{y}_2\,, \tag{2.32}$$

then (2.31) is satisfied with $K \neq 0$ and $p(x,y) \neq 0$, according to (2.30) and (2.27).

Let us describe these situations. It cannot be simultaneously $\overline{x}_2 = \overline{x}_3 = 0$ (and similarly $\overline{y}_2 = \overline{y}_3 = 0$). Let $\overline{x}_2 \neq 0$. If $\overline{y}_2 = 0$ then (2.32) gives $\overline{x}_3 = \overline{x}_2/2$ with arbitrary $\overline{y}_3 \neq 0$. Conversely, if $\overline{x}_3 = \overline{x}_2/2$ then (2.32) implies $\overline{y}_2 = 0$. In other cases

$$\overline{y}_3 = \frac{(2\overline{x}_2 - \overline{x}_3)\overline{y}_2}{\overline{x}_2 - 2\overline{x}_3} \quad (\overline{y}_2 \neq 0,\ \overline{x}_2 \neq 2\overline{x}_3).$$

The situation $\overline{x}_3 \neq 0$ can be treated similarly with the same results. □

Now we mention briefly some higher-degree polynomials. We shall modify the family of triangular finite elements introduced by Koukal in [7] and [8].

Theorem 2.13. *Let $u \in C^k(\overline{T})$ $(k \ge 1)$. A polynomial $p \in P_{2k+1,2}$ is uniquely determined by conditions*

$$D^\alpha p(P_j) = D^\alpha u(P_j), \quad |\alpha| \le k \quad (j = 1,2,3), \tag{2.33}$$

$$\frac{\partial^r p}{\partial n_a^r}(Q_j^{(r)}) = \frac{\partial^r u}{\partial n_a^r}(Q_j^{(r)}) \quad (j = 1,\dots,r;\ r = 1,\dots,k)\,, \tag{2.34}$$

where the symbol $\partial/\partial n_a$ has the meaning as in Theorem 2.3 and $Q_1^{(r)},\dots,Q_r^{(r)}$ $(1 \le r \le k)$ are the points dividing the side P_2P_3 into $r+1$ parts of the same length.

Theorem 2.14. *Let $u \in C^k(\overline{T})$ $(k \geq 1)$. A polynomial $\Pi u \in P_{2k+1,2}$ is uniquely determined by the conditions*

$$D^\alpha(\Pi u)(P_j) = D^\alpha u(P_j), \quad |\alpha| \leq 1 \quad (j = 1, 2, 3), \tag{2.35}$$

$$\frac{\partial^r(\Pi u)}{\partial s_2^r}(Q_j^{(r)}) = \frac{\partial^r u}{\partial s_2^r}(Q_j^{(r)}) \quad (j = 1, \dots, r;\ r = 1, \dots, k), \tag{2.36}$$

where $\partial/\partial s_2$ denotes the derivative in the direction of the side P_3P_1.

For $k = 1$ the assertions of both theorems are contained in Theorems 2.1 and 2.8. In the case $k \geq 2$ the proof is a modification of the proof of [18, Theorem 17.1].

Generalizing a little the preceding considerations we can prove:

Theorem 2.15. *Let $u \in W^{2k+2,p}(T)$, where $k \geq 1$ and $p \in [1, \infty]$, and let the operator Π be defined by (2.35), (2.36). Then we have for $m = 0, 1$*

$$|u - \Pi u|_{m,p,T} \leq C \frac{h_T^{2k+1}}{\cos(\gamma_T/2)} |u|_{2k+2,p,T}. \tag{2.37}$$

Remark 2.16. A generalization of Theorem 2.3 to the case of interpolation polynomials introduced in Theorem 2.13 is possible. Instead of special Lemmas 2.5–2.7 we can use [16, Theorem 2]. We obtain the estimates

$$|\varphi(P)| \leq C\, M_{2k+2} c^{2k+2}, \qquad \left|\frac{\partial \varphi}{\partial x_j}(P)\right| \leq \frac{C}{\sin\beta}\, M_{2k+2} c^{2k+1},$$

where $P \in \overline{T}$ and $j = 1, 2$.

Remark 2.17. The construction of finite elements introduced in Theorem 2.13 implies the following conjecture: *It is impossible to construct a triangular finite C^1-element which satisfies the maximum angle condition.*

3 Variational crimes and semiregular finite elements in the case of smooth solutions

3.A Formulation of the problem

We shall consider the boundary value problem

$$-\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(k_i(x) \frac{\partial u}{\partial x_i}\right) = f(x), \quad x \in \Omega, \tag{3.1}$$

$$u = 0 \quad \text{on } \Gamma_1, \tag{3.2}$$

$$\sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} n_i(\Omega) = q \quad \text{on } \Gamma_2, \tag{3.3}$$

where Ω is a two-dimensional bounded domain with the boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$, Γ_1 and Γ_2 being the circles with radii R_1 and $R_2 = R_1 + \varrho$, respectively. We assume that the circles Γ_1, Γ_2 have the same center S_0 and that

$$R_1 \gg \varrho. \tag{3.4}$$

The symbols $n_i(G)$ $(i = 1, 2)$ denote the components of the unit outward normal to ∂G.

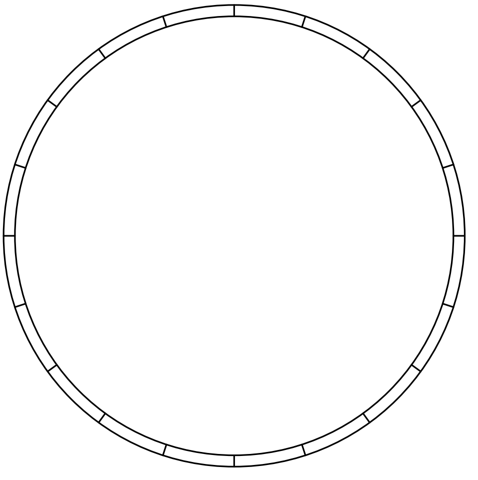

Fig. 3.

A weak solution of problem (3.1)–(3.3) is a solution of the following variational problem (which can be obtained from (3.1)–(3.3) by means of Green's theorem in a standard way).

Problem 3.1. Let Ω be a bounded domain with a Lipschitz continuous boundary $\partial\Omega = \Gamma_1 \cup \Gamma_2$. Let

$$V = \{v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_1\}, \tag{3.5}$$

$$a(w, v) = \sum_{i=1}^{2} \iint_\Omega k_i(x) \frac{\partial w}{\partial x_i} \frac{\partial v}{\partial x_i} \,\mathrm{d}x_1 \mathrm{d}x_2, \tag{3.6}$$

$$L(v) = L^\Omega(v) + L^\Gamma(v) = \iint_\Omega vf \,\mathrm{d}x_1 \mathrm{d}x_2 + \int_{\Gamma_2} vq \,\mathrm{d}s\,, \tag{3.7}$$

where

$$\begin{gathered} k_i \in W^{1,\infty}(\Omega), \quad f \in W^{1,\infty}(\Omega), \\ q = Q\big|_{\Gamma_2}, \quad Q \in C^2(\overline{U}), \end{gathered} \tag{3.8}$$

$$k_i(x) \ge \mu_0 > 0, \tag{3.9}$$

U being a neighbourhood of Γ_2 (i.e., a domain containing Γ_2). Find $u \in V$ such that

$$a(u,v) = L(v) \quad \forall v \in V. \tag{3.10}$$

Assumptions (3.8)–(3.9) guarantee that the symmetric bilinear form (3.6) is bounded and strongly coercive and that the linear form (3.7) is continuous. (Of course, this also holds when $f \in L_2(\Omega)$ and $q \in L_2(\Gamma_2)$. We assume (3.8) because of numerical integration.)

Lemma 3.2. *Let a solution $u \in V$ of Problem 3.1 satisfy $u \in H^2(\Omega)$. Then relation (3.1) holds almost everywhere in Ω and relation (3.3) holds almost everywhere on Γ_2.*

The proof is omitted. Also the following lemma is well-known:

Lemma 3.3. *If (3.9) holds then Problem 3.1 has a unique solution.*

We shall solve Problem 3.1 approximately by the finite element method. To this end let us approximate Γ_2 by a regular polygon Γ_{2h} with vertices $Q_1,\dots,Q_n$ such that every segment Q_iQ_{i+1} has no common point with Γ_1. Let the vertices $P_1,\dots,P_n$ of the polygon Γ_{1h} approximating Γ_1 be obtained in the following way: P_i is the intersection of the segment S_0Q_i with Γ_1. The symbol Ω_h will denote the polygonal domain with the boundary $\partial\Omega_h = \Gamma_{1h} \cup \Gamma_{2h}$.

We divide each segment P_iQ_i by the points $A_1^i, A_2^i, \dots, A_{m-1}^i$ into m parts of the same length in such a way that we have formally $A_0^i = P_i$, $A_m^i = Q_i$. The points A_j^i are the vertices of quadrilaterals into which the domain Ω_h is divided. Such a division of Ω_h will be denoted $\mathcal{D}_h^K$. If we divide each quadrilateral of $\mathcal{D}_h^K$ into two triangles we obtain a division $\mathcal{D}_h^T$ (see Fig. 4). We shall also consider an auxiliary division $\mathcal{D}_h^A$ which will be constructed from $\mathcal{D}_h^K$ by dividing each quadrilateral $A_{m-1}^i A_{m-1}^{i+1} Q_i Q_{i+1}$ into two triangles.

We admit to use narrow quadrilaterals and narrow triangles. This means that we shall have

$$\frac{\varrho}{m} \ll h \tag{3.11}$$

in our considerations, where h is the length of the greatest segment in the division of Ω_h.

We shall assume that $k_i \in W^{1,\infty}(\widetilde{\Omega})$, $f \in W^{1,\infty}(\widetilde{\Omega})$, where $\widetilde{\Omega}$ is such that $\Omega_h \subset \widetilde{\Omega}$ for sufficiently small h. When we consider the functions k_i and f in Ω_h we shall use symbols $\widetilde{k}_i$ and $\widetilde{f}$. In the opposite case the original symbols k_i and f will be used.

The discrete problem is now formulated in an almost standard way. (The expression "almost" concerns the approximation of the term $L^\Gamma(v)$ which needs some space.) Let $\mathcal{D}_h$ denote one of the three divisions $\mathcal{D}_h^K$, $\mathcal{D}_h^T$, $\mathcal{D}_h^A$. We define

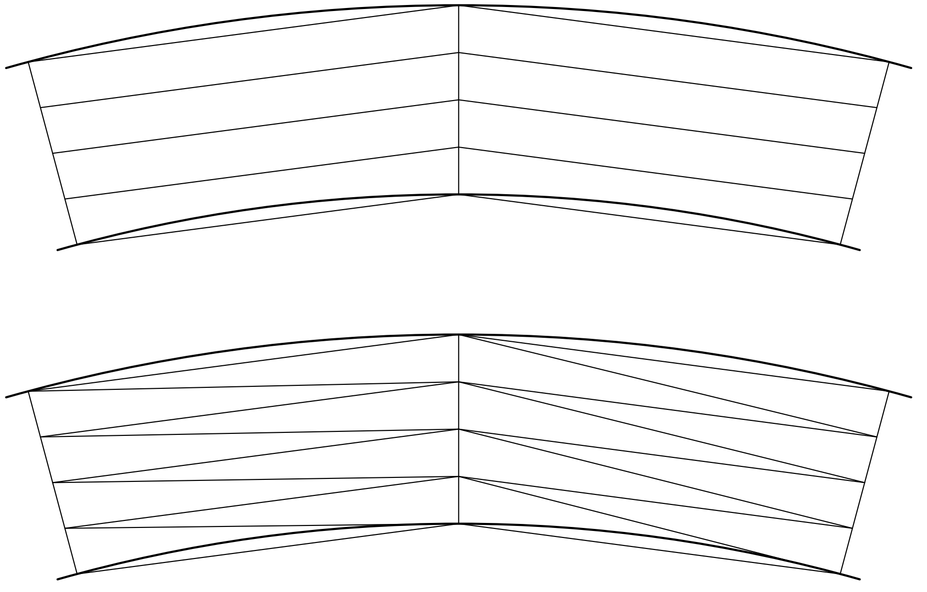

Fig. 4.

spaces

$$X_h = \{v \in C(\overline{\Omega}_h) : v\big|_K = \text{ a four-node isoparametric function } \quad \forall \overline{K} \in \mathcal{D}_h,$$
$$v\big|_T = \text{ a linear polynomial } \quad \forall \overline{T} \in \mathcal{D}_h\} \tag{3.12}$$

and

$$V_h = \{v \in X_h : v = 0 \text{ on } \Gamma_{1h}\}. \tag{3.13}$$

We set for all $v, w \in H^1(\Omega_h)$

$$\widetilde{a}_h(v,w) = \sum_{i=1}^{2} \iint_{\Omega_h} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \tag{3.14}$$

and

$$\widetilde{L}_h^{\Omega}(v) = \iint_{\Omega_h} v \widetilde{f} \, \mathrm{d}x_1 \mathrm{d}x_2 \quad \forall v \in X_h. \tag{3.15}$$

To define $\widetilde{L}_h^{\Gamma}(v)$ is more complicated. Therefore, we omit it and refer only to [21].

The symbols $a_h(v,w)$, $L_h^{\Omega}(v)$ and $L_h^{\Gamma}(v)$, where $v,w \in X_h$, will denote the approximations of $\widetilde{a}_h(v,w)$, $\widetilde{L}_h^{\Omega}(v)$ and $\widetilde{L}_h^{\Gamma}(v)$, respectively, when using numerical integration. For example, in the case of $\mathcal{D}_h^T$ we have for all $v,w \in X_h$

$$a_h(v,w) = \sum_{\overline{T}\in\mathcal{D}_h^T} \sum_{i=1}^{2} \sum_{j=1}^{N_T} 2\omega_{T_0,j}\widetilde{k}_i(x_{T,j})\ \frac{\partial v}{\partial x_i}\bigg|_T \frac{\partial w}{\partial x_i}\bigg|_T \mathrm{mes}_2 T,$$

where $x_{T,j}$ are the integration points on a triangle T and $\omega_{T_0,j}$ the corresponding coefficients of the given integration formulas (prescribed on the reference triangle $\overline{T}_0$).

Now we can define the approximate problem:

Problem 3.4. Find $u_h \in V_h$ such that

$$a_h(u_h, v) = L_h(v) \quad \forall v \in V_h. \tag{3.16}$$

3.B An abstract error estimate

Definition 3.5. Let $u \in H^2(\Omega)$. We define $Q_h u \in X_h$ by

$$\begin{aligned} Q_h u\,\big|_{\overline{K}\in\mathcal{D}_h} &= Q_K u = \text{the four-node isoparametric interpolant of } u,\\ Q_h u\,\big|_{\overline{T}\in\mathcal{D}_h} &= I_T u = \text{the linear interpolant of } u, \end{aligned}$$

where $\mathcal{D}_h$ is one of the divisions $\mathcal{D}_h^K$, $\mathcal{D}_h^T$, $\mathcal{D}_h^A$.

Lemma 3.6. *Let Γ_0 be the circle with a center S_0 and radius $R_0 = R_1 - \varrho$. Let $\widetilde{\Omega}$ be a bounded domain such that $\partial\widetilde{\Omega} = \Gamma_0 \cup \Gamma_2$. There exists a linear and bounded extension operator $E : H^k(\Omega) \to H^k(\widetilde{\Omega})$ such that the constant C appearing in the inequality*

$$\|E(v)\|_{k,\widetilde{\Omega}} \le C\|v\|_{k,\Omega} \quad \forall v \in H^k(\Omega)$$

does not depend on R_1/ϱ and v. The operator E is also a linear and bounded extension operator from $H^{k-i}(\Omega)$ into $H^{k-i}(\widetilde{\Omega})$ $(1 \le i \le k)$.

Lemma 3.6 follows from the considerations introduced in [13, pp. 20–22].

Theorem 3.7. *Let $u \in H^2(\Omega)$, $\widetilde{u} := E(u)$ and let the condition*

$$\|v\|_{1,\Omega_h}^2 \le C a_h(v,v) \quad \forall v \in V_h, \quad \forall h \in (0,h_0) \tag{3.17}$$

be satisfied, where the constant C does not depend on v and h and where h_0 is sufficiently small. Then Problem 3.4 has a unique solution $u_h \in V_h$ and there *exists a positive constant C_0 independent of $u \in H^2(\Omega)$ and $w \in V_h$ such that*

$$C_0^{-1}\|\widetilde{u} - u_h\|_{1,\Omega_h} \leq \|Q_h u - \widetilde{u}\|_{1,\Omega_h} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|a_h(Q_h u, w) - \widetilde{a}_h(Q_h u, w)|}{\|w\|_{1,\Omega_h}} +$$

$$+ \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Gamma}(w) - L_h^{\Gamma}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)|}{\|w\|_{1,\Omega_h}}. \tag{3.18}$$

Theorem 3.7 is proved in [21]. Our first aim is to prove that condition (3.17) is satisfied. This will be done in subsection 3.D, where we also give estimates of the second, third and fourth terms appearing on the right-hand side of (3.18). These terms express the error of numerical integration.

The estimate of the first term, which expresses the interpolation error, is introduced in subsection 3.C. This estimate follows from the known interpolation theorems. The fifth term, which expresses the error due to the approximation of the boundary, will be estimated in subsection 3.E.

3.C The interpolation error

The estimate of the first term appearing on the right-hand side of (3.18) follows from Theorems 1.3 and 1.4:

Theorem 3.8. *We have*

$$\|Q_h u - \widetilde{u}\|_{1,\Omega_h} \leq Ch\|u\|_{2,\Omega},$$

where the constant C is independent of h, u and the division $\mathcal{D}_h$.

3.D The effect of numerical integration

The effect of numerical integration must be analyzed more carefully than in the case of regular elements. In the case of triangles the result is that the numerical integration does not depend on the geometry of triangles and that the degrees of precision of quadrature formulas sufficient for the rate of convergence $O(h)$ are the same as in the regular case (except for the integration along the boundary Γ_{2h} – see Theorem 3.18). The proofs of the assertions presented in this subsection can be found in [21].

First we mention the analysis of the numerical integration on quadrilaterals. Let $\overline{K}$ be a quadrilateral whose greatest side lies on the axis x_1 and let it have the vertices

$$P_1(h, 0),\ P_2(0, 0),\ P_3(\delta\cos\beta, \delta\sin\beta),\ P_4(h - \varepsilon\cos\alpha, \varepsilon\sin\alpha)$$

where $\varepsilon = \operatorname{dist}(P_1, P_4)$, $\delta = \operatorname{dist}(P_2, P_3)$ and α and β are the angles at P_1 and P_2, respectively. As each quadrilateral belonging to $\mathcal{D}_h$ has parallel long sides we have

$$b := \frac{\varrho}{m} = \varepsilon\sin\alpha = \delta\sin\beta.$$

Let $\overline{K}_0$ be the reference square lying in the coordinate system ξ_1, ξ_2 and having the vertices $P_1^*(1,0)$, $P_2^*(0,0)$, $P_3^*(0,1)$, $P_4^*(1,1)$. If we denote

$$\varepsilon_3 = \delta \cos\beta, \quad \varepsilon_4 = \varepsilon \cos\alpha, \quad \varepsilon^* = \varepsilon_3 + \varepsilon_4,$$

then the one-to-one mapping of $\overline{K}_0$ onto $\overline{K}$ has the form

$$x_1 = h\xi_1 + \varepsilon_3\xi_2 - \varepsilon^*\xi_1\xi_2, \quad x_2 = b\xi_2. \tag{3.19}$$

If the side P_1P_2 makes an angle φ with the axis x_1 and the vertex P_2 has coordinates x_{10}, x_{20} then (3.19) is substituted by the mapping

$$\begin{aligned} x_1 = x_1^K(\xi_1,\xi_2) &\equiv x_{10} + (h\xi_1 + \varepsilon_3\xi_2 - \varepsilon^*\xi_1\xi_2)\cos\varphi - b\xi_2 \sin\varphi, \\ x_2 = x_2^K(\xi_1,\xi_2) &\equiv x_{20} + (h\xi_1 + \varepsilon_3\xi_2 - \varepsilon^*\xi_1\xi_2)\sin\varphi + b\xi_2 \cos\varphi. \end{aligned} \tag{3.20}$$

Both transformations (3.19) and (3.20) have the same Jacobian

$$J_K = (h - \varepsilon^*\xi_2)b.$$

It should be noted that for $n \gg 1$ we have

$$\varepsilon_i \approx \frac{1}{2n}\big(2\pi(R_1 + \Delta + \frac{\varrho}{m}) - 2\pi(R_1 + \Delta)\big) = \frac{\pi\varrho}{nm} \quad (i = 3,4;\ 0 \le \Delta \le \varrho(1 - 1/m)).$$

Further

$$h \approx \frac{2\pi R_1}{n}.$$

The last two relations imply in this case

$$\varepsilon_i = \sigma_i b, \quad \sigma_i \le Ch \quad (i = 3,4). \tag{3.21}$$

Let us denote

$$(1) := 2, \quad (2) := 1, \quad \kappa_{ij} = (-1)^{i+j}. \tag{3.22}$$

Then we can write (omitting the subscript K at J)

$$\frac{\partial \xi_i}{\partial x_j} = \kappa_{ij}\frac{1}{J}\frac{\partial x_{(j)}}{\partial \xi_{(i)}} \quad (i,j = 1,2) \tag{3.23}$$

and the theorem on transformation of an integral yields

$$E_K\Big(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i}\frac{\partial w}{\partial x_i}\Big) = E_{K_0}\Big(\sum_{i,r,s=1}^{2} \widetilde{k}_i^* \chi_{irs}\frac{\partial v^*}{\partial \xi_r}\frac{\partial w^*}{\partial \xi_s}\Big) \tag{3.24}$$

where

$$E_K(F) := \iint_K F(x_1, x_2)\,\mathrm{d}x_1\mathrm{d}x_2 - \sum_{j=1}^{N_K} \omega_{K_0,j} F(x_{K,j}) |J_K(\xi_{1j}, \xi_{2j})|, \tag{3.25}$$

$$F^*(\xi_1, \xi_2) := F(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2)),$$

$$E_{K_0}(F) := \iint_{K_0} F^*(\xi_1, \xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2 - \sum_{j=1}^{N_K} \omega_{K_0,j} F^*(\xi_{1j}, \xi_{2j}), \tag{3.26}$$

$$\chi_{irs} = \kappa_{ir}\kappa_{is} \frac{1}{J} \frac{\partial x_{(i)}}{\partial \xi_{(r)}} \frac{\partial x_{(i)}}{\partial \xi_{(s)}}$$

with $[\xi_{1j}, \xi_{2j}]$ the integration points on $\overline{K}_0$.

Theorem 3.9. *Let*

$$E_{K_0}(p) = 0 \quad \forall p \in \mathcal{P}_2,$$

where $\mathcal{P}_k$ denotes the set of polynomials in two variables of degree not greater than k. Then we have

$$\left| E_K \left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \right| \le Ch \max_{i=1,2} \|\widetilde{k}_i\|_{1,\infty,K} |v|_{1,K} |w|_{1,K} \quad \forall v,\, w \in X_h.$$

As the Jacobian J of both transformations (3.19) and (3.20) is the same the proof in both cases is very similar.

Remark 3.10. In the cases when relation (3.21) is not satisfied (however, the long sides are parallel) the assertion of Theorem 3.9 can be proved provided

$$E_{K_0}(p) = 0 \quad \forall p \in \mathcal{P}_4.$$

Remark 3.11. The case of a quadrilateral K with parallel long sides is a special case of quadrilaterals K satisfying the condition

$$|\varepsilon \sin \alpha - \delta \sin \beta| \le Cbh. \tag{3.27}$$

It can be proved that the results of Theorem 3.9 and Remark 3.10 can be extended to the case (3.27).

The effect of numerical integration in the case of narrow triangles must be analyzed more carefully than in the case of regular triangles. Let $\overline{T}$ be an arbitrary triangle lying in the plane x_1, x_2 and let $\overline{T}_0$ be the reference triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$ lying in the plane ξ_1, ξ_2. Let

$$x_1 = x_1(\xi_1, \xi_2), \quad x_2 = x_2(\xi_1, \xi_2) \tag{3.28}$$

be the linear transformation which maps $\overline{T}_0$ one-to-one onto $\overline{T}$ (for its form see, for example, (2.25), (2.26)) and let $\xi_1 = \xi_1(x_1, x_2)$, $\xi_2 = \xi_2(x_1, x_2)$ be its inverse.

Lemma 3.12. *Let $v \in C^1(\overline{T})$ and let*

$$v^*(\xi_1, \xi_2) = v(x_1(\xi_1, \xi_2), x_2(\xi_1, \xi_2)).$$

Then we have

$$\left\| \sum_{r=1}^{2} \frac{\partial v^*}{\partial \xi_r} \frac{\partial \xi_r}{\partial x_i} \right\|_{0,T_0} \leq C|J|^{-1/2} |v|_{1,T},$$

where J is the Jacobian of (3.28).

The error functionals E_T and E_{T_0} on a triangle $\overline{T}$ and the reference triangle $\overline{T}_0$, respectively, are defined in a similar way as E_K and E_{K_0} (see (3.25) and (3.26)), their expression is only simpler. Using Lemma 3.12 we can prove the following theorem.

Theorem 3.13. *Let $\overline{T}$ be an arbitrary triangle (not necessarily satisfying the maximum angle condition). Let*

$$E_{T_0}(p) = 0 \quad \forall p \in \mathcal{P}_0.$$

Then we have

$$\left| E_T \left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) \right| \leq Ch \max_{i=1,2} |\widetilde{k}_i|_{1,\infty,T} |v|_{1,T} |w|_{1,T} \quad \forall v,\, w \in X_h.$$

For $v, w \in V_h$ we have

$$a_h(v, w) = \widetilde{a}_h(v, w) - \{\widetilde{a}_h(v, w) - a_h(v, w)\},$$

$$\widetilde{a}_h(v, w) - a_h(v, w) = \sum_{\overline{K} \in \mathcal{D}_h} E_K \left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right) + \sum_{\overline{T} \in \mathcal{D}_h} E_T \left(\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \right).$$

Using these relations we obtain from Theorems 3.9 and 3.13 (details are similar as in the proof of [18, Theorem 11.8]; we use in addition the discrete Friedrichs' inequality of the type [18, (29.1)] (for its proof see Appendix) which together with (3.9) implies $\|v\|^2_{1,\Omega_h} \leq C\widetilde{a}_h(v, v)\ \forall v \in V_h$):

Corollary 3.14. *If the forms $a_h(v, w)$, where $v, w \in X_h$, are computed from $\widetilde{a}_h(v, w)$ by means of quadrature formulas required in Theorems 3.9 and 3.13, then condition (3.17) is satisfied.*

Theorem 3.15. *Let*

$$E_{K_0}(p) = 0 \quad \forall p \in \mathcal{P}_2, \qquad E_{T_0}(p) = 0 \quad \forall p \in \mathcal{P}_0.$$

Then we have for $u \in H^2(\Omega)$

$$\sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|a_h(Q_h u, w) - \widetilde{a}_h(Q_h u, w)|}{\|w\|_{1,\Omega_h}} \leq Ch \max_{i=1,2} \|\widetilde{k}_i\|_{1,\infty,\widetilde{\Omega}} \|u\|_{2,\Omega}\,, \tag{3.29}$$

where the constant C does not depend on u, $\widetilde{k}_i$, and h.

Proof. Relation (3.29) follows from Theorems 3.9, 3.13 and 1.3, 1.4. Details are the same as in the proof of [18, Theorem 11.12]. □

Theorem 3.16. *Let*

$$E_{K_0}(p) = 0 \quad \forall p \in \mathcal{P}_2 \text{ (or } \forall p \in \mathcal{Q}_1),$$
$$E_{T_0}(p) = 0 \quad \forall p \in \mathcal{P}_0\,,$$

where $\mathcal{Q}_1$ is the set of all bilinear polynomials. Then we have

$$\sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Omega}(w) - L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} \leq Ch\|\widetilde{f}\|_{1,\infty,\widetilde{\Omega}}\sqrt{\mathrm{mes}_2 \Omega},$$

where the constant C does not depend on $\widetilde{f}$ and h.

In order to estimate the effect of numerical integration along Γ_2 we introduce the following error functionals:

$$E_r(F) := \int_0^{l_r} F(\xi_r)\mathrm{d}\xi_r - \sum_{j=1}^{N_r} l_r \beta_{r,j} F(s_{r,j}),$$
$$E_0(F^*) := \int_0^1 F^*(t)\mathrm{d}t - \sum_{j=1}^{N_r} \beta_{r,j} F^*(t_j),$$

where $s_{r,j}$ are integration points on $[0, l_r]$, $\beta_{r,j}$ the corresponding coefficients of the given integration formula and

$$F^*(t) := F(l_r t), \quad t \in I \equiv [0,1].$$

Hence

$$E_r(F) = l_r E_0(F^*).$$

When considering the line integrals we need also the trace inequalities which are introduced in the following lemma.

Lemma 3.17. *We have*

$$\|v\|_{0,\partial\Omega} \leq \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega), \tag{3.30}$$

$$\|v\|_{0,\partial\Omega_h} \leq \frac{C}{\sqrt{\varrho}}\|v\|_{1,\Omega_h} \quad \forall v \in H^1(\Omega_h), \tag{3.31}$$

where the constant C does not depend on v, h and ϱ.

The proofs of (3.30) and (3.31) are similar to [12, pp. 15–16]).

Theorem 3.18. *Let*

$$E_0(p) = 0 \quad \forall p \in \mathcal{P}_2.$$

Then we have

$$\sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|\widetilde{L}_h^{\Gamma}(w) - L_h^{\Gamma}(w)|}{\|w\|_{1,\Omega_h}} \leq \frac{C}{\sqrt{\varrho}} h^2 M_2(q) \sqrt{\mathrm{mes}_1 \Gamma_2},$$

where the constant C does not depend on q, ϱ and h and where $M_2(q)$ depends on the first and second derivatives of the function Q at the points of Γ_2 (as to the relation between q and Q see (3.8)).

3.E The error of the approximation of the boundary

The estimate of the last term in (3.18) will be divided into several lemmas.

Notation 3.19. We denote

$$\tau_h = \Omega_h - \overline{\Omega}, \quad \omega_h = \Omega - \overline{\Omega}_h. \tag{3.32}$$

Further, let $w \in X_h$. The symbol $\overline{w}$ is called the natural extension of w and denotes the function $\overline{w} : \overline{\Omega}_h \cup \overline{\Omega} \to R^1$ such that $\overline{w} = w$ on Ω_h and

$$\overline{w}\big|_{\overline{T}^{\,\mathrm{id}} - \overline{T}} = p\big|_{\overline{T}^{\,\mathrm{id}} - \overline{T}},$$

where $p \in \mathcal{P}_1$ satisfies $p\big|_{\overline{T}} = w\big|_{\overline{T}}$. ($\overline{T}^{\,\mathrm{id}} \subset \Omega$ is the curved triangle which is approximated by $\overline{T}$.)

Lemma 3.20. *Let $u \in H^2(\Omega)$. Then we have for $w \in V_h$*

$$\begin{aligned} |\widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w)| \leq{}& |L^{\Gamma}(\overline{w}) - \widetilde{L}_h^{\Gamma}(w)| + \\ &+ \left| \iint_{\omega_h} \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(k_i \frac{\partial u}{\partial x_i}\Big) \overline{w} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| + \\ &+ \left| \iint_{\omega_h} \sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 \right| + \\ &+ \left| \iint_{\tau_h} \Big(\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i}\Big) + \widetilde{f} \Big) w \, \mathrm{d}x_1 \mathrm{d}x_2 \right|. \end{aligned} \tag{3.33}$$

Proof. Using the definitions of $\widetilde{a}_h(\widetilde{u}, w)$, $\widetilde{L}_h(w)$ and Green's theorem we obtain

$$\begin{aligned} \widetilde{a}_h(\widetilde{u}, w) - \widetilde{L}_h(w) ={}& \iint_{\Omega_h} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} \frac{\partial w}{\partial x_i} \, \mathrm{d}x_1 \mathrm{d}x_2 - \\ &- \widetilde{L}_h^{\Omega}(w) - \widetilde{L}_h^{\Gamma}(w) = \int_{\Gamma_{2h}} \sum_{i=1}^{2} \widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i} n_i(\Omega_h) w \, \mathrm{d}s - \\ &- \iint_{\Omega_h} \Big(\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\Big(\widetilde{k}_i \frac{\partial \widetilde{u}}{\partial x_i}\Big) + \widetilde{f} \Big) w \, \mathrm{d}x_1 \mathrm{d}x_2 - \widetilde{L}_h^{\Gamma}(w). \end{aligned}$$

To the right-hand side let us add zero in the form

$$-\int_{\Gamma_2}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}n_i(\Omega)\overline{w}\,\mathrm{d}s+L^{\Gamma}(\overline{w})=0.$$

If we denote $\Delta=\overline{T}^{\,\mathrm{id}}-T$ and use Lemma 3.2 then we can write

$$\begin{aligned}\widetilde{a}_h(\widetilde{u},w)-\widetilde{L}_h(w)=&-\sum_{\Delta\subset\omega_h}\int_{\partial\Delta}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}n_i(\Delta)\overline{w}\,\mathrm{d}s-\\&-\iint_{\tau_h}\Big(\sum_{i=1}^{2}\frac{\partial}{\partial x_i}\Big(\widetilde{k}_i\frac{\partial\widetilde{u}}{\partial x_i}\Big)+\widetilde{f}\Big)w\,\mathrm{d}x_1\mathrm{d}x_2+L^{\Gamma}(\overline{w})-\widetilde{L}_h^{\Gamma}(w).\end{aligned}$$

Transforming the first term on the right-hand side by means of Green's theorem we obtain (3.33). □

The third term on the right-hand side is most disagreeable. It is estimated in the following lemma:

Lemma 3.21. *Let $u\in H^2(\Omega)$ and $\widetilde{k}_i\in W^{1,\infty}(\Omega)$ $(i=1,2)$. Then*

$$\Big|\iint_{\omega_h}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}\frac{\partial\overline{w}}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2\Big|\le Ch^2\frac{\sqrt{m}}{\varrho}\max_{i=1,2}\|k_i\|_{1,\infty,\Omega}\|u\|_{2,\Omega}\|w\|_{1,\Omega_h}. \quad (3.34)$$

If in addition

$$u\in H^2(\Omega)\cap W^{1,\infty}(\Omega), \quad (3.35)$$

then

$$\Big|\iint_{\omega_h}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}\frac{\partial\overline{w}}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2\Big|\le Ch^2\sqrt{\frac{m}{\varrho}}\max_{i=1,2}\|k_i\|_{1,\infty,\Omega}\|u\|_{1,\infty,\Omega}\|w\|_{1,\Omega_h}. \quad (3.36)$$

Proof. We have

$$\Big|\iint_{\omega_h}\sum_{i=1}^{2}k_i\frac{\partial u}{\partial x_i}\frac{\partial\overline{w}}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2\Big|\le\max_{i=1,2}\|k_i\|_{1,\infty,\Omega}|u|_{1,\omega_h}|\overline{w}|_{1,\omega_h}. \quad (3.37)$$

Assumption (3.35) gives

$$|u|_{1,\omega_h}\le Ch|u|_{1,\infty,\Omega}. \quad (3.38)$$

Let us denote $\Delta=T^{\mathrm{id}}-\overline{T}$. Then

$$\begin{aligned}|\overline{w}|^2_{1,\omega_h}&=\sum_{\Delta\subset\omega_h}\mathrm{mes}_2\Delta|(\nabla w\,\big|_T)|^2\le C\sum_{\Delta\subset\omega_h}h_T^3|(\nabla w\,\big|_T)|^2=\\&=C\frac{m}{\varrho}\sum_{\Delta\subset\omega_h}h_T^3\frac{\varrho}{m}|(\nabla w\,\big|_T)|^2\le C\frac{m}{\varrho}h^2\sum_{\Delta\subset\omega_h}|w|^2_{1,T}\le C\frac{m}{\varrho}h^2|w|^2_{1,\Omega_h}\end{aligned}$$

because
$$\frac{\varrho}{m} h_T |(\nabla w\,|_T)|^2 \le C|w|^2_{1,T}.$$
Hence
$$|w|_{1,\omega_h} \le Ch\sqrt{\frac{m}{\varrho}}|w|_{1,\Omega_h}. \tag{3.39}$$
Combining (3.37)–(3.39) we obtain (3.36). For the proof of (3.34) see [21]. □

Estimate (3.36) cannot be improved. Thus, if we want to obtain the rate of convergence $O(h)$ we must assume that
$$C_1 h^2 \le \frac{\varrho}{m} \quad (C_1 > 0). \tag{3.40}$$
Assumption (3.40) is also necessary in estimating the first term on the right-hand side of (3.33) if we want to obtain in it the rate of convergence $O(h)$ (see [21]).

3.F The final result

All preceding results yield the following theorem:

Theorem 3.22. *Let us consider a division $\mathcal{D}_h^T$ (or $\mathcal{D}_h^A$). Let $u \in H^2(\Omega)$, $\widetilde{f} \in W^{1,\infty}(\widetilde{\Omega})$, $\widetilde{k}_i \in W^{1,\infty}(\widetilde{\Omega})$ $(i = 1, 2)$. Let assumptions (3.8)$_{3,4}$, (3.9), (3.40) and assumptions concerning the degrees of precision of the quadrature formulas (see Theorems 3.9, 3.13, 3.15, 3.16 and 3.18) be satisfied. Then*
$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \le \frac{C}{\sqrt{\varrho}} h\,, \tag{3.41}$$
where the constant C does not depend on u, ϱ, m, h and the division $\mathcal{D}_h^T$ (or $\mathcal{D}_h^A$).

If in addition $u \in W^{1,\infty}(\Omega)$ (see (3.35)) then
$$\|\widetilde{u} - u_h\|_{1,\Omega_h} \le Ch\,, \tag{3.42}$$
where again the constant C does not depend on u, ϱ, m, h and the division $\mathcal{D}_h^T$ (or $\mathcal{D}_h^A$).

Theorem 3.23. *If we use divisions $\mathcal{D}_h^K$ for the definition of the spaces X_h then the assertions of Theorem 3.22 remain without changes.*

For the proof see [21, pp. 390–392].

Now we mention results in the case of the boundary value problem of equation (3.1) with boundary conditions opposite to conditions (3.2) and (3.3):
$$u = 0 \quad \text{on } \Gamma_2, \tag{3.43}$$
$$\sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} n_i(\Omega) = q \quad \text{on } \Gamma_1. \tag{3.44}$$

In this case Problem 3.4 and all results up to relation (3.32) inclusive remain without changes, except for Lemma 3.2, where (3.3) is replaced by (3.44), and except for the definition of $\mathcal{D}_h^A$: we divide into two triangles each quadrilateral $P_i P_{i+1} A_1^i A_1^{i+1}$. Doing some additional considerations (see [21, pp. 393–397]) we obtain the following theorems:

Theorem 3.24. *Let the assumptions of Theorem 3.22 be satisfied except for the additional assumption $u \in W^{1,\infty}(\Omega)$ which is substituted by $\widetilde{u} \in W^{1,\infty}(\widetilde{\Omega})$. Then estimates (3.41) and (3.42) are again valid.*

Theorem 3.25. *If we use divisions $\mathcal{D}_h^K$ for the definition of the spaces X_h then the assertions of Theorem 3.24 remain without changes.*

Remark 3.26. Modifying considerations of [12, Chapter 4] we can prove the following regularity results: Let $j \geq 1$. If $k_i \in C^{j-1,1}(\overline{\Omega})$, $f \in W_2^{j-1}(\Omega)$, $q \in C^{j-1,1}(\Gamma_r)$ ($r = 1$ or 2) then $u \in H^{j+1}(\Omega)$. This means that the assumption guaranteeing (3.42) can be satisfied.

4 Composite domains in magnetostatical problems

In this section we restrict ourselves for a greater simplicity to triangular elements. We shall study the situation indicated in Fig. 5, where the circle consists of three subdomains, the middle one being very narrow. We shall see that in such a case requirement (3.40) can be omitted.

Problem 4.1. Let Ω be a simply connected domain with a Lipschitz continuous boundary $\partial\Omega$ such that
$$\overline{\Omega} = \overline{\Omega}^R \cup \overline{\Omega}^A \cup \overline{\Omega}^S$$
where R, S and A stand for rotor, stator and air, respectively, and Ω^R, Ω^S and Ω^A are domains with Lipschitz continuous boundaries. Let

$$V = \{v \in H^1(\Omega) : v = 0 \quad \text{on } \Gamma_1\}, \tag{4.1}$$

$$\left.\begin{aligned} a(w,v) &= \sum_{i=1}^{2} \iint_{\Omega} \nu(|\nabla w|^2) \frac{\partial w}{\partial x_i}\frac{\partial v}{\partial x_i}\,\mathrm{d}x_1\mathrm{d}x_2, \\ \nu \equiv \nu_0 \text{ in } \Omega^A, &\quad \nu \equiv \nu_0\nu_r^R \text{ in } \Omega^R, \quad \nu \equiv \nu_0\nu_r^S \text{ in } \Omega^S, \end{aligned}\right\} \tag{4.2}$$

$$L(v) = L^{\Omega}(v) + L^{\Gamma}(v) = \iint_{\Omega} vf\,\mathrm{d}x_1\mathrm{d}x_2 + \int_{\Gamma_2} vq\,\mathrm{d}s\,, \tag{4.3}$$

where $f \in L_2(\Omega)$, $q \in L_2(\Gamma_2)$. Find $u \in H^1(\Omega)$ such that

$$u - z \in V, \tag{4.4}$$
$$a(u,v) = L(v) \quad \forall v \in V, \tag{4.5}$$

where $z \in W^{1,p}(\Omega)$ ($p > 2$) satisfies $\operatorname{tr} z = \overline{u}$ on Γ_1. (We note that as usual $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\operatorname{mes}_1 \Gamma_1 > 0$.) □

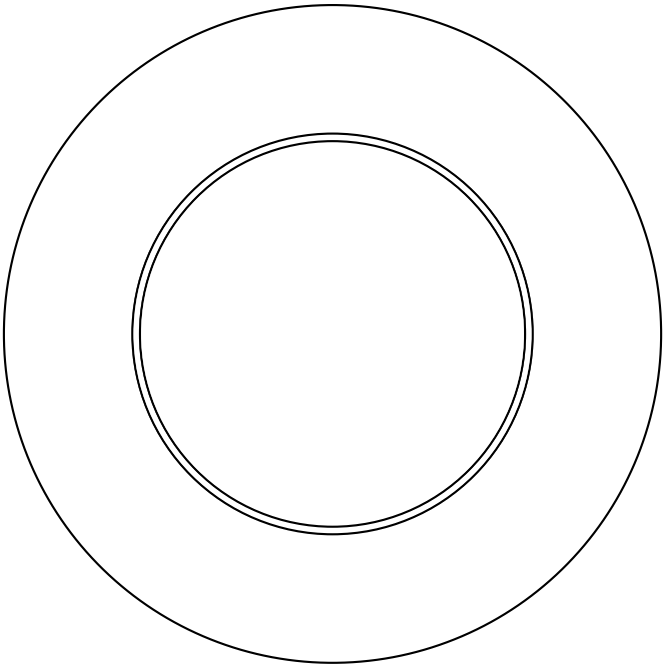

Fig. 5.

Problem 4.1 corresponds to a two-dimensional magnetostatical problem; its connection with Maxwell's equations is explained, for example, in [20] — here we only note that $u = u(x, y)$ has the physical meaning of the z-component of the magnetic potential vector $\overrightarrow{A} = (0, 0, u)$, the positive function $\nu = \nu(s)$ is the magnetic reluctivity, $f = f(x, y)$ is the z-component of the external current density vector $\overrightarrow{J_e} = (0, 0, f)$ and $\overline{u}$ and q are functions appearing on the right-hand sides of the Dirichlet and Neumann boundary conditions, respectively.

We have $\nu_r^M \in C^\infty([0, \infty))$. Using the expression for ν_r^M, which is introduced, e.g., in [10], [11], we can prove (similarly as in [18, Example 33.3]) that there exist positive constants β_1^M, β_2^M $(M = R, S)$ such that

$$\beta_1^M \leq \frac{\mathrm{d}}{\mathrm{d}s}(s\nu_r^M(s^2)) \leq \beta_2^M \quad \forall s \in [0, \infty), \quad M = R,\, S. \tag{4.6}$$

Property (4.6) has an important consequence: if we integrate (4.6) in $[0, t]$ $(t > 0)$ then we obtain

$$\beta_1^M \leq \nu_r^M(t^2) \leq \beta_2^M \quad \forall t \in (0, \infty).$$

This result and the continuity of ν_r^M give

$$\beta_1^M \leq \nu_r^M(s^2) \leq \beta_2^M \quad \forall s \in [0, \infty). \tag{4.7}$$

Making use of (4.6), (4.7) we can prove that Problem 4.1 has a unique solution $u \in H^1(\Omega)$ (see [20, Lemma 2 and Theorem 3]).

In order to obtain a discrete solution of Problem 4.1 by the finite element method we triangulate the closed domain $\overline{\Omega}$ in such a way that the triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ is a union of triangulations $\mathcal{T}_h^R$, $\mathcal{T}_h^S$ and $\mathcal{T}_h^A$ of $\overline{\Omega}^R$, $\overline{\Omega}^S$ and $\overline{\Omega}^A$, respectively. On the contrary to the standard theories we assume that the minimum angle condition

$$\vartheta_h^M := \min_{\overline{T} \in \mathcal{T}_h^M} \vartheta_T \geq \vartheta_0 > 0 \quad \forall h \in (0, h_0), \tag{4.8}$$

where ϑ_T is the magnitude of the minimum angle of $\overline{T}$, is satisfied only for $M = R,\, S$. As the domain Ω^A is very narrow the triangulations $\mathcal{T}_h^A$ are supposed to satisfy the *maximum angle condition*

$$\gamma_T \leq \gamma_0 < \pi \quad \forall \overline{T} \in \mathcal{T}_h^A, \quad \forall h \in (0, h_0), \tag{4.9}$$

where γ_T is the magnitude of the maximum angle of $\overline{T}$.

Assumption 4.2. In order to simplify our considerations we shall assume that Ω^S, Ω^A and Ω^R are such that $\partial\Omega^S = \partial K_1 \cup \partial K_2$, $\partial\Omega^A = \partial K_2 \cup \partial K_3$ and $\partial\Omega^R = \partial K_3$, where ∂K_1, ∂K_2 and ∂K_3 are circles with the same center S_0 and radii R_1, R_2 and R_3, respectively, which satisfy the relations

$$R_1 > R_2 > R_3 > 0, \quad R_3 = R_2 - \varrho, \quad R_1 - R_2 \gg \varrho, \quad R_3 \gg \varrho$$

where $\varrho > 0$ is fixed (see Fig. 5). □

The discrete problem is formulated in a standard way. We define the spaces

$$X_h = \{v \in C(\overline{\Omega}_h) : v\big|_T = \text{ a linear polynomial } \quad \forall T \in \mathcal{T}_h\}, \tag{4.10}$$

$$V_h = \{v \in X_h : v = 0 \text{ on } \overline{\Gamma}_{1h}\} \tag{4.11}$$

and the set

$$W_h = \{v \in X_h : v(P_i) = \overline{u}(P_i) \quad \forall P_i \in \sigma_h \cap \overline{\Gamma}_1\}\,, \tag{4.12}$$

where $\overline{\Omega}_h$ is the union of the closed triangles $\overline{T} \in \mathcal{T}_h$, $\overline{\Gamma}_{1h}$ is the part of $\partial\Omega_h$ approximating $\overline{\Gamma}_1$ and σ_h is the set of all nodes of $\mathcal{T}_h$. Further we set

$$a_h(v,w) = \sum_{M=R,A,S} \sum_{i=1}^{2} \iint_{\Omega_h^M} \nu^M(|\nabla v|^2) \frac{\partial v}{\partial x_i} \frac{\partial w}{\partial x_i} \,\mathrm{d}x_1 \mathrm{d}x_2 \quad \forall v, w \in H^1(\Omega_h), \tag{4.13}$$

which gives

$$a_h(v,w) = \sum_{M=R,S} \sum_{\overline{T} \in \mathcal{T}_h^M} \sum_{i=1}^{2} \nu_0 \nu_r^M(|\nabla(v\big|_T)|^2) \left.\frac{\partial v}{\partial x_i}\right|_T \left.\frac{\partial w}{\partial x_i}\right|_T \mathrm{mes}_2 T \, + \\ + \sum_{\overline{T} \in \mathcal{T}_h^A} \sum_{i=1}^{2} \nu_0 \left.\frac{\partial v}{\partial x_i}\right|_T \left.\frac{\partial w}{\partial x_i}\right|_T \mathrm{mes}_2 T \quad \forall v, w \in X_h. \tag{4.14}$$

Finally, we set

$$L_h(v) = L_h^{\Omega}(v) + L_h^{\Gamma}(v) \quad \forall v \in X_h\,, \tag{4.15}$$

where $L_h^{\Omega}(v)$ and $L_h^{\Gamma}(v)$ are the approximations of the forms

$$\widetilde{L}_h^{\Omega}(v) = \iint_{\Omega_h} vf \,\mathrm{d}x_1\mathrm{d}x_2\,, \qquad \widetilde{L}_h^{\Gamma}(v) = \int_{\Gamma_{2h}} q_h v \,\mathrm{d}s \tag{4.16}$$

by means of quadrature formulas of first degree of precision. (Details and the definition of the function q_h are introduced in [4], [18] and [21].) Using (4.10)–(4.15) we define:

Problem 4.3. Find $u_h \in W_h$ such that

$$a_h(u_h, v) = L_h(v) \quad \forall v \in V_h. \tag{4.17}$$

It can be proved similarly as in [4], [17] or [18] that every discrete problem has a unique solution u_h. The main result of this section is the following theorem.

Theorem 4.4. *Let the solution $u \in H^1(\Omega)$ of Problem 4.1 satisfy*

$$u_M \in H^2(\Omega^M) \quad (M = R,\, S,\, A), \tag{4.18}$$

where $u_M := u\big|_{\Omega^M}$. Let $f \in W^{1,\infty}(\Omega)$ and $q \in C^1(\overline{\Gamma}_2)$. Then we have for all $h \in (0, h_0)$

$$\|u_h - u\|_{1,\Omega_h} \le C\,h\,, \tag{4.19}$$

where $u \in H^1(\Omega)$ is the solution of Problem 4.1, $\|\cdot\|_{1,\Omega_h}$ is the norm in the space $H^1(\Omega_h)$ and C is a constant independent of $h := \max_{\overline{T}\in\mathcal{T}_h} h_T$ and ϱ.

Assumption (4.18) is guaranteed if $\Gamma_2 = \emptyset$ and $\overline{u}$ is sufficiently smooth.

The proof of Theorem 4.4 is based on the following abstract error estimate which can be proved in the same way as [4, Theorem 3.3.1] or [18, Theorem 38.5]:

$$\|u - u_h\|_{1,\Omega_h} \le C\left\{\inf_{v\in W_h} \|u - v\|_{1,\Omega_h} + \sup_{\substack{w\in V_h\\ w\neq 0}} \frac{|a_h(u,w) - L_h(w)|}{\|w\|_{1,\Omega_h}}\right\}, \tag{4.20}$$

where the constant C does not depend on h and ϱ. The two terms on the right-hand side of (4.20) will be estimated in Theorems 4.8 and 4.13.

The following lemma is a reformulation of Lemma 3.6:

Lemma 4.5. *Let ∂K_4 be the circle with the center S_0 and radius $R_4 = R_2 - 2\varrho$, where ϱ is the same as in Assumption 4.2. Let $\widetilde{\Omega}^S$, $\widetilde{\Omega}^A$ be bounded domains such that $\partial\widetilde{\Omega}^S = \partial K_1 \cup \partial K_4$, $\partial\widetilde{\Omega}^A = \partial K_2 \cup \partial K_4$. There exist linear and bounded extension operators $E_M : H^2(\Omega^M) \to H^2(\widetilde{\Omega}^M)$ $(M = S,\ A)$ such that the constant C_M appearing in the inequality*

$$\|E_M(v)\|_{2,\widetilde{\Omega}^M} \le C_M \|v\|_{2,\Omega^M} \quad \forall v \in H^2(\Omega^M)$$

does not depend on R_2/ϱ and v.

Remark 4.6. As Lemma 4.5 is used in the proof of Theorem 4.8 the polygonal domains Ω_h^A must be situated between the circles ∂K_2 and ∂K_4. We derive now the expression for the minimum number of vertices of such a polygonal domain in the case $\varrho/R_2 < 10^{-1}$.

Let A_1 be an arbitrary point of the circle ∂K_2 and let t be one of the two tangents to the circle ∂K_3 which pass through the point A_1. Let $B = t \cap \partial K_3$, $A_2 = \{t \cap \partial K_2\} - \{A_1\}$. If $\varrho/R_2 < 10^{-1}$ then we can neglect the terms depending on ϱ^3 and find

$$d_1 = \operatorname{dist}(A_1, B) = (2\varrho R_2 - \varrho^2)^{1/2}, \quad d_2 = \operatorname{dist}(A_1, A_2) = 2d_1.$$

Let us approximate ∂K_2 by a regular polygon with vertices $P_1, \dots, P_n$ where

$$n = n_2 = \left[\frac{2\pi R_2}{d_2}\right] + 1 = \left[\frac{\pi R_2}{(2\varrho R_2 - \varrho^2)^{1/2}}\right] + 1.$$

Let the vertices $Q_1, \dots, Q_n$ of the polygon ∂K_3^h approximating ∂K_3 be obtained in the following way: Q_i is the intersection of the segment $S_0 P_i$ with ∂K_3.

For example, if $\varrho = 1\,\text{mm}$ and $R_2 = 50\,\text{mm}$ then $n_2 = 16$. This is a surprisingly small number. Of course, it is better to use the relation

$$n = n_1 = \left[\frac{2\pi R_2}{d_1}\right] + 1.$$

In the case $\varrho = 1\,\text{mm}$, $R_2 = 50\,\text{mm}$ we have $n_1 = 32$.

If we divide every quadrilateral $P_i P_{i+1} Q_i Q_{i+1}$ into two triangles we obtain a triangulation which satisfies (from a practical point of view) the maximum angle condition only: For $n = n_1$ the minimum angle is less than 6 degrees and for $n = n_2$ less than 3 degrees. □

Lemma 4.7. *If the solution $u \in H^1(\Omega)$ of Problem 4.1 satisfies assumption (4.18) then*

$$N_i^M(u) := \nu^M(|\nabla u_M|^2) \frac{\partial u_M}{\partial x_i} \in H^1(\Omega^M) \quad (M = R, S, A). \tag{4.21}$$

Consequently,

$$\sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left(\nu^M(|\nabla u_M|^2)\frac{\partial u_M}{\partial x_i}\right) + f_M = 0 \quad \text{a.e. in } \Omega^M \quad (M = R, S, A), \tag{4.22}$$

$$q = \sum_{i=1}^{2} \nu^S(|\nabla u_S|^2)\frac{\partial u_S}{\partial x_i} n_i(\Omega^S) \quad \text{a.e. on } \Gamma_2, \tag{4.23}$$

where $f_M = f|_{\Omega^M}$ *and the symbols* $n_i(G)$ $(i = 1, 2)$ *denote the components of the unit outward normal to* ∂G*. Finally,*

$$\nu_r^M \frac{\partial u_M}{\partial n}\bigg|_{\partial K_j} = \frac{\partial u_A}{\partial n}\bigg|_{\partial K_j} \qquad \text{a.e. on } \partial K_j \quad (M = R, S), \tag{4.24}$$

where $j = 2$ *for* $M = S$ *and* $j = 3$ *for* $M = R$ *and* $\partial/\partial n$ *is the normal derivative (the orientation of* n *can be chosen arbitrarily).*

Theorem 4.8. *Under the assumptions of Theorem 4.4 we have*

$$\inf_{v \in W_h} \|u - v\|_{1,\Omega_h} \le C\,h\left\{\sum_{M=R,S} (1 + \sup|\nu_r^M|)\|u_M\|_{2,\Omega^M} + \|u_A\|_{2,\Omega^A}\right\}, \tag{4.25}$$

where the constant C *does not depend on both* h *and* ϱ*.*

For the proofs of Lemma 4.7 and Theorem 4.8 see [20, Lemma 12 and Theorem 13].

Notation 4.9. a) We denote

$$\omega_h^M := \Omega^M - \overline{\Omega}_h^M, \quad \tau_h^M := \Omega_h^M - \overline{\Omega}^M.$$

b) The natural extension $\overline{w}_M$ of $w_M := w\big|_{\Omega_h^M}$ from $\overline{\Omega}_h^M$ onto $\overline{\Omega}_h^M \cup \overline{\Omega}^M$ is the function $\overline{w}_M : \overline{\Omega}_h^M \cup \overline{\Omega}^M \to R^1$ satisfying $\overline{w}_M = w_M$ on $\overline{\Omega}_h^M$ and

$$\overline{w}_M\big|_{T^{\rm id}} = p\big|_{T^{\rm id}} \quad \text{on} \quad T^{\rm id} \supset T,$$

where p is the polynomial of first degree satisfying $p\big|_T = w\big|_T$ and $T^{\rm id}$ is the ideal curved triangle associated with T (it is also called the exact curved triangle). (For more detail see [18] or [4].)

c) The natural extension $\overline{w}$ of $w \in X_h$ is the function $\overline{w} : \Omega \to R^1$ such that $\overline{w} = w$ on $\overline{\Omega}_h$ and $\overline{w} = \overline{w}_S$ on ω_h^S.

Lemma 4.10. *We have*

$$\|v\|_{0,\tau_h^M} \le C(h\|v\|_{0,\partial K_{i+1}} + h^2|v|_{1,\tau_h^M}) \quad \forall v \in H^1(\tau_h^M) \quad (M = S,\ A),$$

where $i = 1$ *and* $i = 2$ *for* $M = S$ *and* $M = A$*, respectively, and where the constant* C *does not depend on both* h *and* ϱ*.*

Lemma 4.10 follows from the proof of [18, Lemma 28.3].

Lemma 4.11. *We have for all* $w \in X_h$

$$\|\overline{w}_M\|_{0,\varepsilon_h^M} \le C\, h \|w\|_{1,\Omega_h} \quad (\varepsilon = \tau, \omega;\ M = R, S), \tag{4.26}$$

$$|\overline{w}_M|_{1,\varepsilon_h^M} \le C\, h^{1/2} |w|_{1,\Omega_h} \quad (\varepsilon = \tau, \omega;\ M = R, S). \tag{4.27}$$

Proof. As $\mathcal{T}_h^R$, $\mathcal{T}_h^S$ satisfy the minimum angle condition estimates (4.26), (4.27) follow from [4, Lemma 3.3.12].

Lemma 4.12. *We have for all* $w \in X_h$

$$|L_h^\Omega(w) - \widetilde{L}_h^\Omega(w)| \le C\, h \|f\|_{1,\infty,\Omega} \|w\|_{1,\Omega_h}, \tag{4.28}$$

$$|L_h^\Gamma(w) - \widetilde{L}_h^\Gamma(w)| \le C\, h (\mathrm{mes}_1 \Gamma_2)^{1/2} |q|_{1,\infty,\Gamma_2} \|w\|_{1,\Omega_h}, \tag{4.29}$$

$$|\widetilde{L}_h^\Gamma(w) - L^\Gamma(\overline{w})| \le C\, h^{3/2} \|q\|_{0,\Gamma_2} \|w\|_{1,\Omega_h}. \tag{4.30}$$

For the proof of (4.28), (4.29) and (4.30) see, for example, [3, Theorem 4.5.1], [18, Lemma 30.1] and [4, Lemma 3.3.13], respectively.

Theorem 4.13. *Under the assumptions of Theorem 4.4 we have for all* $w \in V_h$

$$\begin{aligned} |a_h(u,w) - L_h(w)| \le C\, h \{ &\|f\|_{1,\infty,\Omega} + (\mathrm{mes}_1 \Gamma_2)^{1/2} \|q\|_{1,\infty,\Gamma_2} + \\ &+ (1 + \sup |\nu_r^S|) \|u_S\|_{2,\Omega^S} + \sum_{M=A,R} \|u_M\|_{2,\Omega^M} + \\ &+ \sum_{i=1}^{2} \left\| \partial N_i^S(u)/\partial x_i \right\|_{0,\Omega^S} \} \|w\|_{1,\Omega_h}, \end{aligned} \tag{4.31}$$

where $N_i^S(u)$ *is defined in* (4.21).

Proof. Instead of S, A and R we shall write 1, 2 and 3, respectively. We have

$$|a_h(u,w) - L_h(w)| \le |a_h(u,w) - \widetilde{L}_h(w)| + |\widetilde{L}_h(w) - L_h(w)|, \tag{4.32}$$

where

$$\widetilde{L}_h(w) = \widetilde{L}_h^\Omega(w) + \widetilde{L}_h^\Gamma(w). \tag{4.33}$$

After a longer computation we obtain (see [20, pp. 413–415])

$$a_h(u,w) - \widetilde{L}_h(w) = D_1 + \sum_{j=1}^{2} (D_2^{(j,j)} - D_2^{(j+1,j)}) - D_3 - D_4\,, \tag{4.34}$$

where

$$\begin{aligned}
D_1 &= L^{\Gamma}(\overline{w}) - \widetilde{L}_h^{\Gamma}(w),\\
D_2^{(k,j)} &= \sum_{i=1}^{2} \iint_{\tau_h^j} \nu^k(|\nabla u_{j+1}|^2) \frac{\partial u_{j+1}}{\partial x_i} \frac{\partial w}{\partial x_i}\, \mathrm{d}x_1 \mathrm{d}x_2,\\
D_3 &= \sum_{i=1}^{2} \iint_{\omega_h^1(2)} \overline{w} \frac{\partial}{\partial x_i} \left(\nu^1(|\nabla u_1|^2) \frac{\partial u_1}{\partial x_i} \right) \mathrm{d}x_1 \mathrm{d}x_2,\\
D_4 &= \sum_{i=1}^{2} \iint_{\omega_h^1(2)} \nu^1(|\nabla u_1|^2) \frac{\partial u_1}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i}\, \mathrm{d}x_1 \mathrm{d}x_2,
\end{aligned}$$

where $\omega_h^1(2)$ denotes the part of ω_h^1 which is adjacent to Γ_2.

The estimate of $|D_1|$ is given in (4.30). The term $D_2^{(k,j)}$ is of the same type as the term appearing in Lemma 3.21. However, the presence of the domains $\Omega^R \equiv \Omega^3$, $\Omega^S \equiv \Omega^1$ enable us to avoid requirement (3.40). It follows from (4.7) that

$$|D_2^{(k,j)}| \leq K |u_{j+1}|_{1,\tau_h^j} |w|_{1,\tau_h^j}. \tag{4.35}$$

As $u_{j+1} \in H^2(\Omega^{j+1})$ we have by Lemma 4.10

$$|u_{j+1}|_{1,\tau_h^j} \leq C \sum_{i=1}^{2} \left(h \left\| \frac{\partial u_{j+1}}{\partial x_i} \right\|_{0,\partial K_{j+1}} + h^2 \left| \frac{\partial u_{j+1}}{\partial x_i} \right|_{1,\tau_h^j} \right). \tag{4.36}$$

The trace theorem yields

$$\left\| \frac{\partial u_3}{\partial x_i} \right\|_{0,\partial K_3} \leq C \|u_3\|_{2,\Omega^3}. \tag{4.37}$$

Owing to the fact that $u \in C(\overline{\Omega})$ we have

$$u_1 \big|_{\partial K_2} = u_2 \big|_{\partial K_2}.$$

This relation implies that

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial t} \quad \text{a.e. on } \partial K_2,$$

where $\partial/\partial t$ is the tangential derivative. Combining this result with (4.24) (where $j = 2$) and using the trace theorem on Ω^1 we derive

$$\left\| \frac{\partial u_2}{\partial x_i} \right\|_{0,\partial K_2} \leq C(1 + \sup |\nu_r^1|) \|u_1\|_{2,\Omega^1}. \tag{4.38}$$

Estimates (4.35)–(4.38) give

$$\sum_{j=1}^{2} (|D_2^{(j,j)}| + |D_2^{(j+1,j)}|) \leq Ch \Big\{ (1 + \sup |\nu_r^1|) \|u_1\|_{2,\Omega^1} + \sum_{j=2}^{3} \|u_j\|_{2,\Omega^j} \Big\} |w|_{1,\Omega_h}. \tag{4.39}$$

Relation (4.21), the Schwarz inequality and Lemma 4.11 imply

$$|D_3| \le C\,h\biggl(\sum_{i=1}^{2}\biggl\|\frac{\partial}{\partial x_i}\Bigl(\nu^1(|\nabla u_1|^2)\frac{\partial u_1}{\partial x_i}\Bigr)\biggr\|_{0,\Omega^1}\biggr)\|w\|_{1,\Omega_h}. \tag{4.40}$$

Finally, as $\mathcal{T}_h^1$ satisfies the minimum angle condition and $u_1 \in H^2(\Omega^1)$ (see (4.18)) we have by (4.7), (4.27), Lemma 4.10 (which holds also for ω_h^1 with ∂K_1 instead of ∂K_{i+1}) and the trace inequality

$$|D_4| \le K|u_1|_{1,\omega_h^1}|\overline{w}|_{1,\omega_h^1} \le C\,h^{3/2}\|u_1\|_{2,\Omega^1}\|w\|_{1,\Omega_h}. \tag{4.41}$$

Relations (4.34), (4.30), (4.39)–(4.41) give the bound of the first term on the right-hand side of (4.32). The estimate of the second term on the right-hand side of (4.32) follows from Lemma 4.12. Hence we obtain (4.31). □

Theorem 4.4 follows now from (4.20) and Theorems 4.8 and 4.13.

5 General convergence theorem

On the contrary to Section 3 we shall assume $u \in H^1(\Omega)$ only and we shall prove the convergence (without any rate of convergence) under a stronger assumption than (3.40):

$$C_1 h^{2-\delta} \le \frac{\varrho}{m} \le C_2 h^{2-\delta}\,, \tag{5.1}$$

where

$$0 < \delta < 1 \tag{5.2}$$

is a given number which can be arbitrarily small and $C_1[\mathrm{m}^{1-\delta}]$, $C_2[\mathrm{m}^{1-\delta}]$ are positive constants. The abstract error estimate has in the case $u \in H^1(\Omega)$ the form:

Theorem 5.1. *Let condition (3.17) be satisfied. Then Problem 3.4 has a unique solution $u_h \in V_h$ and we have*

$$C_0^{-1}\|\widetilde{u}-u_h\|_{1,\Omega_h} \le \inf_{v\in V_h}\biggl(\|v-\widetilde{u}\|_{1,\Omega_h} + \sup_{\substack{w\in V_h\\ w\neq 0}}\frac{|a_h(v,w)-\widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega_h}}\biggr) + \\ + \sup_{\substack{w\in V_h\\ w\neq 0}}\frac{|\widetilde{L}_h^{\Omega}(w)-L_h^{\Omega}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w\in V_h\\ w\neq 0}}\frac{|\widetilde{L}_h^{\Gamma}(w)-L_h^{\Gamma}(w)|}{\|w\|_{1,\Omega_h}} + \sup_{\substack{w\in V_h\\ w\neq 0}}\frac{|\widetilde{a}_h(\widetilde{u},w)-\widetilde{L}_h(w)|}{\|w\|_{1,\Omega_h}}, \tag{5.3}$$

where C_0 is a positive constant, $u \in H^1(\Omega)$ is the solution of Problem 3.1 and $\widetilde{u} = E(u)$ with $E : H^1(\Omega) \to H^1(\widetilde{\Omega})$ (see Lemma 3.6 where $k = 1$).

In what follows we restrict ourselves to the case of triangular elements with linear polynomials. First we generalize interpolation results for Zlámal's simplest ideal triangular finite element (see [25] and also [18]).

Let $\overline{T} \in \mathcal{D}_h^T$ be an arbitrary triangle with two vertices lying on $\partial\Omega$. We shall denote them by $P_2(x_1^{(2)}, x_2^{(2)})$, $P_3(x_1^{(3)}, x_2^{(3)})$ in such a way that

$$\operatorname{dist}(P_1, P_2) = \frac{\varrho}{m}, \tag{5.4}$$

$P_1(x_1^{(1)}, x_2^{(1)})$ being the vertex lying in Ω. Thus the smallest angle α_T of $\overline{T}$, which tends to zero with $h \to 0$, lies at P_3. The angles lying at P_1 and P_2 will be denoted by β_T and γ_T, respectively. Both these angles tend to $\pi/2$ with $h \to 0$.

Setting

$$\overline{x}_2 = x_1^{(2)} - x_1^{(1)},\ \overline{x}_3 = x_1^{(3)} - x_1^{(1)},\ \overline{y}_2 = x_2^{(2)} - x_2^{(1)},\ \overline{y}_3 = x_2^{(3)} - x_2^{(1)}$$

we can write the transformation, which maps the triangle $\overline{T}_0$ with vertices $R_1(0,0)$, $R_2(1,0)$ and $R_3(0,1)$ one-to-one onto $\overline{T}$, in the form

$$\begin{aligned} x_1 &= x_1^{(0)}(\xi_1, \xi_2) \equiv x_1^{(1)} + \overline{x}_2\xi_1 + \overline{x}_3\xi_2, \\ x_2 &= x_2^{(0)}(\xi_1, \xi_2) \equiv x_2^{(1)} + \overline{y}_2\xi_1 + \overline{y}_3\xi_2. \end{aligned} \tag{5.5}$$

We have for the triangles lying along $\partial\Omega$

$$2\mathrm{mes}_2 T = \operatorname{dist}(P_1, P_2)\operatorname{dist}(P_2, P_3)\sin\gamma_T.$$

From here, from (5.1), (5.4) and from the maximum angle condition we easily obtain

$$C_3 h_T^{3-\delta} \le \mathrm{mes}_2 T \le C_4 h_T^{3-\delta}, \tag{5.6}$$

h_T being the length of the greatest side of $\overline{T}$ and C_3, C_4 positive constants.

Now we remind some results introduced in [18, Section 22]. Let λ_h and λ be the segment P_2P_3 and the part of $\partial\Omega$ approximated by P_2P_3, respectively. Let

$$x_1 = \varphi_\lambda(\xi_2), \quad x_2 = \psi_\lambda(\xi_2), \quad \xi_2 \in [0,1], \tag{5.7}$$

be a parametric representation of λ defined on $[0,1]$ with the property

$$\varphi_\lambda(0) = x_1^{(2)},\ \varphi_\lambda(1) = x_1^{(3)},\ \psi_\lambda(0) = x_2^{(2)},\ \psi_\lambda(1) = x_2^{(3)}.$$

We define the functions $\Phi_\lambda(\xi_2)$, $\Psi(\xi_2)$ on $[0,1]$ by

$$\begin{aligned} \Phi_\lambda(\xi_2) &= [\varphi_\lambda(\xi_2) - x_1^{(2)} - \overline{x}_{32}\xi_2]/(1-\xi_2), \quad \xi_2 \in [0,1), \\ \Phi_\lambda(1) &= -\varphi_\lambda'(1) + \overline{x}_{32}, \quad \Phi_\lambda^{(j)}(1) = -\frac{1}{j+1}\varphi_\lambda^{(j+1)}(1), \\ \Psi_\lambda(\xi_2) &= [\psi_\lambda(\xi_2) - x_2^{(2)} - \overline{y}_{32}\xi_2]/(1-\xi_2), \quad \xi_2 \in [0,1), \\ \Psi_\lambda(1) &= -\psi_\lambda'(1) + \overline{y}_{32}, \quad \Psi_\lambda^{(j)}(1) = -\frac{1}{j+1}\psi_\lambda^{(j+1)}(1), \end{aligned}$$

where $\overline{x}_{32} = x_1^{(3)} - x_1^{(2)}$, $\overline{y}_{32} = x_2^{(3)} - x_2^{(2)}$. If $\varphi_\lambda, \psi_\lambda \in C^{(n+1)}([0,1])$ then, according to [18, Section 22], $\Phi_\lambda, \Psi_\lambda \in C^n([0,1])$ and

$$\begin{aligned} &\Phi_\lambda(\xi_2) = O(h_T^2), \quad \Phi_\lambda^{(j)}(\xi_2) = O(h_T^{j+1}), \quad \xi_2 \in [0,1], \\ &\Psi_\lambda(\xi_2) = O(h_T^2), \quad \Psi_\lambda^{(j)}(\xi_2) = O(h_T^{j+1}), \quad \xi_2 \in [0,1], \end{aligned} \tag{5.8}$$

where $j = 1, \ldots, n$. The symbol $\overline{T}_\lambda^{\mathrm{id}}$ will denote the curved triangle with two straight sides P_1P_2, P_1P_3 and the curved side λ.

Theorem 5.2. *Let the boundary $\partial\Omega$ of the domain Ω be piecewise of class C^{k+1}. Then for $h \in (0, h_0)$, where h_0 is sufficiently small, we have:*

a) The transformation

$$\begin{aligned} x_1 &= x_1^\lambda(\xi_1,\xi_2) \equiv x_1^{(1)} + \overline{x}_2\xi_1 + \overline{x}_3\xi_2 + \xi_1\Phi_\lambda(\xi_2), \\ x_2 &= x_2^\lambda(\xi_1,\xi_2) \equiv x_2^{(1)} + \overline{y}_2\xi_1 + \overline{y}_3\xi_2 + \xi_1\Psi_\lambda(\xi_2) \end{aligned} \tag{5.9}$$

maps one-to-one the reference triangle $\overline{T}_0$, which lies in the ξ_1,ξ_2-plane and has the vertices $R_1(0,0)$, $R_2(1,0)$, $R_3(0,1)$, onto the ideal triangle $\overline{T}_\lambda^{\mathrm{id}}$ with vertices $P_i(x_1^{(i)}, x_2^{(i)})$ ($i = 1,2,3$ - a local notation) and curved side λ, which has parametric equations (5.7), in such a way that

$$R_i \leftrightarrow P_i \quad (i=1,2,3), \qquad R_1R_j \leftrightarrow P_1P_j \quad (j=2,3), \qquad R_2R_3 \leftrightarrow \lambda \tag{5.10}$$

and $T_0 \equiv \operatorname{int}\overline{T}_0 \leftrightarrow \operatorname{int}\overline{T}_\lambda^{\mathrm{id}} \equiv T_\lambda^{\mathrm{id}}$.

b) The Jacobian $J_\lambda(\xi_1,\xi_2)$ of transformation (5.9) is different from zero on $\overline{T}_0$ and it holds for $(\xi_1,\xi_2) \in \overline{T}_0$:

$$C_5 h_T^{3-\delta} \le |J_\lambda(\xi_1,\xi_2)| \le C_6 h_T^{3-\delta} \quad (C_i = \text{const} > 0). \tag{5.11}$$

c) Both mapping (5.9) and its inverse mapping are of class C^k and for $(\xi_1,\xi_2) \in \overline{T}_0$ we have

$$\frac{\partial x_i^\lambda}{\partial \xi_1} = O(h_T^{2-\delta}), \quad \frac{\partial x_i^\lambda}{\partial \xi_2} = O(h_T) \quad (i=1,2), \tag{5.12}$$

$$\frac{\partial^2 x_i^\lambda}{\partial \xi_j \partial \xi_k} = O(h_T^2) \quad (i,j,k = 1,2), \tag{5.13}$$

$$\frac{\partial \xi_1^\lambda}{\partial x_i} = O(h_T^{-2+\delta}), \quad \frac{\partial \xi_2^\lambda}{\partial x_i} = O(h_T^{-1}) \quad (i=1,2), \tag{5.14}$$

where

$$\xi_1 = \xi_1^\lambda(x_1,x_2), \quad \xi_2 = \xi_2^\lambda(x_1,x_2) \tag{5.15}$$

is the inverse mapping to mapping (5.9).

d) Let $\widetilde{S}_1$, $\widetilde{S}_2$ be arbitrary points of $\overline{T}_0$ and S_1, S_2 their images in transformation (5.9). Let ε be the distance between $\widetilde{S}_1$, $\widetilde{S}_2$ and let η be the distance between S_1, S_2. Then

$$C_7 \varepsilon h_T^{2-\delta} \le \eta \le C_8 \varepsilon h_T\,, \tag{5.16}$$

where C_7, C_8 are positive constants independent of ε and h_T.

Proof. A) First we prove assertions concerning $J(\xi_1, \xi_2)$. Using the relations

$$|\overline{x}_2| = O(h_T^{2-\delta}),\ |\overline{y}_2| = O(h_T^{2-\delta}),\ |\overline{x}_3| = O(h_T),\ |\overline{y}_3| = O(h_T), \tag{5.17}$$

we obtain from (5.9) and (5.8)

$$J_\lambda(\xi_1, \xi_2) = [\overline{x}_2 + \varPhi_\lambda(\xi_2)][\overline{y}_3 + \xi_1 \varPsi'_\lambda(\xi_2)] - \\ - [\overline{x}_3 + \xi_1 \varPhi'_\lambda(\xi_2)][\overline{y}_2 + \varPsi_\lambda(\xi_2)] = 2\mathrm{mes}_2 T + O(h_T^3).$$

This result together with (5.6) imply both $J_\lambda(\xi_1, \xi_2) \neq 0$ on $\overline{T}_0$ and estimates (5.11).

B) The proof of inequalities (5.16) follows the same lines as part (c) of the proof of [18, Theorem 22.4]. Instead of [18, Lemma 22.2] we use the fact that at least one of the estimates

$$|\alpha_1 \overline{x}_2 + \alpha_2 \overline{x}_3| \geq Ch_T^{2-\delta}, \quad |\alpha_1 \overline{y}_2 + \alpha_2 \overline{x}_3| \geq Ch_T^{2-\delta} \tag{5.18}$$

holds, where α_1, α_2 are real numbers satisfying

$$\alpha_1^2 + \alpha_2^2 = 1. \tag{5.19}$$

If $\alpha_1 = 0$ or $\alpha_2 = 0$ then assertion (5.18) is evident. Let $\alpha_1 \neq 0$, $\alpha_2 \neq 0$. First we consider the case

$$\operatorname{sign} \alpha_1 = \operatorname{sign} \alpha_2. \tag{5.20}$$

Then the expression

$$V_1 = \frac{1}{|\alpha_1 + \alpha_2|} [(\alpha_1 \overline{x}_2 + \alpha_2 \overline{x}_3)^2 + (\alpha_1 \overline{y}_2 + \alpha_2 \overline{y}_3)^2]^{1/2}$$

is the length of the segment $P_1 P_{23}$, where

$$P_{23} = \big((|\alpha_1| x_1^{(2)} + |\alpha_2| x_1^{(3)})/|\alpha_1 + \alpha_2|, (|\alpha_1| x_2^{(2)} + |\alpha_2| x_2^{(3)})/|\alpha_1 + \alpha_2|\big)$$

is a point of the segment $P_2 P_3$. If $\beta_T \leq \pi/2$ then $V_1 > P_1 P_2$. As $P_1 P_2 \geq Ch_T^{2-\delta}$, according to (5.1), assertion (5.18) follows because by (5.19) and (5.20) we have $|\alpha_1 + \alpha_2| > 1$.

If $\beta_T > \pi/2$ then $\beta_T = \omega_T$ where ω_T is the maximum angle of T. We have $V_1 \geq d$ where d is the distance of the vertex P_1 from the segment $P_2 P_3$. As α_T is small the angle made by $P_1 P_2$ and the segment of the length d is less than $\omega_T/2$. Hence $d > P_1 P_2 \cos(\omega_T/2)$ and assertion (5.18) follows, according to the maximum angle condition.

Now let

$$\operatorname{sign} \alpha_1 = -\operatorname{sign} \alpha_2 \tag{5.21}$$

and let the point P^* be such that $P_1 = \frac{1}{2}(P_3 + P^*)$. This gives $P^* = (x_1^*, x_2^*) = 2P_1 - P_3 = (2x_1^{(1)} - x_1^{(3)}, 2x_2^{(1)} - x_2^{(3)})$ and

$$V_2 = \frac{1}{|\alpha_1| + |\alpha_2|}[(|\alpha_1|\overline{x}_2 - |\alpha_2|\overline{x}_3)^2 + (|\alpha_1|\overline{y}_2 - |\alpha_2|\overline{y}_3)^2]^{1/2}$$

is the length of the segment $P_1P_{23}^*$, where

$$P_{23}^* = \big((|\alpha_1|x_1^{(2)} + |\alpha_2|x_1^*)/(|\alpha_1| + |\alpha_2|), (|\alpha_1|x_2^{(2)} + |\alpha_2|x_2^*)/(|\alpha_1| + |\alpha_2|)\big)$$

is a point of the segment P_2P^*. Let T^* be the triangle with vertices P_1, P_2, P^*. In T^* the angle at P_1 is equal to $\pi - \beta_T$. If $\pi - \beta_T \leq \pi/2$, then $V_2 \geq P_1P_2 \geq Ch_T^{2-\delta}$.

If $\pi - \beta_T > \pi/2$ then $\pi - \beta_T = \omega_T + \alpha_T$, where $\omega_T = \gamma_T$. We have $V_2 \geq d^*$ with d^* the distance of the vertex P_1 from the segment P_2P^*. As the angle α_T^* at P^* is small, we have $d^* > P_1P_2\cos(\omega_T/2 + \alpha_T/2)$ and assertion (5.18) follows, according to the maximum angle condition, because α_T is small and β_T is not small.

C) Setting $\xi_2 = 0$ in (5.9) we obtain a parametric representation of P_1P_2:

$$x_1 = x_1^{(1)} + \overline{x}_2\xi_1, \quad x_2 = x_2^{(1)} + \overline{y}_2\xi_1, \quad \xi_1 \in [0, 1].$$

Setting $\xi_1 = 0$ in (5.9) we obtain a parametric representation of P_1P_3:

$$x_1 = x_1^{(1)} + \overline{x}_3\xi_2, \quad x_2 = x_2^{(1)} + \overline{y}_3\xi_2, \quad \xi_2 \in [0, 1].$$

Thus segments P_1P_2 and P_1P_3 are images of segments R_1R_2 and R_1R_3, respectively, in transformation (5.9).

Relations $\xi_1 = 1 - t$, $\xi_2 = t$ ($t \in [0, 1]$) form a parametric representation of the segment R_2R_3. In this case we obtain from (5.9) and the definitions of the functions Φ_λ, Ψ_λ:

$$x_1 = x_1^\lambda(1 - t, t) = \varphi(t), \quad x_2 = x_2^\lambda(1 - t, t) = \psi(t), \quad t \in [0, 1].$$

This means that the arc λ is the image of the segment R_2R_3 in transformation (5.9).

Consequently, the Jordan curve $\partial T_\lambda^{\mathrm{id}}$ is the image of the Jordan curve ∂T_0 in transformation (5.9).

Owing to inequalities (5.16) mapping (5.9) is injective. As (5.9) is also continuous on $\overline{T}_0$ it is a homeomorphism. A homeomorphism maps the interior of the Jordan curve onto the interior of its image.

If f is a homeomorphism then f is bijective and f^{-1} is continuous. Thus relations (5.10) and $\mathrm{int}\,\overline{T}_0 \leftrightarrow \mathrm{int}\,\overline{T}_\lambda^{\mathrm{id}}$ hold and mapping (5.15) is continuous.

D) Owing to [18, Lemma 22.1] mapping (5.9) is of class C^k. The validity of relations (5.12), (5.13) follows immediately from (5.9), (5.8) and (5.17).

It remains to prove the assertions concerning the inverse mapping (5.15). In part C we proved that $\xi_i^\lambda(x_1, x_2)$ are continuous on $\overline{T}_\lambda^{\mathrm{id}}$.

Using (3.22), (3.23) together with (5.11) and (5.12) we obtain (5.14) and the continuity of the first derivatives. The continuity of higher derivatives can be proved similarly as in [18, p. 184]. □

Theorem 5.3. *Let the boundary $\partial\Omega$ be piecewise of class C^3. Let the polynomial $w^*(\xi_1,\xi_2)$ of degree not greater than one be uniquely determined by the conditions*

$$w^*(R_i) = g_i \quad (i = 1, 2, 3).$$

Then the function $\widetilde{w} : \overline{T}_\lambda^{\mathrm{id}} \to R^1$ defined by the relations

$$\widetilde{w}(x_1,x_2) := w^*(\xi_1^\lambda(x_1,x_2), \xi_2^\lambda(x_1,x_2)), \quad (x_1,x_2) \in \overline{T}_\lambda^{\mathrm{id}},$$

where $\xi_i^\lambda(x_1,x_2)$ are the functions from (5.15), has the following properties:

a) it satisfies the relation

$$w^*(\xi_1,\xi_2) = \widetilde{w}(x_1^\lambda(\xi_1,\xi_2), x_2^\lambda(\xi_1,\xi_2)), \quad (\xi_1,\xi_2) \in \overline{T}_0$$

and is uniquely determined by the conditions

$$\widetilde{w}(P_i) = g_i \quad (i = 1, 2, 3); \tag{5.22}$$

b) $\widetilde{w} \in C^2(\overline{T}_\lambda^{\mathrm{id}})$;

c) the function values on both straight sides P_1P_j are polynomials in one variable of degree not greater than one uniquely determined by the parameters g_1 and g_j prescribed at P_1 and P_j, respectively;

d) if both parameters g_2, g_3 prescribed at P_2, $P_3 \in \lambda$ are equal to zero then $\widetilde{w}(x_1,x_2) = 0$ for all $(x_1,x_2) \in \lambda$.

The proof is the same as the proof of [18, Theorem 23.1].

Definition 5.4. The function $\widetilde{w} : \overline{T}_\lambda^{\mathrm{id}} \to R^1$ from Theorem 5.3 is called the ideal triangular finite C^0-element of the type $(L,1)$ (where L stands for Lagrange) belonging to $\overline{T}_\lambda^{\mathrm{id}}$ and is uniquely determined by conditions (5.22). The set of all such finite elements is briefly denoted by $(\overline{T}_\lambda^{\mathrm{id}}, L, 1)$.

Theorem 5.5. *Let the boundary $\partial\Omega$ be piecewise of class C^3. Let $u \in H^2(T_\lambda^{\mathrm{id}})$, where the curved side λ of $\overline{T}_\lambda^{\mathrm{id}}$ is not approximated by the shortest side of $\overline{T}$, and let $u_I \in (\overline{T}_\lambda^{\mathrm{id}}, L, 1)$ be the ideal triangular finite C^0-element uniquely determined by the conditions*

$$u_I(P_j) = u(P_j) \quad (j = 1, 2, 3). \tag{5.23}$$

Then

$$\|u_I - u\|_{0,T_\lambda^{\mathrm{id}}} \le Ch^2 \|u\|_{0,T_\lambda^{\mathrm{id}}}, \quad |u_I - u|_{1,T_\lambda^{\mathrm{id}}} \le Ch_T^\delta \|u\|_{2,T_\lambda^{\mathrm{id}}}, \tag{5.24}$$

where C is a constant independent of h_T, $\overline{T}_\lambda^{\mathrm{id}}$ and u.

Proof. We have, according to the theorem on transformation of an integral and Theorem 5.2,

$$\|u - u_I\|_{0,T_\lambda^{\mathrm{id}}}^2 \le Ch_T^{3-\delta} \|u^* - u_I^*\|_{0,T_0}^2. \tag{5.25}$$

Considering in the same way as in the proof of [18, Theorem 10.5] we obtain (cf. [18, (10.12)])

$$\|u^* - u_I^*\|_{0,T_0}^2 \le |u^*|_{2,T_0}^2. \tag{5.26}$$

Using again Theorem 5.2 and the theorem on transformation of an integral we find that

$$\left|\frac{\partial u^*}{\partial \xi_i}\right|_{1,T_0}^2 \le \frac{C}{h_T^{3-\delta}} h_T^4 \|u\|_{2,T_\lambda^{\mathrm{id}}}^2 \quad (i = 1, 2). \tag{5.27}$$

Combining (5.25)–(5.27) we obtain $(5.24)_1$.

Further,

$$|u_I - u|_{1,T_\lambda^{\mathrm{id}}}^2 = \iint_{T_\lambda^{\mathrm{id}}} \left\{ \left(\frac{\partial}{\partial x_1}(u_I - u)\right)^2 + \left(\frac{\partial}{\partial x_2}(u_I - u)\right)^2 \right\} \mathrm{d}x_1 \mathrm{d}x_2 \le$$

$$\le C h_T^{3-\delta} \left(h_T^{-4+2\delta} \left\| \frac{\partial}{\partial \xi_1}(u_I^* - u^*) \right\|_{0,T_0}^2 + h_T^{-2} \left\| \frac{\partial}{\partial \xi_2}(u_I^* - u^*) \right\|_{0,T_0}^2 \right). \tag{5.28}$$

Similarly as in [6]

$$\left\| \frac{\partial}{\partial \xi_i}(u_I^* - u^*) \right\|_{0,T_0}^2 \le C \left| \frac{\partial u^*}{\partial \xi_i} \right|_{1,T_0}^2 \qquad (i = 1, 2). \tag{5.29}$$

Combining (5.28), (5.29) and (5.27) we obtain $(5.24)_2$. □

Remark 5.6. In the case of the minimum angle condition we have $\delta = 1$ and Theorem 5.5 is identical with [18, Theorem 25.3] where $n = 1$.

Remark 5.7. If the curved side λ of $\overline{T}_\lambda^{\mathrm{id}}$ is approximated by the shortest side of $\overline{T}$ then h_T^δ, which appears on the right-hand side of $(5.24)_2$, is substituted by h_T.

Definition 5.8. a) Let $\mathcal{T}_h^{\mathrm{id}}$ be the ideal triangulation of $\overline{\Omega}$ corresponding to the triangulation $\mathcal{D}_h^T$. (We obtain $\mathcal{T}_h^{\mathrm{id}}$ by replacing the triangles $\overline{T} \in \mathcal{D}_h^T$ lying along $\partial\Omega$ by corresponding ideal triangles.) The symbol M_h denotes the set of ideal triangles $\overline{T}_\lambda^{\mathrm{id}} \in \mathcal{T}_h^{\mathrm{id}}$ lying along the part of $\partial\Omega$ where the homogeneous Dirichlet condition is prescribed.

b) The function $\widehat{w} \in H^1(\Omega)$ is said to be associated with a given function $w \in X_h$ if:

(i) $\widehat{w} \in C(\overline{\Omega})$;
(ii) $\widehat{w}(P_i) = w(P_i)$ at all nodal points P_i of $\mathcal{D}_h^T$;
(iii) $\widehat{w}$ is linear on each triangle $\overline{T} \in \mathcal{D}_h^T \cap \mathcal{T}_h^{\mathrm{id}}$ and on each ideal triangle $\overline{T}_\lambda^{\mathrm{id}} \notin M_h$;
(iv) if $\overline{T}_\lambda^{\mathrm{id}} \in M_h$, then

$$\widehat{w}\,\big|_{\overline{T}_\lambda^{\mathrm{id}}} = \widetilde{w}\,\big|_{\overline{T}_\lambda^{\mathrm{id}}},$$

where $\widetilde{w}$ is defined in Definition 5.4.

Now we are prepared to estimate the fifth term appearing on the right-hand side of (5.3) in the case when $u \in H^1(\Omega)$ only.

Lemma 5.9. *For all $w \in V_h$ and $U \in H^1(\widetilde{\Omega})$ satisfying $U = u$ in Ω we have*

$$\begin{aligned}
|\widetilde{L}_h(w) - \widetilde{a}_h(U,w)| \le |\widetilde{L}_h^{\Gamma}(w) - L^{\Gamma}(\overline{w})| &+ \\
+ \sum_{\overline{T}_\lambda^{\mathrm{id}} \in M_h} \left| \iint_{\overline{T}_\lambda^{\mathrm{id}}} \left\{ (\overline{w} - \widehat{w}) f + \sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} \frac{\partial(\widehat{w} - \overline{w})}{\partial x_i} \right\} \mathrm{d}x_1 \mathrm{d}x_2 \right| &+ \\
+ \left| \iint_{\tau_h} \left\{ -\sum_{i=1}^{2} \widetilde{k}_i \frac{\partial U}{\partial x_i} \frac{\partial w}{\partial x_i} + w \widetilde{f} \right\} \mathrm{d}x_1 \mathrm{d}x_2 \right| &+ \\
+ \left| \iint_{\omega_h} \left\{ \sum_{i=1}^{2} k_i \frac{\partial u}{\partial x_i} \frac{\partial \overline{w}}{\partial x_i} - \overline{w} f \right\} \mathrm{d}x_1 \mathrm{d}x_2 \right|&. \qquad (5.30)
\end{aligned}$$

Proof. We have

$$\widetilde{L}_h(w) = (\widetilde{L}_h^{\Omega}(w) - L^{\Omega}(\widehat{w})) + (\widetilde{L}_h^{\Gamma}(w) - L^{\Gamma}(\widehat{w})) + L(\widehat{w}),$$

where $\widehat{w} \in V$ is associated with $w \in V_h$ in the sense of Definition 5.8. It holds $a(u, \widehat{w}) = L(\widehat{w})$. Hence

$$-\widetilde{a}_h(U,w) = (a(u,\widehat{w}) - \widetilde{a}_h(U,w)) - L(\widehat{w}).$$

The rest of the proof is straightforward (see, for example, the proof of [18, Theorem 38.9]). □

Theorem 5.10. *We have*

$$|\widetilde{L}_h(w) - \widetilde{a}_h(\tilde{u}, w)| \le C h^{\delta/2} \|w\|_{1,\Omega_h} \quad \forall w \in V_h\,, \qquad (5.31)$$

where the constant C does not depend on h and w and where the extension $\tilde{u}$ of u has the same meaning as in Theorem 5.1.

Proof. A) Let us denote the terms appearing on the right-hand side of (5.30) by $D_1, \ldots, D_4$. By [21, Lemmas 29, 37] and assumption (5.1) we have

$$D_1 \le C h \|q\|_{0,\Gamma_1} \|w\|_{1,\Omega_h}. \qquad (5.32)$$

Now we estimate D_2. Let B_h be the union of triangles of $\mathcal{D}_h^T$ lying along the part Γ_j of $\partial\Omega$ on which the homogeneous Dirichlet boundary condition is prescribed. Using this notation we have in the case $j = 1$, according to the Cauchy inequality,

$$D_2 \le \left(\|f\|_{0,B_h - \tau_h} + \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\widetilde{\Omega}} |u|_{1,B_h - \tau_h} \right) \left(\sum_{T_\lambda^{\mathrm{id}} \in M_h} \|\widehat{w} - \overline{w}\|_{1,T_\lambda^{\mathrm{id}}}^2 \right)^{1/2} \qquad (5.33)$$

and in the case $j = 2$

$$D_2 \le \left(\|f\|_{0,B_h \cup \omega_h} + \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\widetilde{\Omega}} |u|_{1,B_h \cup \omega_h} \right) \left(\sum_{T_\lambda^{\mathrm{id}} \in M_h} \|\widehat{w} - \overline{w}\|_{1,T_\lambda^{\mathrm{id}}}^2 \right)^{1/2}. \qquad (5.34)$$

The function $\widehat{w}\big|_{\overline{T}^{\mathrm{id}}_\lambda}$, where $\overline{T}^{\mathrm{id}}_\lambda \in M_h$, interpolates the function $\overline{w}\big|_{\overline{T}^{\mathrm{id}}_\lambda}$ on $\overline{T}^{\mathrm{id}}_\lambda$. Thus Theorem 5.5 and the linearity of $\overline{w}\big|_{\overline{T}^{\mathrm{id}}_\lambda}$ give

$$\|\widehat{w}-\overline{w}\|_{1,T^{\mathrm{id}}_\lambda} \le Ch^\delta_T \|\overline{w}\|_{2,T^{\mathrm{id}}_\lambda} = Ch^\delta_T \|\overline{w}\|_{1,T^{\mathrm{id}}_\lambda}.$$

Hence in the case $j=2$ (i.e., in the case $u=0$ on Γ_2)

$$\sum_{T^{\mathrm{id}}_\lambda \in M_h} \|\widehat{w}-\overline{w}\|^2_{1,T^{\mathrm{id}}_\lambda} \le Ch^{2\delta} \sum_{T^{\mathrm{id}}_\lambda \in M_h} \|\overline{w}\|^2_{1,T^{\mathrm{id}}_\lambda} \le Ch^{2\delta}\|\overline{w}\|^2_{1,\Omega} \le$$
$$\le Ch^{2\delta}\{\|\overline{w}\|^2_{1,\Omega_h} + \|\overline{w}\|^2_{1,\omega_h}\} \quad (5.35)$$

and in the case $j=1$

$$\sum_{T^{\mathrm{id}}_\lambda \in M_h} \|\widehat{w}-\overline{w}\|^2_{1,T^{\mathrm{id}}_\lambda} \le Ch^{2\delta}\|\overline{w}\|^2_{1,\Omega_h}. \quad (5.36)$$

If $j=2$, then relations [21, (74), (75)] and $w=0$ on Γ_{2h} yield

$$\|\overline{w}\|_{1,\omega_h} \le Ch\sqrt{\frac{m}{\varrho}}|w|_{1,\Omega_h}.$$

Using (5.1) we obtain

$$\sqrt{\frac{m}{\varrho}} \le Ch^{\delta/2-1}. \quad (5.37)$$

Hence

$$\|\overline{w}\|^2_{1,\omega_h} \le Ch^\delta |w|^2_{1,\Omega_h}$$

and (5.35) implies that also in the case $j=2$ estimate (5.36) holds. Thus for $j=1,2$, according to (5.33), (5.34),

$$D_2 \le Ch^\delta \|w\|_{1,\Omega_h}\,, \quad (5.38)$$

where

$$C \le \|f\|_{0,\Omega} + \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\widetilde{\Omega}} |u|_{1,\Omega}.$$

As to the estimate of D_3 we start from the expression, which follows from the third term on the right-hand side of (5.30) with $U=\tilde{u}$:

$$D_3 \le \max_{i=1,2} \|\widetilde{k}_i\|_{0,\infty,\widetilde{\Omega}} |\tilde{u}|_{1,\tau_h} |w|_{1,\tau_h} + \|\tilde{f}\|_{0,\tau_h}\|w\|_{0,\tau_h}. \quad (5.39)$$

Using (5.37) and considering similarly as in part B of the proof of [21, Lemma 25] we can derive

$$|w|_{1,\tau_h} \le Ch^{\delta/2}|w|_{1,\Omega_h}. \quad (5.40)$$

Further

$$\|w\|_{0,\tau_h}^2 \le Ch^2 \left(\|w\|_{0,\Gamma_{1h}}^2 + Ch^2|w|_{1,\tau_h}^2\right) \le C\frac{h^2}{\varrho}\|w\|_{1,\Omega_h}^2. \tag{5.41}$$

The first inequality follows from the proof of [18, Lemma 28.3] and the second from (3.31) and (5.40). Finally,

$$\|\widetilde{f}\|_{0,\tau_h} \le \|\widetilde{f}\|_{0,\infty,\widetilde{\Omega}}\sqrt{\mathrm{mes}_2\tau_h} \le Ch\|\tilde{f}\|_{0,\infty,\tilde{\Omega}}\sqrt{\mathrm{mes}_1\Gamma_1}. \tag{5.42}$$

Combining (5.39)–(5.42) we find that

$$D_3 \le Ch^{\delta/2}\|w\|_{1,\Omega_h}, \tag{5.43}$$

where the constant C does not depend on h and w. Similarly,

$$D_4 \le Ch^{\delta/2}\|w\|_{1,\Omega_h}. \tag{5.44}$$

Relations (5.32), (5.38), (5.43), (5.44) together with Lemma 5.9 yield estimate (5.31). □

Now we shall analyze the first term on the right-hand side of (5.3). We start with the following finite element density theorem.

Lemma 5.11. *Let $V = \{w \in H^1(\Omega) : \mathrm{tr}\, w = 0$ on $\Gamma_j\}$. For every pair $\varepsilon > 0$, $w \in V$ we can find $w_\varepsilon \in C^\infty(\overline{\Omega}) \cap V$ and $h_{\varepsilon,w} > 0$ such that for all $h \in (0,h)_{\varepsilon,w}$ we have*

$$\|\widetilde{w} - I_h w_\varepsilon\|_{1,\Omega_h} < \varepsilon \tag{5.45}$$

where $\widetilde{v} \in H^k(\widetilde{\Omega})$ is the extension of $v \in H^k(\Omega)$ according to Lemma 3.6 and $I_h v \in X_h \equiv \{w \in C(\overline{\Omega}_h) : w\big|_T \in (T,L,1)\ \forall T \in \mathcal{T}_h\}$ is the interpolant of $v \in C(\overline{\Omega})$ defined by $(I_h v)(P_i) = v(P_i)\ \forall P_i$.

Proof. By [18, Theorem $\mathcal{P}$.92] the set $C^\infty(\overline{\Omega}) \cap V$ is dense in V. Hence, there exists a function $w_\varepsilon \in C^\infty(\overline{\Omega}) \cap V$ such that

$$\|w - w_\varepsilon\|_{1,\Omega} < \varepsilon/(2C_1) \tag{5.46}$$

where C_1 is the constant from the inequality

$$\|\widetilde{v}\|_{1,\widetilde{\Omega}} \le C_1\|v\|_{1,\Omega} \quad \forall v \in H^1(\Omega). \tag{5.47}$$

We shall consider $\widetilde{w}$ in $H^1(\widetilde{\Omega})$ and $\widetilde{w_\varepsilon}$ in $H^2(\widetilde{\Omega})$. As the extension $\widetilde{w_\varepsilon}$ is equal to the extension of w_ε from $H^1(\Omega)$ (see Lemma 3.6), we have, according to the linearity of extension operators, $\widetilde{w} - \widetilde{w_\varepsilon} = (w - w_\varepsilon)^\sim$; thus (5.46) and (5.47) yield

$$\|\widetilde{w} - \widetilde{w_\varepsilon}\|_{1,\widetilde{\Omega}} < \varepsilon/2. \tag{5.48}$$

The triangular inequality gives

$$\|\widetilde{w} - I_h w_\varepsilon\|_{1,\Omega_h} \leq \|\widetilde{w} - \widetilde{w_\varepsilon}\|_{1,\Omega_h} + \|\widetilde{w_\varepsilon} - I_h w_\varepsilon\|_{1,\Omega_h}. \tag{5.49}$$

Now we estimate the terms on the right-hand side of (5.49). By (5.48) we have

$$\|\widetilde{w} - \widetilde{w_\varepsilon}\|_{1,\Omega_h} < \varepsilon/2. \tag{5.50}$$

As to the second term, we have

$$I_h w_\varepsilon = I_h \widetilde{w_\varepsilon}$$

because $\Omega \subset \widetilde{\Omega}$. This fact, the interpolation theorem for semiregular triangular linear elements (see Theorem 1.3) and the extension theorem (see Lemma 3.6) yield

$$\|\widetilde{w_\varepsilon} - I_h w_\varepsilon\|_{1,\Omega_h} \leq Ch\|\widetilde{w_\varepsilon}\|_{2,\Omega_h} \leq C_2 Ch\|w_\varepsilon\|_{2,\Omega}.$$

Thus there exists such an $h_{\varepsilon,w}$ that

$$\|\widetilde{w_\varepsilon} - I_h w_\varepsilon\|_{1,\Omega_h} < \varepsilon/2 \quad \forall h \in (0, h_{\varepsilon,w}). \tag{5.51}$$

Combining relations (5.49)–(5.51) we obtain (5.45). □

Theorem 5.12. *We have*

$$\lim_{h\to 0} \left\{ \inf_{v \in V_h} \|v - \widetilde{u}\|_{1,\Omega_h} \right\} = 0. \tag{5.52}$$

Proof. By Lemma 5.11, for a given $\varepsilon > 0$ we can find $u_\varepsilon \in C^\infty(\Omega) \cap V$ and $h_{\varepsilon,u} > 0$ such that

$$\|\widetilde{u} - I_h u_\varepsilon\|_{1,\Omega_h} < \varepsilon \quad \forall h \in (0, h_{\varepsilon,u}).$$

As $I_h u_\varepsilon \in V_h$ we have

$$\inf_{v \in V_h} \|v - \widetilde{u}\|_{1,\Omega_h} \leq \|\widetilde{u} - I_h u_\varepsilon\|_{1,\Omega_h}.$$

Both inequalities imply (5.52). □

Theorem 5.13. *We have for all $h \in (0, h_0)$*

$$IS := \inf_{v \in V_h} \sup_{\substack{w \in V_h \\ w \neq 0}} \frac{|a_h(v,w) - \widetilde{a}_h(v,w)|}{\|w\|_{1,\Omega_h}} \leq Ch(1 + \|u\|_{1,\Omega}),$$

where $u \in H^1(\Omega)$ is the solution of the continuous variational problem and the constant C does not depend on h and u.

Proof. Let $\varepsilon = 1$ and let us set

$$v = I_h u_\varepsilon \in V_h\,, \tag{5.53}$$

where, according to Lemma 5.11,

$$\|\widetilde{u} - I_h u_\varepsilon\|_{1,\Omega_h} < \varepsilon = 1 \quad \forall h \in (0, h_{\varepsilon,u}). \tag{5.54}$$

Using (5.53) and Theorem 3.13 we find

$$IS \le Ch\|I_h u_\varepsilon\|_{1,\Omega_h}. \tag{5.55}$$

Triangular inequality, extension theorem and relation (5.54) imply

$$\|I_h u_\varepsilon\|_{1,\Omega_h} \le \|\widetilde{u}\|_{1,\Omega_h} + \|\widetilde{u} - I_h u_\varepsilon\|_{1,\Omega_h} \le \|\widetilde{u}\|_{1,\widetilde{\Omega}} + 1 \le C\|u\|_{1,\Omega} + 1.$$

Combining this result with (5.55) we obtain the assertion of Theorem 5.13. □

The third and fourth terms appearing on the right-hand side of (5.3) are estimated in Theorems 3.16 and 3.18, respectively. Thus using the preceding results we obtain

Theorem 5.14. *Let us consider the set of divisions $\{\mathcal{D}_h^T\}$ $(h \in (0, h_0))$ introduced in Section 3. Let assumptions of Problem 3.1 and assumptions concerning the degrees of precision of quadrature formulas on a triangle and its side (see Theorems 3.13 and 3.18) be satisfied. If inequalities (5.1) hold then*

$$\lim_{h\to 0} \|\widetilde{u} - u_h\|_{1,\Omega_h} = 0$$

where u_h is the solution of Problem 3.4 belonging to $\mathcal{D}_h^T$, $u \in H^1(\Omega)$ is the solution of Problem 3.1 and $\widetilde{u} = E(u) \in H^1(\widetilde{\Omega})$ its extension in the sense of Lemma 3.6 with $k = 1$.

6 Appendix: Discrete Friedrichs' inequality

In [21] the inequality

$$\|v\|_{1,\Omega_h} \le C|v|_{1,\Omega_h} \quad \forall v \in V_h \quad \forall h < h_0 \tag{6.1}$$

was used without proof. As the proof differs from the proof, which was presented in [18] in the case of regular finite elements, we introduce the following lemma which is sufficient for the considerations in [21] and this paper.

Lemma 6.1. *Let Ω be a domain considered in Sections 3 and 5 and let (3.40) be satisfied, i.e. let*

$$C_1 h^2 \le \frac{\varrho}{m} \quad (C_1 > 0).$$

Then inequality (6.1) holds.

Proof. a) The case of the Dirichlet boundary condition (3.2). In this case

$$V_h = \{v \in X_h : \ v = 0 \text{ on } \varGamma_{1h}\}.$$

Let $\overline{v}$ be the natural extension of v and let $\widetilde{\varOmega}$ be the bounded domain with boundary $\partial\widetilde{\varOmega} = \varGamma_2 \cup \varGamma_3$ where $\varGamma_3$ is the circle with the centre S_0 and radius $R_3 < R_1$. We set $\overline{v} \equiv 0$ in the bounded set U_h with the boundary $\partial U_h = \varGamma_3 \cup \varGamma_{1h}$. According to the Friedrichs inequality

$$\|\overline{v}\|^2_{0,\widetilde{\varOmega}} \leq C|\overline{v}|^2_{1,\widetilde{\varOmega}}. \tag{6.2}$$

As $\varOmega_h \in \widetilde{\varOmega}$ we have

$$\|v\|^2_{0,\varOmega_h} \leq \|\overline{v}\|^2_{0,\widetilde{\varOmega}}. \tag{6.3}$$

It remains to prove

$$|\overline{v}|^2_{1,\widetilde{\varOmega}} \leq C|v|^2_{1,\varOmega_h}. \tag{6.4}$$

We have

$$|\overline{v}|^2_{1,\widetilde{\varOmega}} = |v|^2_{1,\varOmega_h} + |\overline{v}|^2_{1,\omega_h}. \tag{6.5}$$

First we consider the case of the division $\mathcal{D}_h^T$. (For the definition of $\mathcal{D}_h^T$ and other types of divisions see the text following Lemma 3.3.) Let $\lambda_h \subset \varGamma_{2h}$ be the segment Q_jQ_{j+1} which approximates the arc $\lambda \subset \varGamma_2$. Similarly as in the proof of [21, Lemma 33] we can prove that

$$\mathrm{dist}\,(Q_j^*, \varGamma_2) \leq \frac{1}{8}\frac{\varrho}{m} \equiv \frac{1}{8}b,$$

where Q_j^* is the mid-point of λ_h. Thus

$$\mathrm{mes}_2\mathcal{P}_h \leq \frac{1}{4}\mathrm{mes}_2 T,$$

where $\mathcal{P}_h$ is the bounded domain with the boundary $\partial\mathcal{P}_h = \lambda \cup \lambda_h$ and T the triangle adjacent to $\mathcal{P}_h$. As v is piecewise linear we have

$$|\overline{v}|^2_{1,\mathcal{P}_h} \leq \frac{1}{4}|v|^2_{1,T}.$$

Hence

$$|\overline{v}|^2_{1,\omega_h} \leq \frac{1}{4}|v|^2_{1,\varOmega_h}.$$

Inserting this result into (6.5) we obtain estimate (6.4) with $C = 5/4$. The same result can be obtained in the case of the division $\mathcal{D}_h^A$.

In the case of the division $\mathcal{D}_h^K$ we use the result for $\mathcal{D}_h^A$ and estimate [21, (91)].

Combining (6.2)–(6.4) we arrive at

$$\|v\|_{0,\Omega_h}^2 \leq C|v|_{1,\Omega_h}^2 \quad \forall v \in V_h.$$

Hence (6.1) follows.

b) The case of the Dirichlet boundary condition $v = 0$ on Γ_2. In this case

$$V_h = \{v \in X_h : \ v = 0 \text{ on } \Gamma_{2h}\}$$

and we define the quasinatural extension $\overline{v}$ of $v \in V_h$ by

$$\overline{v} = v \text{ on } \Omega_h, \quad \overline{v} = 0 \text{ on } \omega_h. \tag{6.6}$$

The Friedrichs inequality gives

$$\|\overline{v}\|_{0,\Omega}^2 \leq C|\overline{v}|_{1,\Omega}^2. \tag{6.7}$$

Relations (6.6) imply

$$|\overline{v}|_{1,\Omega}^2 \leq |v|_{1,\Omega_h}^2. \tag{6.8}$$

If we prove

$$\|\overline{v}\|_{0,\Omega}^2 \geq C\|v\|_{0,\Omega_h}^2 \quad (C > 0)\,, \tag{6.9}$$

then (6.1) follows from (6.7)–(6.9).

Let us consider the case of $\mathcal{D}_h^K$. Transformation (3.20) maps one-to-one the reference square $\overline{K}_0$ with vertices $P_1^*(1,0)$, $P_2^*(0,0)$, $P_3^*(0,1)$, $P_4^*(1,1)$ onto the quadrilateral $\overline{K}$ with vertices P_1, P_2, P_3, P_4 where P_1, P_2 lie on Γ_1 and P_3P_4 is parallel to P_1P_2. Let $S_1 \in P_1P_4$, $S_2 \in P_2P_3$, let S_1S_2 be parallel to P_1P_2 and let

$$\operatorname{dist}(P_1P_2, S_1S_2) = \frac{1}{8}b.$$

Then, according to [21, Lemma 33], the arc $\lambda \subset \Gamma_1$ which is approximated by $\lambda_h = P_1P_2$ lies in Δ, where Δ denotes the quadrilateral with vertices P_1, P_2, S_2, S_1. Let us assume that we proved

$$\|v\|_{0,\Delta}^2 \leq \frac{3}{4}\|v\|_{0,K}^2. \tag{6.10}$$

Then

$$\|v\|_{0,K-\mathcal{P}_h}^2 \geq \|v\|_{0,K-\Delta}^2 = \|v\|_{0,K}^2 - \|v\|_{0,\Delta}^2 = \frac{1}{4}\|v\|_{0,K}^2,$$

where $\mathcal{P}_h$ is the bounded domain with the boundary $\partial\mathcal{P}_h = \lambda \cup \lambda_h$. Hence (6.9) follows with $C = \frac{1}{4}$.

Let us prove (6.10). According to the definition, the function $v(x,y)$ is on every quadrilateral $\overline{K}$ such that

$$\widetilde{v}(\xi,\eta) \equiv v(x^K(\xi,\eta), y^K(\xi,\eta)) = \sum_{i=1}^{4} B_i p_i(\xi,\eta),$$

where

$$p_1 = \xi(1-\eta),\ p_2 = (\xi-1)(\eta-1),\ p_3 = (1-\xi)\eta,\ p_4 = \xi\eta$$

and $B_i = v(P_i)$ $(i = 1, \dots, 4)$. The functions $x^K(\xi,\eta)$, $y^K(\xi,\eta)$ are the right-hand sides of transformation (3.20).

The quadrilateral Δ is the image of the rectangle Δ_0 with vertices P_1^*, P_2^*, S_2^*, S_1^* in transformation (3.20), where $S_1^* = [1, \frac{1}{8}]$, $S_2^* = [0, \frac{1}{8}]$. First we prove

$$\iint_{\Delta_0} [\widetilde{v}(\xi,\eta)]^2 \,\mathrm{d}\xi\mathrm{d}\eta \le \frac{1}{2} \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 \,\mathrm{d}\xi\mathrm{d}\eta. \tag{6.11}$$

Let us express the integrals

$$J_1 = \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 \,\mathrm{d}\xi\mathrm{d}\eta = \int_0^1 \left\{ \int_0^1 \left(\sum_{i=1}^4 B_i p_i(\xi,\eta) \right)^2 \mathrm{d}\eta \right\} \mathrm{d}\xi,$$

$$J_2 = \iint_{\Delta_0} [\widetilde{v}(\xi,\eta)]^2 \,\mathrm{d}\xi\mathrm{d}\eta = \int_0^1 \left\{ \int_0^{1/8} \left(\sum_{i=1}^4 B_i p_i(\xi,\eta) \right)^2 \mathrm{d}\eta \right\} \mathrm{d}\xi$$

as the quadratic forms of $B_1, \dots, B_4$. Let us denote $A = B_2$, $B = B_1$, $C = B_4$, $D = B_3$. Then

$$\begin{aligned} &4608(J_1 - 2J_2) = \\ &\quad = (174A + 87B + 117C + 134D)^2/174 + (130,5B + 175,5C)^2/130,5 + \\ &\qquad + (195,31035C + 97,655175D)^2/195,31035 + 146,48277D^2, \end{aligned}$$

from which estimate (6.11) follows.

The Jacobian J of transformation (3.20) is of the form

$$J = (h - \varepsilon^*\eta)b,$$

where, according to (3.21) and (3.40), $b = O(h^2)$, $\varepsilon^* = O(h^3)$. Thus using (6.11) and the relation

$$\iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 \eta \,\mathrm{d}\xi\mathrm{d}\eta = \eta_0 \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 \,\mathrm{d}\xi\mathrm{d}\eta \quad (0 < \eta_0 < 1),$$

which is a consequence of the mean-value theorem, we obtain

$$\begin{aligned} \|v\|_{0,\Delta}^2 &= \iint_{\Delta_0} [\widetilde{v}(\xi,\eta)]^2 (h - \varepsilon^*\eta) b \,\mathrm{d}\xi\mathrm{d}\eta \le \\ &\le \iint_{\Delta_0} [\widetilde{v}(\xi,\eta)]^2 hb \,\mathrm{d}\xi\mathrm{d}\eta \le \frac{1}{2} \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 hb \,\mathrm{d}\xi\mathrm{d}\eta \le \\ &\le \frac{3}{4} \left\{ \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 hb \,\mathrm{d}\xi\mathrm{d}\eta - \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 \varepsilon^* \eta_0 b \,\mathrm{d}\xi\mathrm{d}\eta \right\} = \\ &\qquad = \frac{3}{4} \iint_{K_0} [\widetilde{v}(\xi,\eta)]^2 (h - \varepsilon^*\eta) b \,\mathrm{d}\xi\mathrm{d}\eta = \frac{3}{4} \|v\|_{0,K}^2, \end{aligned}$$

which proves (6.10).

In the case of division $\mathcal{D}_h^T$ the proof of (6.9) is similar but simpler: Let T be a triangle with vertices P_1, P_2 lying on Γ_1 and let Q_1 and Q_2 be the mid-points of the sides P_1P_3 and P_2P_3, respectively. Let T^* denote the triangle with vertices Q_1, Q_2, P_3. Then

$$\|v\|^2_{0,T-\mathcal{P}_h} \geq \|v\|^2_{0,T^*}$$

and it is relatively easy to compute that

$$\|v\|^2_{0,T^*} \geq \frac{1}{64}\|v\|^2_{0,T} \; .$$

The last two inequalities imply (6.9) with $C = 1/64$. □

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