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Abstract Differential Equations of Arbitrary (Fractional) Orders

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Abstract. The arbitrary (fractional) order integral operator is a singular integral operator, and the arbitrary (fractional) order differential operator is a singular integro-differential operator. And they generalize (interpolate) the integral and differential operators of integer orders. The topic of fractional calculus (derivative and integral of arbitrary orders) is enjoying growing interest not only among Mathematicians, but also among physicists and engineers (see [1]–[18]).
Let α be a positive real number. Let$X$ be a Banach space and A be a linear operator defined on X with domain $D(A)$.
In this lecture we are concerned with the different approaches of the definitions of the fractional differential operator D^α and then (see [5,6,7]) study the existence, uniqueness, and continuation (with respect to α) of the solution of the initial value problem of the abstract differential equation

$$D^\alpha u(t) = Au(t) + f(t), \qquad D = \frac{d}{dt},\ t > 0, \tag{1}$$

where A is either bounded or closed with domain dense in X.
Fractional-order differential-difference equations, fractional-order diffusion-wave equation and fractional-order functional differential equations will be given as applications.

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1 Introduction

Let X be a Banach space. Let $L_1(I, X)$ be the class of (Lebesgue) integrable functions on the interval $I = [a, b]$, $0 < a < b < \infty$,

This is the final form of the paper.

Definition 1. Let $f(x) \in L_1(I, X)$, $\beta \in R^+$. The fractional (arbitrary order) integral of the function $f(x)$ of order β is (see [1]–[11]) defined by

$$I_a^\beta f(x) = \int_a^x \frac{(x-s)^{\beta-1}}{\Gamma(\beta)} f(s) ds. \tag{2}$$

When $a = 0$ and $X = R$ we can write $I_0^\beta f(x) = f(x) * \phi_\beta(x)$, where $\phi_\beta(x) = \frac{x^{\beta-1}}{\Gamma(\beta)}$ for $x > 0$, $\phi_\beta(x) = 0$ for $x \leq 0$ and $\phi_\beta \to \delta(x)$ (the delta function) as $\beta \to 0$ (see [11]).

Now the following lemma can be easily proved

Lemma 2. *Let β and $\gamma \in R^+$. Then we have*

(i) $I_a^\beta : L_1(I, X) \to L_1(I, X)$, *and if* $f(x) \in L_1(I, X)$, *then* $I_a^\gamma I_a^\beta f(x) = I_a^{\gamma+\beta} f(x)$.

(ii) $\lim_{\beta \to n} I_a^\beta f(x) = I_a^n f(x)$, *uniformly on* $L_1(I, X)$, $n = 1, 2, 3, \ldots$, *where* $I_a^1 f(x) = \int_a^x f(s) ds$.

For the fractional order derivative we have (see [1]–[10] and [15]) mainly the following two definitions.

Definition 3. The (Riemann-Liouville) fractional derivative of order $\alpha \in (0, 1)$ of $f(x)$ is given by

$$\frac{d^\alpha f(x)}{dx^\alpha} = \frac{d}{dx} I_a^{1-\alpha} f(x), \tag{3}$$

Definition 4. The fractional derivative D^α of order $\alpha \in (0, 1]$ of the function $f(x)$ is given by

$$D_a^\alpha f(x) = I_a^{1-\alpha} D f(x), \qquad D = \frac{d}{dx}\,. \tag{4}$$

This definition is more convenient in many applications in physics, engineering and applied sciences (see [15]). Moreover, it generalizes (interpolates) the definition of integer order derivative. The following lemma can be directly proved.

Lemma 5. *Let $\alpha \in (0, 1)$. If $f(x)$ is absolutely continuous on $[a, b]$, then*

(i) $D_a^\alpha f(x) = \frac{d^\alpha f(x)}{dx^\alpha} + \frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)} f(a)$

(ii) $\lim_{\alpha \to 1} D_a^\alpha f(x) = D f(x) \neq lim_{\alpha \to 1} \frac{d^\alpha f(x)}{dx^\alpha}$.

(iii) *If* $f(x) = k$, k *is a constant, then* $D_a^\alpha k = 0$, *but* $\frac{d^\alpha k}{dx^\alpha} \neq 0$.

Definition 6. The finite Weyl fractional integral of order $\beta \in R^+$ of $f(t)$ is

$$W_b^{-\beta} f(t) = \frac{1}{\Gamma(\beta)} \int_t^b (s-t)^{\beta-1} f(s)\, ds\,, \qquad t \in (0, b), \tag{5}$$

and the finite Weyl fractional derivative of order $\alpha \in (n-1, n)$ of $f(t)$ is

$$W_b^\alpha f(t) = W_b^{-(n-\alpha)} (-1)^n D^n f(t), \qquad D^n f(t) \in L_1(I, X). \tag{6}$$

The author [6] stated this definition and proved that if $f(t) \in C(I, X)$, then

$$\lim_{\beta \to p} W_b^{-\beta} f(t) = W_b^{-p} f(t), \qquad p = 1, 2, \ldots, \qquad W_b^{-1} f(t) = \int_t^b f(s) ds, \quad (7)$$

and if $g(t) \in C^n(I, X)$ with $g^{(j)}(b) = 0,\ j = 0, 1, \ldots, (n-1)$, then

$$\lim_{\alpha \to q} W_b^{\alpha} g(t) = (-1)^q D^q g(t), \qquad q = 0, 1, \ldots, (n-1), \qquad W_b^0 g(t) = g(t). \quad (8)$$

2 Ordinary Differential Equations

Let A be a bounded operator defined on X, consider the initial value problem

$$\begin{cases} D_a^{\alpha} u(t) = Au(t) + f(t), \quad t \in (a, b], \quad \alpha \in (0, 1], \\ \quad u(a) = u_o. \end{cases} \quad (9)$$

Definition 7. By a solution of (9) we mean a function $u(t) \in C(I, X)$ that satisfies (9).

Theorem 8. *Let $u_o \in X$ and $f(t) \in C^1(I, X)$. If $||A|| \leq \frac{\Gamma(1+\alpha)}{b^{\alpha}}$, then (9) has the unique solution*

$$u_{\alpha}(t) = T_a^{\alpha}(t) u_o + I_a^{\alpha} T_a^{\alpha}(t) f(t) \in C^1((a, b], X), \quad (10)$$

where

$$T_a^{\alpha} g(t) = \sum_{k=o}^{\infty} I_a^{k\alpha} A^k g(t), \qquad g(t) \in L_1(I, X). \quad (11)$$

And

(1) $T_a^{\alpha}(a) u_o = u_o$,
(2) $D_a^{\alpha} T_a^{\alpha}(t) u_o = A T_a^{\alpha}(t) u_o$,
(3) $\lim_{\alpha \to 1} T_a^{\alpha}(t) u_o = e^{(t-a)A} u_o$.

Moreover

$$\lim_{\alpha \to 1} u_{\alpha}(t) = e^{(t-a)A} u_o + \int_a^t e^{(t-s)A} f(s)\, ds\,. \quad (12)$$

Proof. See [8].

As an application let $0 < \beta \leq \alpha \leq 1$ and consider the two (forward and backward) initial value problems of the fractional-order differential-difference equation

$$(P) \begin{cases} D_a^{\alpha} u(t) + C D_a^{\beta} u(t-r) = Au(t) + Bu(t-r), \quad t > a, \\ u(t) = g(t), \quad t \in [a-r, a], \quad r > 0, \end{cases} \quad (13)$$

$$(Q)\begin{cases} W_b^\alpha u(t) + CW_b^\beta u(t+r) = Au(t) + Bu(t+r), \quad t < b, \quad \alpha \geq \beta, \\ u(t) = g(t), \quad t \in [b, b+r], \quad r > 0, \end{cases} \tag{14}$$

where A , B and C are bounded operators defined on X.

Theorem 9. *Let $g(t) \in C^1([a-r,a],X)$. If $||A|| \leq \frac{\Gamma(1+\alpha)}{b^\alpha}$, then the problem (P) has a unique solution $u(t) \in C((a,b],X)$, $Du(t) \in C(I_{nr},X)$ and $D^\alpha_{a+nr}u(t) \in C(I_{nr},X)$, where $I_{nr} = (a, a+nr]$.*
Moreover if $C = 0$ then $u(t) \in C^1(I,X)$ and $D^\alpha_a u(t) \in C(I,X)$.

Proof. See [8].

Theorem 10. *Let $u(t)$ be the solution of (P). If the assumptions of Theorem 9 are satisfied, then there exist two positive constants k_1 and k_2 such that*

$$||u(t)|| \leq k_1 e^{(t-a)k_2}, \tag{15}$$

i.e., the solution of (P) is exponentially bounded.

Proof. See [8].

The same results can be proved for the problem (Q) (see [8]).

3 Fractional-Order Functional Differential Equation

Consider the two initial value problems

$$D^\alpha_a x(t) = f(t, x(m(t))), \quad x(a) = x_o, \quad \alpha \in (0,1], \tag{16}$$
$$W^\alpha_b y(t) = f(t, y(m(t))), \quad y(b) = y_o, \quad \alpha \in (0,1], \tag{17}$$

with the following assumptions

(i) $f : (a,b) \times R^+ \to R^+ = [0,\infty)$, satisfies Carathéodory conditions and there exists a function $c \in L^1$ and a constant $k \geq 0$ such that $f(t,x(t)) \leq c(t) + k|x|$, for all $t \in (a,b)$ and $x \in R^+$. Moreover, $f(t,x(t))$ is assumed to be nonincreasing (nondecreasing) on the set $(a,b) \times R^+$ with respect to t and nondecreasing with respect to x,

(ii) $m : (a,b) \to (a,b)$ is increasing, absolutely continuous and there exists a constant $M > 0$ such that $m' \geq M$ for almost all $t \in (a,b)$,

(iii) $k/M < 1$.

Theorem 11. *Let the assumptions* (i)–(iii) *be satisfied. If x_o and y_o are positive constants, then the problem* (16) *has at least one solution $x(t) \in L^1$ which is a.e. nondecreasing (and so $Dx(t) \in L^1$) and the problem* (17) *has at least one solution $y(t) \in L^1$ which is a.e. nonincreasing (and so $Dy(t) \in L^1$).*

Proof. See [9].

4 Fractional-Order Evolution Equations

Let A be a closed linear operator defined on X with domain $D(A)$ dense in X and consider the two initial value problems

$$\begin{cases} D^{\gamma}u(t) = Au(t), \quad t \in (0,b], \quad \gamma \in (0,1], \\ \quad u(0) = u_o, \end{cases} \tag{18}$$

$$\begin{cases} D^{\beta}u(t) = Au(t), \quad t \in (0,b], \quad \beta \in (1,2], \\ \quad u(0) = u_o, \quad u_t(0) = u_1. \end{cases} \tag{19}$$

Remark 12. Some special cases of these two equations have been studied by some authors (see [12] and [16] e.g.).

Definition 13. By a solution of the initial value problem (18) we mean a function $u_{\gamma}(t) \in L_1(I, D(A))$ for $\gamma \in (0,1]$ which satisfies the problem (18). The solution $u_{\beta}(t)$ of the problem (19) is defined in a similar way.

Consider now the following assumption

(1) Let A generates an analytic semi-group $\{T(t),\ t > 0\}$ on X. In particular $\Lambda = \{\lambda \in C : |arg\lambda| < \pi/2 + \delta_1\}$, $0 < \delta_1 < \pi/2$ is contained in the resolvent set of A and $||(\lambda I - A)^{-1}|| \le M/|\lambda|$, $Re\lambda > 0$ on Λ_1, for some constant $M > 0$, where C is the set of complex numbers.

Theorem 14. *Let $u_1, u_o \in D(A^2)$. If A satisfies assumption (1), then there exists a unique solution $u_{\gamma}(t) \in L_1(I, D(A))$ of (18) given by*

$$u_{\gamma}(t) = u_o - \int_0^t r_{\gamma}(s) e^s u_o ds, \qquad Du_{\gamma}(t) \in D(A), \tag{20}$$

and a unique solution $u_{\beta}(t) \in L_1(I, D(A))$ of (19) given by

$$u_{\beta}(t) = u_o + tu_1 - \int_0^t r_{\beta}(s) e^s (u_o + (t-s)u_1) ds, \qquad D^2 u_{\beta}(t) \in D(A). \tag{21}$$

Here $r_{\gamma}(t)$ and $r_{\beta}(t)$ are the resolvent operators of the the two integral equations

$$u_{\gamma}(t) = u_o + \int_0^t \phi_{\gamma}(t-s) A u_{\gamma}(s) ds, \tag{22}$$

$$u_{\beta}(t) = u_o + tu_1 + \int_0^t \phi_{\beta}(t-s) A u_{\beta}(s) ds, \tag{23}$$

respectively.

Proof. See [6].

Now one of the main results in this paper is the following continuation theorem. To the best of my knowledge, this has not been studied before.

Theorem 15. *Let the assumptions of Theorem 14 be satisfied with $u_1 = 0$, then*

$$\lim_{\gamma\to 1^-} u_\gamma(t) = \lim_{\beta\to 1^+} u_\beta(t) = T(t)u_o, \tag{24}$$

$$\lim_{\gamma\to 1^-} D^\gamma u_\gamma(t) = \lim_{\beta\to 1^+} D^\beta u_\beta(t) = AT(t)u_o = Du(t), \tag{25}$$

where $\{T(t),\ t \geq 0\}$ is the semigroup generated by the operator A and so $u(t) = T(t)u_o$ is the solution of the problem

$$\begin{cases} \dfrac{du(t)}{dt} = Au(t), \quad t > 0 \\ u(0) = u_o. \end{cases} \tag{26}$$

Proof. See [6].

5 Fractional-Order Diffusion-Wave Equation

Let $X = R^n$ and $u(t,x) : R^n \times I \to R^n,\ I = (0,T]$.

Definition 16. The fractional **D-W** (diffusion-wave) equation is the equation (see [7])

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = Au(x,t), \quad t > 0, \tag{27}$$

and the fractional diffusion-wave problem is the Cauchy problem

$$\textbf{(D-W)} \begin{cases} \dfrac{\partial^\alpha u(x,t)}{\partial t^\alpha} = Au(x,t), \quad t > 0, \quad x \in R^n, \quad 0 < \alpha \leq 2, \\ u(x,0) = u_o(x), \quad u_t(x,0) = 0, \quad x \in R^n. \end{cases} \tag{28}$$

From the properties of the fractional calculus we can prove (see [7])

Theorem 17 (Continuation of the problem). *If the solution of the* **(D-W)** *problem exists, then as $\alpha \to 1$ the* **(D-W)** *problem reduces to the diffusion problem*

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = Au(x,t), \quad t > 0, \quad x \in R^n, \\ u(x,0) = u_o(x), \quad x \in R^n, \end{cases} \tag{29}$$

and as $\alpha \to 2$ the **(D-W)** *problem reduces to the wave problem*

$$\begin{cases} \dfrac{\partial^2 u(x,t)}{\partial t^2} = Au(x,t), \quad t > 0, \quad x \in R^n, \\ u(x,0) = u_o(x), \quad u_t(x,0) = 0, \quad x \in R^n. \end{cases} \tag{30}$$

Proof. See [7].

Theorem 18. *Let $u_o \in D(A^2)$. If A satisfies the condition* (**1**) *with $X = R^n$, then the* **(D-W)** *problem has a unique solution $u_\alpha(x,t) \in L_1(I, D(A))$ and this solution is continuous with respect to $\alpha \in (0,2]$. Moreover*

$$\lim_{\alpha \to 1} u_\alpha(x,t) = u_1(x,t) \quad \text{and} \quad \lim_{\alpha \to 2^-} u_\alpha(x,t) = u_2(x,t), \tag{31}$$

where $u_1(x,t)$ and $u_2(x,t)$ are the solutions of (29) *and* (30), *respectively.*

Proof. See [7].

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