EQUADIFF 9

Michal Beneš

Analysis of equations in the phase-field model

In: Zuzana Došlá and Jaromír Kuben and Jaromír Vosmanský (eds.): Proceedings of Equadiff 9, Conference on Differential Equations and Their Applications, Brno, August 25-29, 1997, [Part 3] Papers. Masaryk University, Brno, 1998. CD-ROM. pp. 17--35.

Persistent URL: http://dml.cz/dmlcz/700304

EQUADIFF 9 CD ROM, Brno 1997
Masaryk University

PAPERS
pp. 17–35

# Analysis of Equations in the Phase-Field Model

Michal Beneš

Department of Mathematics, Czech technical University, Trojanova 13,
120 00 Prague, Czech Republic
Email: benes@kmdec.fjfi.cvut.cz
WWW: http://kmdec.fjfi.cvut.cz/~benes

**Abstract.** The article presents basic numerical analysis of equations in the phase-field model which is performed using a FDM semi-discrete scheme. The compactness technique allows to prove convergence of the scheme. Simultaneously, existence and uniqueness of weak solution to the original system is shown. Additionally, the asymptotical behaviour of the solution with respect to the small parameter $\xi$ is studied. Both temperature and phase fields converge in certain sense if $\xi \to 0$. The phase field gives rise to a step-wise function indicating the presence of different phases.

**AMS Subject Classification.** 80A22, 82C26, 35A40

**Keywords.** phase-field model, method of lines, compactness method

## 1 Introduction

The paper contains several remarks concerning basic analysis of the standard form of phase-field model. The system of equations in question reads as follows:

$$\begin{aligned} \frac{\partial u}{\partial t} &= \nabla^2 u + L\frac{\partial p}{\partial t} \quad , \\ \alpha\xi^2\frac{\partial p}{\partial t} &= \xi^2\nabla^2 p + f_0(p) - \beta u \xi \quad , \end{aligned} \tag{1}$$

with initial conditions

$$u\,|_{t=0} = u_0 \;, \quad p\,|_{t=0} = p_0 \quad ,$$

and with boundary conditions of Dirichlet type

$$u\,|_{\partial\Omega} = u_\Omega \;, \quad p\,|_{\partial\Omega} = p_\Omega \quad ,$$

where $L$, $\alpha$, $\beta$, $\xi$ are positive constants, $\Omega$ is a bounded domain in $\mathrm{R}^n$ and $f_0$ derivative of a quartic potential. For the sake of simplicity, we will consider rectangular form of $\Omega$ in 2D, $f_0(p) = ap(1-p)(p-\frac{1}{2})$ with $a > 0$ and homogeneous boundary conditions.

*This is the final form of the paper.*

Such a system of equations has been studied by many authors throughout last decade (see, e.g. [5], [5], [1], [8], [16], [13], [10]). In the physical context, the system (1) is treated as a regularization of the modified Stefan problem describing microstructure formation in solidification of a pure substance if $\xi \to 0$, see [7], [2]:

$$\frac{\partial u}{\partial t} = \nabla^2 u \qquad \text{in } \Omega_s \text{ and } \Omega_l \quad , \tag{2}$$

$$u\mid_{\partial\Omega} = u_\Omega \quad , \tag{3}$$

$$u\mid_{t=0} = u_0 \quad , \tag{4}$$

$$\frac{\partial u}{\partial n}\mid_s - \frac{\partial u}{\partial n}\mid_l = -Lv_\Gamma \quad , \tag{5}$$

$$6\sqrt{\frac{2}{a}}\beta u = -\kappa + \alpha v_\Gamma \quad , \tag{6}$$

$$\Omega_s(t)\mid_{t=0} = \Omega_{so} \quad , \tag{7}$$

where $\Omega_s$, $\Omega_l$ are solid and liquid phases, respectively, $L$ is latent heat per unit volume, melting point is 0, $u$ temperature field. Discontinuity of heat flux on $\Gamma(t)$ is described by the Stefan condition (5), the formula (6) is the Gibbs-Thompson relation on $\Gamma(t)$. The parameter $\alpha$ is the coefficient of attachment kinetics. Following [2], the relation of (1) and (2)–(7) is studied using asymptotical analysis. The article presents the following results: convergence of the semi-discrete scheme, existence and uniqueness of the original system of equations, and convergence towards the sharp-interface state.

## 2 Interpolation theory for grid functions

The analysis of the system (1) concerning the existence and uniqueness of the weak solution is performed using a semi-discrete scheme based on finite differences. The following notations are introduced (see [15]):

$$\mathbf{h} = (h_1, h_2)\,,\ h_1 = \frac{L_1}{N_1},\ h_2 = \frac{L_2}{N_2},\ \mathbf{x}_{ij} = [x^1_{ij}, x^2_{ij}],\ u_{ij} = u(\mathbf{x}_{ij}), \tag{8}$$

$$\omega_{\mathbf{h}} = \{[ih_1, jh_2] \mid i = 1,\dots,N_1 - 1;\ j = 1,\dots,N_2 - 1\} \quad , \tag{9}$$

$$\bar\omega_{\mathbf{h}} = \{[ih_1, jh_2] \mid i = 0,\dots,N_1;\ j = 0,\dots,N_2\} \quad , \tag{10}$$

$$\gamma_h = \bar\omega_{\mathbf{h}} - \omega_{\mathbf{h}} \quad , \tag{11}$$

$$u_{\bar x_1,ij} = \frac{u_{ij} - u_{i-1,j}}{h_1} \quad , \qquad u_{x_1,ij} = \frac{u_{i+1,j} - u_{ij}}{h_1} \quad , \tag{12}$$

$$u_{\bar x_2,ij} = \frac{u_{ij} - u_{i,j-1}}{h_2} \quad , \qquad u_{x_2,ij} = \frac{u_{i,j+1} - u_{ij}}{h_2} \quad , \tag{13}$$

$$u_{\bar x_1 x_1,ij} = \frac{1}{h_1^2}\left(u_{i+1,j} - 2u_{ij} + u_{i-1,j}\right) \quad , \tag{14}$$

and

$$\bar{\nabla}_h u = [u_{\bar{x}_1}, u_{\bar{x}_2}], \quad \nabla_h u = [u_{x_1}, u_{x_2}], \quad \Delta_h u = u_{\bar{x}_1 x_1} + u_{\bar{x}_2 x_2} \quad , \tag{15}$$

If $\mathcal{H}_\mathbf{h} = \{f \mid f : \bar{\omega}_\mathbf{h} \to \mathrm{R}\}$ is a set of grid functions, the following notations will be used $(f, g \in \mathcal{H}_h)$ :

$$\|f\|_{ph} = \left( \sum_{i,j=1}^{N_1-1, N_2-1} h_1 h_2 |f_{ij}|^p \right)^{\frac{1}{p}} \quad \text{for} \quad p > 1 \quad , \tag{16}$$

$$(f, g)_h = \sum_{i,j=1}^{N_1-1, N_2-1} h_1 h_2 f_{ij} g_{ij} \quad , \quad \|f\|_h^2 = (f, f)_h \quad , \tag{17}$$

$$(f^1, g^1\rfloor = \sum_{i=1,j=1}^{N_1, N_2-1} h_1 h_2 f^1_{ij} g^1_{ij} \quad , \quad \|f^1\rfloor\|^2 = (f^1, f^1\rfloor \quad , \tag{18}$$

$$(f^2, g^2\rceil = \sum_{i=1,j=1}^{N_1-1, N_2} h_1 h_2 f^2_{ij} g^2_{ij} \quad , \quad \|f^2\rceil\|^2 = (f^2, f^2\rceil \quad , \tag{19}$$

$$(\mathbf{f}, \mathbf{g}] = (f^1, g^1\rfloor + (f^2, g^2\rceil \quad , \quad \|\mathbf{f}]\|^2 = (\mathbf{f}, \mathbf{f}] \quad , \tag{20}$$

where $\mathbf{f} = [f^1, f^2]$ and $\mathbf{g} = [g^1, g^2]$.

Referring to [2], we recall the following formulas

– Green formulas

$$(f, g_{\bar{x}_1 x_1})_h = -(f_{\bar{x}_1}, g_{\bar{x}_1}\rfloor + \sum_{j=1}^{N_2-1} (f g_{\bar{x}_1} |_{N_1,j} - f g_{x_1} |_{0,j}) h_2, \tag{21}$$

and

$$(f, g_{\bar{x}_2 x_2})_h = -(f_{\bar{x}_2}, g_{\bar{x}_2}\rceil + \sum_{i=1}^{N_1-1} (f g_{\bar{x}_2} |_{i,N_2} - f g_{x_2} |_{i,0}) h_1, \tag{22}$$

In a natural way, we define the space

$$l_p(\omega_h) = \{\mathcal{H}_h \mid \| \cdot \|_{ph}\} \quad . \tag{23}$$

– Poincaré inequality. Let $u \in l_2(\omega_h)$ and $u |_{\gamma_h} = 0$. Then

$$\|u\|_h^2 \le C(\Omega)[\ \|u_{\bar{x}_1}\rfloor\|^2 + \|u_{\bar{x}_2}\rceil\|^2\ ] \quad . \tag{24}$$

We continue by introducing an extension of grid functions, so that they are defined almost everywhere on $\Omega$. Such extensions are studied by the usual technique of $\mathrm{L}_p$ and $\mathrm{H}^k$ spaces. The approach of [14] is adopted for the equations in question. The limiting process requires a refinement of the FDM grid $\bar{\omega}_h$, if $\mathbf{h} \to 0$. For this purpose, a proper metric should be chosen. If we intent to use the compactness technique, a mapping converting a grid function $f_h : \bar{\omega}_h \to \mathrm{R}$ into a function $f : \Omega \to \mathrm{R}$ is needed. Then, the norm of $\mathrm{L}_p$ spaces will serve as a metric for convergence of the numerical scheme.

**Definition 1.** Be $\bar{\omega}_h$ an uniform rectangular grid imposed on a domain $\Omega \subset \mathrm{R}^2$. Let $\mathbf{h} = [h_1, h_2]$ is the mesh size. Then, the dual grid is a set

$$\bar{\omega}_h^* = \Big\{ \Sigma_{ij} \subset \bar{\Omega} \mid \Sigma_{ij} = \left( x_i^1 - \frac{h_1}{2}, x_i^1 + \frac{h_1}{2} \right) \times \left( x_j^2 - \frac{h_2}{2}, x_j^2 + \frac{h_2}{2} \right) \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \Big\} .$$

The dual simplicial grid is a set

$$\bar{\omega}_h^{*s} = \bar{\omega}_h^{*\triangleleft} \cup \bar{\omega}_h^{*\triangleright} \quad , \tag{25}$$

with

$$\begin{aligned} \bar{\omega}_h^{*\triangleleft} &= \{ \Sigma_{ij}^{\triangleleft} \subset \bar{\Omega} \mid \Sigma_{ij}^{\triangleleft} = [x_{i,j}, x_{i-1,j}, x_{i,j-1}]_\kappa \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \} \quad , \\ \bar{\omega}_h^{*\triangleright} &= \{ \Sigma_{ij}^{\triangleright} \subset \bar{\Omega} \mid \Sigma_{ij}^{\triangleright} = [x_{i-1,j-1}, x_{i-1,j}, x_{i,j-1}]_\kappa \cap \bar{\Omega} \text{ for } [x_i^1, x_j^2] \in \bar{\omega}_h \} \quad , \end{aligned}$$

where $[\ ]_\kappa$ denotes the convex hull.

*Remark 2.* Consequently, $\bigcup_{\Sigma \in \bar{\omega}_h^*} \Sigma = \bar{\Omega}$ - the system $\bar{\omega}_h^*$ covers the domain $\Omega$. Each (rectangular) set $\Sigma \in \bar{\omega}_h^*$ has the point $[x_i^1, x_j^2]$ in its center. Similarly, the system $\bar{\omega}_h^{*s}$ also covers $\bar{\Omega}$.

**Definition 3.** Let $\mathcal{H}_h$ be a set of grid functions on $\bar{\omega}_h$. Define the following mappings:

- $\mathcal{Q}_h : \mathcal{H}_h \to \mathcal{C}(\bar{\Omega})$ such that for each $u \in \mathcal{H}_h$

  $$(\mathcal{Q}_h u)(x^1, x^2) = u_{i-1,j-1} + \nabla_h u_{h,i-1,j-1} \cdot [x^1 - x_{i-1,j-1}^1, x^2 - x_{i-1,j-1}^2] \quad ,$$

  if $[x^1, x^2] \in \Sigma_{ij}^{\triangleright}$, $\Sigma_{ij}^{\triangleright} \in \bar{\omega}_h^{*\triangleright}$;

  $$(\mathcal{Q}_h u)(x^1, x^2) = u_{ij} + \bar{\nabla}_h u_{ij} \cdot [x^1 - x_{ij}^1, x^2 - x_{ij}^2] \quad ,$$

  if $[x^1, x^2] \in \Sigma_{ij}^{\triangleleft}$, $\Sigma_{ij}^{\triangleleft} \in \bar{\omega}_h^{*\triangleleft}$.
- $\mathcal{S}_h : \mathcal{H}_h \to L_1(\Omega)$ such that for each $u \in \mathcal{H}_h$

  $$(\mathcal{S}_h u)(x^1, x^2) = u_{ij} \quad ,$$

  if $[x^1, x^2] \in \Sigma_{ij}$, $\Sigma_{ij} \in \bar{\omega}_h^*$;
- $\mathcal{P}_h : \mathcal{C}(\bar{\Omega}) \to \mathcal{H}_h$ such that for each $u \in \mathcal{C}(\bar{\Omega})$

  $$(\mathcal{P}_h u)_{ij} = u(\mathbf{x}_{ij}) \quad ,$$

  if $\mathbf{x}_{ij} \in \bar{\omega}_h$.

*Remark 4.* The operator $\mathcal{P}_h$ is linear and continuous from $\mathcal{C}(\bar{\Omega})$ to $\mathcal{H}_h$, and can be extended to $\mathrm{H}^1(\Omega)$ via density argument. $\mathcal{Q}_h u$ is a continuous piecewise linear function, $\nabla(\mathcal{Q}_h u)$ exists a.e. in $\Omega$. We proceed by determining basic properties of the above defined maps as proven in [2]:

1. If $u, v\,|_{\gamma_h} = 0$ the scalar product coincides with the scalar product in $l_2(\omega_h)$

$$\int_\Omega \mathcal{S}_h u \mathcal{S}_h v dx = (u, v)_h \quad . \tag{26}$$

2. Let $\omega_h$ is a grid on the domain $\Omega$ with the mesh $\mathbf{h}$, let $u, v \in \mathcal{H}_h$ is such that $u, v\,|_{\gamma_h} = 0$. Then

$$(\nabla(\mathcal{Q}_h u), \nabla(\mathcal{Q}_h v)) = (\bar{\nabla}_h u, \bar{\nabla}_h v] \quad . \tag{27}$$

3. Let $\omega_h$ is a grid on the domain $\Omega$ with the mesh $\mathbf{h}$, let $u \in \mathcal{H}_h$. Then

$$\|\mathcal{Q}_h u\|_{\mathrm{L}_2(\Omega)} \leq \|\mathcal{S}_h u\|_{\mathrm{L}_2(\Omega)} \quad . \tag{28}$$

4. Let $\omega_h$ is a grid on the domain $\Omega$ with the mesh $\mathbf{h}$, let $u \in \mathcal{H}_h$, $u\,|_{\gamma_h} = 0$. Then

$$\int_\Omega |\mathcal{Q}_h u - \mathcal{S}_h u|^2 d\mathbf{x} \leq \frac{|\mathbf{h}|^2}{6} \|\bar{\nabla}_h u]|^2 \quad , \tag{29}$$

   if $u\,|_{\gamma_h} = 0$.
5. Let $p \in \mathcal{C}^{0,\nu}(\Omega)$, $\nu \in (0, 1)$. Then,

$$\mathcal{S}_h(\mathcal{P}_h p) \to p \quad \text{in } \mathrm{L}_s(\Omega), \text{ if } \mathbf{h} \to 0 \quad , \tag{30}$$

   for $s > 1$.
6. Let $u \in \mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$. Then

$$\mathcal{Q}_h(\mathcal{P}_h u) \to u \tag{31}$$

   in $\mathrm{H}^1(\Omega)$, if $\mathbf{h} \to 0$.
7. Let $p \in \mathcal{C}^2(\Omega)$ and $p\,|_{\partial\Omega} = 0$. Then

$$\nabla(\mathcal{Q}_h(\mathcal{P}_h p)) \to \nabla p \quad , \tag{32}$$

   in $\mathrm{L}_2(\Omega)$, if $\mathbf{h} \to 0$.

## 3 Main result

In this section, we give a proof of existence and uniqueness of the solution to (1) regardless on values of coefficients. Compared to [5], we get a more general result. Similar procedure has been presented in [3].

**Definition 5.** Consider a bounded domain $\Omega \subset \mathrm{R}^2$, $T > 0$. The classical solution of the system of phase-field equations is a couple of functions

$$[u, p] : \langle 0, T\rangle \times \bar{\Omega} \to \mathrm{R}^2 \quad ,$$

satisfying the equations

$$
\begin{aligned}
\frac{\partial u}{\partial t} &= \nabla^2 u + L\frac{\partial p}{\partial t} \quad \text{in} \quad (0,T)\times\Omega \\
u\,|_{\partial\Omega} &= 0 \quad , \; t\in(0,T) \quad , \\
u\,|_{t=0} &= u_0 \quad \text{in } \Omega \quad , \\
\alpha\xi^2\frac{\partial p}{\partial t} &= \xi^2\nabla^2 p + f_0(p) - \beta\xi u \quad \text{in} \quad (0,T)\times\Omega \quad , \\
p\,|_{\partial\Omega} &= 0 \quad , \quad t\in(0,T) \quad , \\
p\,|_{t=0} &= p_0 \quad \text{in } \Omega \quad .
\end{aligned}
\tag{33}
$$

*Remark 6.* The form of the phase-field equations is referred to [5]. For the sake of simplicity, we consider a 2-D rectangular domain and homogeneous boundary condition. Obviously, the extension to higher dimensions, and to other boundary conditions is possible. Let $[u,p]$ is a classical solution such that $u,p\in \mathrm{C}^2(\langle 0,T\rangle\times\bar{\Omega})$ and let $v,q\in\mathcal{D}(\Omega)$. Multiplying the first one of equations (1) by $v$ and the second one by $q$ (scalar product in $\mathrm{L}_2(\Omega)$), and using the Green formula, we get

$$
\begin{aligned}
\frac{d}{dt}(u,v) + (\nabla u,\nabla v) &= L\frac{d}{dt}(p,v) \quad \text{a.e. in} \quad (0,T) \quad , \\
u(0) &= u_0 \quad , \\
\alpha\xi^2\frac{d}{dt}(p,q) + \xi^2(\nabla p,\nabla q) &= (f_0(p),q) - \beta\xi(u,q) \quad \text{a.e. in} \quad (0,T) \quad , \\
p(0) &= p_0 \quad .
\end{aligned}
\tag{34}
$$

This leads to the next definition:

**Definition 7.** Weak solution of the boundary-value problem for the phase-field equations is a couple of functions $[u,p]$ from $(0,T)$ to $[\mathrm{H}_0^1(\Omega)]^2$ such that it satisfies (34) for each $q,v\in\mathrm{H}_0^1(\Omega)$.

The term $f_0(p)$ requires that $p\in\mathrm{L}_4(\Omega)$ for almost all $t\in(0,T)$. As $\Omega\subset\mathrm{R}^2$, it suffices to take $p\in\mathrm{H}_0^1(\Omega)$ for almost all $t\in(0,T)$ due to the continuous imbedding into $\mathrm{L}_q(\Omega)$ for each $q\in(1,+\infty)$. If

$$[u,p]\in[\mathrm{L}_\infty(0,T;\mathrm{H}_0^1(\Omega))]^2 \quad ,$$

$[u,p]$ is continuous mapping from $\langle 0,T\rangle$ to $\mathrm{H}^{-1}(\Omega)$, as shown in [11]).

Next statement gives an information about the existence and uniqueness of the solution to (34); the proof by its virtue contains the investigation of convergence of a semi-discrete scheme based on method of lines.

**Theorem 8.** *Consider the problem* (34) *in a rectangular domain* $\Omega=(0,L_1)\times(0,L_2)$, *where*

$$u_0,p_0\in\mathrm{H}^2(\Omega)\cap\mathrm{H}_0^1(\Omega) \quad .$$

*Then, there is a unique solution of the problem* (34) *satisfying*

$$u, p \in \mathrm{L}_\infty(0, T; \mathrm{H}^1_0(\Omega) \cap \mathrm{H}^2(\Omega)) \quad ,$$
$$\partial_t u, \partial_t p \in \mathrm{L}_2(0, T; \mathrm{L}_2(\Omega)) \quad .$$

*Proof.* The proof is constructive. Cover $\Omega$ by an uniform grid with the mesh $\mathbf{h} = [h_1, h_2]$, use the previously introduced notations. Consider the semi-discrete scheme

$$\begin{aligned}
\dot{u}^h &= \Delta_h u^h + L\dot{p}^h \quad \text{on} \quad (0,T)\times\omega_h \quad , \\
u^h\,|_{\gamma_h} &= 0 \quad , \\
u^h\,|_{t=0} &= \mathcal{P}_h u_0 \quad \text{on } \bar{\omega}_h \quad , \\
\alpha\xi^2\dot{p}^h &= \xi^2\Delta_h p^h + f_0(p^h) - \beta\xi u^h \quad \text{in} \quad (0,T)\times\omega_h \quad , \\
p^h\,|_{\gamma_h} &= 0 \quad , \\
p^h\,|_{t=0} &= \mathcal{P}_h p_0 \quad \text{on } \bar{\omega}_h \quad .
\end{aligned} \tag{35}$$

where dot denotes the time derivative. In the proof, the major role is played by the a priori estimate for both equations in question. Multiply the first one of equations (35) by $\dot{u}^h$, and the second one by $\dot{p}^h$; sum over $\omega_h$.

$$\begin{aligned}
\|\dot{u}^h\|_h^2 + \frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h u^h]|^2 &= L(\dot{p}^h, \dot{u}^h)_h \quad , \\
\alpha\xi^2\|\dot{p}^h\|_h^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 &= (f_0(p^h), \dot{p}^h)_h - \beta\xi(u^h, \dot{p}^h)_h \quad .
\end{aligned} \tag{36}$$

Using Schwarz and Young inequalities, we get

$$\begin{aligned}
\frac{1}{2}\|\dot{u}^h\|_h^2 + \frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h u^h]|^2 &\le \frac{1}{2}L^2\|\dot{p}^h\|_h^2 \quad , \\
\frac{1}{2}\alpha\xi^2\|\dot{p}^h\|_h^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 &\le -\frac{d}{dt}(w_0(p^h), 1)_h + \frac{\beta^2}{2\alpha}\|u^h\|_h^2 \quad .
\end{aligned} \tag{37}$$

Combining these estimates, we have

$$\begin{aligned}
\frac{1}{4}\alpha\xi^2\|\dot{p}^h\|_h^2 + \frac{\alpha\xi^2}{4L^2}\|\dot{u}^h\|_h^2 + \frac{\alpha\xi^2}{4L^2}\frac{d}{dt}\|\bar{\nabla}_h u^h]|^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 + \\
+ \frac{d}{dt}(w_0(p^h), 1)_h \le \frac{\beta^2}{2\alpha}\|u^h\|_h^2 \quad .
\end{aligned} \tag{38}$$

Using the discrete Poincaré inequality (24)

$$\|u^h\|_h^2 \le C(\Omega)\|\bar{\nabla}_h u^h]|^2 \quad ,$$

and adding non-negative terms on the right-hand side,

$$\begin{aligned}
&\frac{1}{4}\alpha\xi^2\|\dot{p}^h\|_h^2 + \frac{\alpha\xi^2}{4L^2}\|\dot{u}^h\|_h^2 + \frac{\alpha\xi^2}{4L^2}\frac{d}{dt}\|\bar{\nabla}_h u^h]|^2 + \\
&\qquad + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 + \frac{d}{dt}(w_0(p^h), 1)_h \le \\
&\le \frac{2\beta^2L^2}{\alpha^2\xi^2}C(\Omega)\Big\{\frac{\alpha\xi^2}{4L^2}\|\bar{\nabla}_h u^h]|^2 + \xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2 + (w_0(p^h), 1)_h\Big\} \quad .
\end{aligned} \tag{39}$$

Integrating over $(0,t)$, we have

$$\begin{aligned}&\Big\{\frac{\alpha\xi^2}{4L^2}\|\bar{\nabla}_h u^h]|^2+\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2+(w_0(p^h),1)_h\Big\}(t)\le\\&\Big\{\frac{\alpha\xi^2}{4L}\|\bar{\nabla}_h u^h]|^2+\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2+(w_0(p^h),1)_h\Big\}(0)\exp\Big\{\frac{2\beta^2L^2}{\alpha^2\xi^2}C(\Omega)t\Big\}\quad,\end{aligned}\tag{40}$$

which implies

$$\bar{\nabla}_h u^h,\bar{\nabla}_h p^h\in \mathrm{L}_\infty(0,T;\mathrm{l}_2(\omega_h))\quad,$$
$$p^h\in \mathrm{L}_\infty(0,T;\mathrm{l}_4(\omega_h))\quad.$$

Integrating the preceding result over $(0,T)$ again, we get

$$\begin{aligned}\int_0^T\Big\{\frac{\alpha\xi^2}{4L^2}\|\bar{\nabla}_h u^h]|^2+\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2+(w_0(p^h),1)_h\Big\}(t)dt\le\\ \le\Big\{\frac{\alpha\xi^2}{4L}\|\bar{\nabla}_h u^h]|^2+\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2+\\ +(w_0(p^h),1)_h\Big\}(0)\frac{1}{\frac{2\beta^2L^2}{\alpha^2\xi^2}C(\Omega)}\Big\{\exp\Big(\frac{2\beta^2L^2}{\alpha^2\xi^2}C(\Omega)T\Big)-1\Big\}\quad,\end{aligned}\tag{41}$$

which implies

$$\bar{\nabla}_h u^h,\bar{\nabla}_h p^h\in \mathrm{L}_2(0,T;\mathrm{l}_2(\omega_h))\quad,$$
$$p^h\in \mathrm{L}_2(0,T;\mathrm{l}_4(\omega_h))\quad.$$

Extending these results into the continuum of $\Omega$, we see that $\nabla\mathcal{Q}_h(\mathcal{P}_h p_0)$ and $\nabla\mathcal{Q}_h(\mathcal{P}_h u_0)$ are bounded in $\mathrm{L}_2(\Omega)$ (by (31)), and $\mathcal{S}_h(\mathcal{P}_h p_0)$ is bounded in $\mathrm{L}_4(\Omega)$ (by (30)). Therefore

$$\nabla\mathcal{Q}_h u^h,\nabla\mathcal{Q}_h p^h\in \mathrm{L}_\infty(0,T;\mathrm{L}_2(\Omega))\quad,$$
$$\mathcal{S}_h p^h\in \mathrm{L}_\infty(0,T;\mathrm{L}_4(\Omega))\quad,$$

from which,

$$\nabla\mathcal{Q}_h u^h,\nabla\mathcal{Q}_h p^h\in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))\quad,$$
$$\mathcal{S}_h p^h\in \mathrm{L}_2(0,T;\mathrm{L}_4(\Omega))\quad,$$

are bounded independently on $\mathbf{h}$. Moreover, we obtain that

$$\mathcal{S}_h\dot{u}^h,\mathcal{S}_h\dot{p}^h\in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))\quad,$$

are bounded independently on $\mathbf{h}$ as follows from (39). We conclude that

$$\mathcal{Q}_h u^h,\mathcal{Q}_h p^h\in \mathrm{L}_\infty(0,T;\mathrm{H}^1_0(\Omega))\quad,$$
$$\mathcal{Q}_h u^h,\mathcal{Q}_h p^h\in \mathrm{L}_2(0,T;\mathrm{H}^1_0(\Omega))\quad,$$

are bounded independently on $\mathbf{h}$. According to (28),

$$\mathcal{Q}_h\dot{u}^h,\mathcal{Q}_h\dot{p}^h\in \mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))\quad.$$

Passing to a subsequence, we have

- $\mathcal{Q}_{h_n} u^{h_n}, \mathcal{Q}_{h_n} p^{h_n} \rightharpoonup^* u, p$ in $\mathrm{L}_\infty(0,T;\mathrm{H}_0^1(\Omega))$;
- $\mathcal{Q}_{h_n} u^{h_n}, \mathcal{Q}_{h_n} p^{h_n} \rightharpoonup u, p$ in $\mathrm{L}_2(0,T;\mathrm{H}_0^1(\Omega))$;
- $\mathcal{S}_{h_n} \dot{p}^{h_n}, \mathcal{Q}_{h_n} \dot{p}^{h_n} \rightharpoonup \partial_t u, \partial_t p$ in $\mathrm{L}_2(0,T;\mathrm{H}^{-1}(\Omega))$;
- $\mathcal{S}_{h_n} \dot{u}^{h_n}, \mathcal{Q}_{h_n} \dot{u}^{h_n} \rightharpoonup \partial_t u, \partial_t p$ in $\mathrm{L}_2(0,T;\mathrm{H}^{-1}(\Omega))$;
- $\mathcal{S}_{h_n} u^{h_n}, \mathcal{S}_{h_n} p^{h_n} \rightharpoonup u, p$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$.

The non-linear terms in the equation (1) require stronger convergence result. Using the lemma on the compact imbedding, we conclude that $\mathcal{Q}_{h_n} p^{h_n}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$. Relation (29) implies the same result for $\mathcal{S}_{h_n} p^{h_n}$. Denote their common limit as $p$ and the weak limit of $\mathcal{S}_{h_n} \dot{p}^{h_n}$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ as $q_1$. The estimate

$$\begin{aligned}
&\|f_0(\mathcal{S}_h p^h)\|_{\mathrm{L}_{4/3}(\Omega)} \leq \\
&\quad\leq a\Big[\frac{1}{2}\|\mathcal{S}_h p^h\|_{\mathrm{L}_{4/3}(\Omega)} + \frac{3}{2}\|(\mathcal{S}_h p^h)^2\|_{\mathrm{L}_{4/3}(\Omega)} + \|(\mathcal{S}_h p^h)^3\|_{\mathrm{L}_{4/3}(\Omega)}\Big] = \\
&\qquad= a\Big[\frac{1}{2}\|\mathcal{S}_h p^h\|_{\mathrm{L}_{4/3}(\Omega)} + \frac{3}{2}\|\mathcal{S}_h p^h\|^2_{\mathrm{L}_{8/3}(\Omega)} + \|\mathcal{S}_h p^h\|^3_{\mathrm{L}_4(\Omega)}\Big] \quad , \qquad (42)
\end{aligned}$$

justifies the existence of weak limit of $f_0(\mathcal{S}_{h_n} p_{h_n})$ in $\mathrm{L}_2(0,T;\mathrm{L}_{4/3}(\Omega))$ denoted by $q_2$ (dual space).

These limits exist as a consequence of the a priori estimate and of (29), (28). We prove that $q_1 = \partial_t p$, $q_2 = f_0(p)$. First relation is implied by the uniqueness of the limit in $\mathcal{D}'(0,T)$, as

$$\int_0^T (\mathcal{S}_{h_n} \dot{p}^{h_n} - \mathcal{Q}_{h_n} \dot{p}^{h_n}, q)\psi(t)dt = -\int_0^T (\mathcal{S}_{h_n} p^{h_n} - \mathcal{Q}_{h_n} p^{h_n}, q)\dot{\psi}(t)dt \quad ,$$

where $q \in \mathcal{D}(\Omega)$, $\psi \in \mathcal{D}(0,T)$. The remaining equality is proven in the following lemma.

**Lemma 9.** *If $p$ denotes the weak limit of $\mathcal{S}_{h_n} p^{h_n}$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, then*

$$f_0(\mathcal{S}_{h_n} p^{h_n}) \to f_0(p) \text{ weakly in } \mathrm{L}_{\frac{4}{3}}(0,T;\mathrm{L}_{\frac{4}{3}}(\Omega)) \quad .$$

*Proof.* According to the compact imbedding, we have that $\mathcal{S}_{h_n} p^{h_n}$ converges strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ and it can be considered to converge a.e. in this space (see [9]). Furthermore, we observe that as $\mathcal{S}_{h_n} p^{h_n}$ was bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_4(\Omega))$ (see (42), $f_0(\mathcal{S}_{h_n} p^{h_n})$ is bounded in $\mathrm{L}_\infty(0,T;\mathrm{L}_{\frac{4}{3}}(\Omega))$. These two facts together with the Aubin lemma [2] give the final result. □

Before proceeding in the proof, we show more about regularity of $p$.

**Lemma 10.** *Under the assumptions of the theorem, the function $p$ belongs to $\mathrm{H}_0^1(\Omega) \cap \mathrm{H}^2(\Omega)$.*

*Proof.* Multiply the equation of phase by a function $\mathcal{P}_{h_n} q$, where $q \in \mathcal{D}(\Omega)$.

$$\begin{aligned}
&\alpha\xi^2(\dot{p}^{h_n}, \mathcal{P}_{h_n} q)_h + \xi^2(\bar{\nabla}_{h_n} p^{h_n}, \bar{\nabla}_{h_n} \mathcal{P}_{h_n} q] = \\
&\qquad= (f_0(p^{h_n}), \mathcal{P}_{h_n} q)_h - \beta\xi(u^{h_n}, \mathcal{P}_{h_n} q)_h \quad . \qquad (43)
\end{aligned}$$

In terms of $\mathrm{L}_2(\Omega)$, this means that

$$\alpha\xi^2(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}(\mathcal{P}_{h_n}q)) + \xi^2(\nabla(\mathcal{Q}_{h_n}p^{h_n}), \nabla\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}q)) = \\ = (f_0(\mathcal{S}_{h_n}p^{h_n}), \mathcal{S}_{h_n}(\mathcal{P}_{h_n}q)) - \beta\xi(\mathcal{S}_{h_n}u^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n}q) \quad . \tag{44}$$

According to (32), we realize that $\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}q) \overset{n\to\infty}{\to} q$ in $\mathrm{H}^1_0(\Omega)$, and similarly $\mathcal{S}_{h_n}(\mathcal{P}_{h_n}q) \overset{n\to\infty}{\to} q$ in $\mathrm{L}_2(\Omega)$ (see (30)). We can pass to the limit in the sense of $\mathcal{D}'(0,T)$ obtaining

$$\alpha\xi^2(\partial_t p, q) + \xi^2(\nabla p, \nabla q) = (q_2, q) - \beta\xi(u, q) \quad . \tag{45}$$

Consequently, the function $p$ is continuous from $\langle 0, T\rangle$ into $\mathrm{L}_2(\Omega)$. We rewrite the previous equality in the sense of $\mathcal{D}'(\Omega)$,

$$\alpha\xi^2\partial_t p = \xi^2\Delta p + q_2 - \beta\xi u \quad . \tag{46}$$

Note that $q_2 = f_0(p)$ and $p \in \mathrm{L}_\infty(0,T,\mathrm{L}_s(\Omega))$ for any $s > 1$. Consequently, $q_2 \in \mathrm{L}_2(0,T,\mathrm{L}_2(\Omega))$. As $\partial_t p$, $q_2$ belong to $\mathrm{L}_2(\Omega)$, this means that $\Delta p \in \mathrm{L}_2(\Omega)$ for each $t \in (0,T)$. Consequently, we find that $p$ must be in the domain of $\Delta$ — see [11], [2]:

$$p(t) \in D(\Delta) = \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega) \text{ for } t \in (0,T) \quad .$$

□

Next statement investigates the convergence of gradient.

**Lemma 11.** *The sequence $\nabla\mathcal{Q}_{h_n}p^{h_n}$ converges strongly to $\nabla p$ in $L_2((0,T)\times\Omega)$.*

*Proof.* Following the technique of [12], the statement of the lemma is shown. Multiply the equation of phase in (35) by $p^{h_n} - \mathcal{P}_{h_n}p$ and sum over $\omega_h$.

$$\alpha\xi^2(\dot{p}^{h_n}, p^{h_n} - \mathcal{P}_{h_n}p)_h + \xi^2(\bar{\nabla}_{h_n}p^{h_n}, \bar{\nabla}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p)] = \\ = (f_0(p^{h_n}), p^{h_n} - \mathcal{P}_{h_n}p)_h - \beta\xi(u^{h_n}, p^{h_n} - \mathcal{P}_{h_n}p)_h \quad . \tag{47}$$

Rewrite this equality in terms of $\mathrm{L}_2(\Omega)$, and integrate over $(0,T)$.

$$\alpha\xi^2\int_0^T(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt + \\ + \xi^2\int_0^T(\nabla(\mathcal{Q}_{h_n}p^{h_n}), \nabla\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt = \\ = \int_0^T(f_0(\mathcal{S}_{h_n}p^{h_n}), \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt - \\ - \beta\xi\int_0^T(u^{h_n}, \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt. \tag{48}$$

As we have shown that $p \in \mathrm{L}_2(0,T;\mathrm{H}^2(\Omega))$ satisfies (45), it means that $p(t) \in \mathcal{C}^{0,1}(\Omega)$, $t \in (0,T)$, and consequently, $\mathcal{S}_{h_n}(\mathcal{P}_{h_n}p) \to p$, and $\nabla\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p) \to \nabla p$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ (see (30), (31)). We add and subtract a term

$$\xi^2 \int_0^T (\nabla(\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p)), \nabla\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt$$

to the equality (48) knowing that it tends to 0 as

$$\nabla\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p) \to 0 \quad ,$$

weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, if $n \to \infty$. Then, we have

$$\begin{aligned}\xi^2 \int_0^T & (\nabla(\mathcal{Q}_{h_n}p^{h_n} - \mathcal{P}_{h_n}p), \nabla\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt = \\ &= - \alpha\xi^2 \int_0^T (\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt + \\ &\quad + \int_0^T (f_0(\mathcal{S}_{h_n}p^{h_n}), \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt + \\ &\quad - \beta\xi \int_0^T (u^{h_n}, \mathcal{S}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt + \\ &\quad + \xi^2 \int_0^T (\nabla(\mathcal{Q}_{h_n}(\mathcal{P}_{h_n}p)), \nabla\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p))dt \quad . \end{aligned} \tag{49}$$

As all terms in the right hand side tend to 0 if $n \to \infty$, we see that $\nabla(\mathcal{Q}_{h_n}(p^{h_n} - \mathcal{P}_{h_n}p)) \to 0$ in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$, which together with (32) gives the desired result. □

**Passage to the limit.** Take the system of (35) into the consideration, multiply by test functions $\mathcal{P}_{h_n}w$, $\mathcal{P}_{h_n}q$ where $w, q \in \mathcal{D}(\Omega)$. Integrate it over $\omega_h$. Then, we have, in terms of $\mathrm{L}_2(\Omega)$,

$$\begin{aligned} &(\mathcal{S}_{h_n}\dot{u}^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n}w) + (\nabla\mathcal{Q}_{h_n}u^{h_n}, \nabla\mathcal{Q}_{h_n}\mathcal{P}_{h_n}w) = L(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n}w) \quad , \\ &\alpha\xi^2(\mathcal{S}_{h_n}\dot{p}^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n}q) + \xi^2(\nabla\mathcal{Q}_{h_n}p^{h_n}, \nabla\mathcal{Q}_{h_n}\mathcal{P}_{h_n}q) = \\ &\qquad = (f_0(\mathcal{S}_{h_n}p^{h_n}), \mathcal{S}_{h_n}\mathcal{P}_{h_n}q) - \beta\xi(\mathcal{S}_{h_n}u^{h_n}, \mathcal{S}_{h_n}\mathcal{P}_{h_n}q) \quad . \end{aligned} \tag{50}$$

Knowing that

1. $\mathcal{S}_{h_n}\dot{p}^{h_n}$, $\mathcal{S}_{h_n}\dot{u}^{h_n}$ converge weakly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to $\partial_t p$, $\partial_t u$;
2. $\nabla\mathcal{Q}_{h_n}p^{h_n}$, $\nabla\mathcal{Q}_{h_n}u^{h_n}$ converge strongly in $\mathrm{L}_2(0,T;\mathrm{L}_2(\Omega))$ to $\nabla p$, $\nabla u$;
3. $\mathcal{S}_{h_n}\mathcal{P}_{h_n}p_0$, $\mathcal{S}_{h_n}\mathcal{P}_{h_n}u_0$ converges strongly to $p_0$, $u_0$ in $\mathrm{H}_0^1(\Omega)$,

multiply (50) by a scalar function $\psi(t) \in \mathcal{C}^1\langle 0,T\rangle$, for which $\psi(T) = 0$. We integrate by parts. Taking into account all previous results, the fact that

$$\mathcal{S}_{h_n}p^{h_n}(0) = \mathcal{S}_{h_n}\mathcal{P}_{h_n}p_0, \quad \mathcal{S}_{h_n}u^{h_n}(0) = \mathcal{S}_{h_n}\mathcal{P}_{h_n}u_0$$

and the Lebesgue theorem, we are able to pass to the limit.

$$
\begin{aligned}
&(u_0 - Lp_0, w)\psi(0) - \int_0^T (u - Lp, w)\dot{\psi} dt + \int_0^T \psi[(\nabla u, \nabla w) = 0 \quad , \\
&\alpha\xi^2(p_0, q)\psi(0) - \int_0^T \alpha\xi^2(p, q)\dot{\psi} dt + \int_0^T \psi[\xi^2(\nabla p, \nabla q) - \\
&\qquad\qquad\qquad - (f_0(p), q) + \beta\xi(u, q)]dt = 0 \quad . \qquad (51)
\end{aligned}
$$

If $\psi \in \mathcal{D}(0,T)$, we have

$$
\begin{aligned}
\frac{d}{dt}(u - Lp, w) + (\nabla u, \nabla w) = 0 \quad , \\
\alpha\xi^2 \frac{d}{dt}(p, q) + \xi^2(\nabla p, \nabla q) = (f_0(p), q) - \beta\xi(u, q) \quad .
\end{aligned} \qquad (52)
$$

It remains to show that the weak solution satisfies the initial condition. Multiplying (51) by a scalar function $\psi(t) \in \mathcal{C}^1\langle 0, T\rangle$, for which $\psi(T) = 0$, and integrating by parts, we obtain

$$
\begin{aligned}
&(u(0) - Lp(0), w)\psi(0) - \int_0^T (u - Lp, w)\dot{\psi} dt + \int_0^T \psi[(\nabla u, \nabla w) = 0 \quad , \\
&\alpha\xi^2(p(0), q)\psi(0) - \int_0^T \alpha\xi^2(p, q)\dot{\psi} dt + \\
&\qquad + \int_0^T \psi[\xi^2(\nabla p, \nabla q) - (f_0(p), q) + \beta\xi(u, q)]dt = 0 \quad . \qquad (53)
\end{aligned}
$$

Subtracting this equation from (51), we get

$$
(u_0 - Lp_0 - u(0) + Lp(0), w)\psi(0) = 0, \quad (p_0 - p(0), q)\psi(0) = 0 \quad .
$$

From this we see that $u(0) = u_0$, $p(0) = p_0$ in $\mathrm{L}_2(\Omega)$. To prove uniqueness, consider two solutions of the problem (34), denoted as $[u, p]$ and $[v, q]$. Subtracting two systems of equations and denoting $[w, r] = [u - v, p - q]$, multiplying the first equation by $w$ and the second equation by $\dot{r}$ via the semi-discrete scheme, we have

$$
\frac{1}{2}\frac{d}{dt}\|w\|^2 + (\nabla w, \nabla w) = (\dot{r}, w) \quad \text{in} \quad (0,T) \quad , \qquad (54)
$$

$$
w(0) = 0 \quad ,
$$

$$
\alpha\xi^2\|\dot{r}\|^2 + \xi^2\frac{1}{2}\frac{d}{dt}(\nabla r, \nabla r) = (f_0(p) - f_0(q), \dot{r}) - \beta\xi(w, \dot{r}) \quad \text{in} \quad (0,T) \quad ,
$$

$$
r(0) = 0 \quad . \qquad (55)
$$

Denote $\Psi(p,q) = -\frac{1}{2}a + \frac{3}{2}a(p+q) - a(p^2+pq+q^2)$. The existence proof guarantees that there is a constant $\tilde{C}$ such that

$$
\|\Psi(p,q)\|_{\mathrm{L}_4(\Omega)} \leq \tilde{C} \quad ,
$$

(as implied by the continuous imbedding $\mathrm{H}^1_0(\Omega) \subset_{>} \mathrm{L}_q(\Omega)$ for $q \in < 0, +\infty)$). Therefore, we have

$$|(\Psi(p,q)r, \dot{r})| \leq \|\Psi(p,q)\|_{\mathrm{L}_4(\Omega)} \|r\|_{\mathrm{L}_4(\Omega)} \|\dot{r}\|_{\mathrm{L}_2(\Omega)} \leq \tilde{C} \|r\|_{\mathrm{L}_4(\Omega)} \|\dot{r}\|_{\mathrm{L}_2(\Omega)} \quad .$$

Using the Poincaré and Schwarz inequalities, we get

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq \frac{C(\Omega)}{4}\|\dot{r}\|^2$$

$$\alpha\xi^2\|\dot{r}\|^2 + \xi^2 \frac{1}{2}\frac{d}{dt}\|\nabla r\|^2 \leq \tilde{C}\|r\|_{\mathrm{L}_4(\Omega)}\|\dot{r}\|_{\mathrm{L}_2(\Omega)} + \\ + \frac{1}{2}\alpha\xi^2\|\dot{r}\|^2 + \frac{\beta}{2\alpha\xi}\|w\|^2 \quad , \tag{56}$$

or, considering the fact, that there is a constant $C_4 > 0$ such that

$$\|r\|_{\mathrm{L}_4(\Omega)} \leq C_4 \|\nabla r\| \quad ,$$

we obtain

$$\frac{1}{2}\frac{d}{dt}\|w\|^2 \leq \frac{C(\Omega)}{4}\|\dot{r}\|^2$$

$$\alpha\xi^2\|\dot{r}\|^2 + \xi^2 \frac{1}{2}\frac{d}{dt}\|\nabla r\|^2 \leq \frac{1}{4}\alpha\xi^2\|\dot{r}\|^2 + \frac{\tilde{C}^2}{\alpha\xi^2}C_4^2\|\nabla r\|^2 + \\ + \frac{1}{2}\alpha\xi^2\|\dot{r}\|^2 + \frac{\beta}{2\alpha\xi}\|w\|^2 \quad . \tag{57}$$

Combining these inequalities, we have

$$\frac{1}{C(\Omega)}\alpha\xi^2 \frac{1}{2}\frac{d}{dt}\|w\|^2 + \xi^2 \frac{1}{2}\frac{d}{dt}\|\nabla r\|^2 \leq \frac{\tilde{C}^2}{\alpha\xi^2}C_4^2\|\nabla r\|^2 + \frac{\beta}{2\alpha\xi}\|w\|^2 \quad .$$

Such inequality implies, together with the initial conditions, that

$$r(t) = w(t) = 0 \quad \forall t \in (0,T) \quad \text{in} \quad \mathrm{L}_2(\Omega) \quad .$$

□

## 4 Convergence towards the sharp interface model

This paragraph deals with the relation of the phase-field model to a sharp-interface formulation of the Stefan problem. It uses estimates derived above to show certain compactness statements leading to the existence of a step function defining the position of solid domain in time. Consider the weak formulation of the standard phase-field model.

$$\frac{d}{dt}(u,v) + (\nabla u, \nabla v) = L\frac{d}{dt}(p,v) \quad \text{in} \quad (0,T) \quad , \tag{58}$$

$$u(0) = u_0 \quad ,$$

$$\alpha\xi^2\frac{d}{dt}(p,q) + \xi^2(\nabla p, \nabla q) = (f_0(p), q) - \beta\xi(u,q) \quad \text{in} \quad (0,T) \quad ,$$

$$p(0) = p_0 \quad . \tag{59}$$

Main purpose of next investigation will be the dependence on $\xi$. Consider the solution of the semidiscrete scheme (35). We multiply the first equation by $u^h$ and the second one by $\dot{p}^h$.

$$\frac{1}{2}\frac{d}{dt}\|u^h\|_h^2 + \|\bar{\nabla}_h u^h]|^2 = L(\dot{p}^h, u^h)_h \quad , \tag{60}$$

$$\alpha\xi^2\|\dot{p}^h\|_h^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 = -\frac{d}{dt}(w_0(p^h), 1)_h - \beta\xi(u^h, \dot{p}^h)_h \quad . \tag{61}$$

Combining previous equalities, we get

$$\begin{aligned}\alpha\xi^2\|\dot{p}^h\|_h^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 = \\ = -\frac{d}{dt}(w_0(p^h), 1)_h - \frac{\beta\xi}{L}\left\{\frac{1}{2}\frac{d}{dt}\|u^h\|_h^2 + \|\bar{\nabla}_h u^h]|^2\right\} \quad ,\end{aligned} \tag{62}$$

or

$$\alpha\xi^2\|\dot{p}^h\|_h^2 + \xi^2\frac{1}{2}\frac{d}{dt}\|\bar{\nabla}_h p^h]|^2 + \frac{d}{dt}(w_0(p^h), 1)_h + \frac{\beta\xi}{L}\frac{1}{2}\frac{d}{dt}\|u^h\|_h^2 = 0.$$

We integrate over $(0,t)$, which gives

$$\begin{aligned}\left\{\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2 + (w_0(p^h), 1)_h + \frac{\beta\xi}{L}\frac{1}{2}\|u^h\|_h^2\right\}(t) \leq \\ \leq \left\{\xi^2\frac{1}{2}\|\bar{\nabla}_h p^h]|^2 + (w_0(p^h), 1)_h + \frac{\beta\xi}{L}\frac{1}{2}\|u^h\|_h^2\right\}(0) \quad .\end{aligned} \tag{63}$$

Passing to the limit, if $\mathbf{h} \to 0$, which is justified by the proof of the Theorem 8, we get

$$\frac{1}{2}\frac{\beta}{L}\|u_\xi(t)\|^2 + E_\xi[p_\xi](t) \leq \frac{1}{2}\frac{\beta}{L}\|u_\xi(0)\|^2 + E_\xi[p_\xi](0) \quad t \in (0,T) \quad , \tag{64}$$

where we denoted $(p^h \xrightarrow{h\to 0} p_\xi)$,

$$E_\xi[p_\xi](t) = \int_\Omega [\xi\frac{1}{2}|\nabla p_\xi|_E^2 + \frac{1}{\xi}w_0(p_\xi)]d\mathbf{x} \quad .$$

Additionally, there is an estimate for the time derivative, if we integrate (62) over $(0,T)$ and pass to the limit for $\mathbf{h} \to 0$.

$$\alpha\xi\int_0^T \|\partial_t p_\xi\|^2 dt + E_\xi[p_\xi](T) - E_\xi[p_\xi](0) + \frac{\beta}{2L}(\|u(T)\|^2 - \|u(0)\|^2) = 0 \tag{65}$$

Consequently, there is a constant $C_1$ such that

$$\frac{1}{2}\alpha\xi\int_0^T \|\partial_t p\|_h^2 dt + E_\xi[p_\xi](T) \leq E_\xi[p_\xi](0) + C_1 \quad . \tag{66}$$

These estimates allow to use the method proposed by [4] and used in [2]. Define the following monotone function

$$G(s) = \int_0^s |1-(1-2r)^2| dr \quad . \tag{67}$$

We prove next lemma

**Lemma 12.** *Be $p_\xi$ the solution of (34) where $E_\xi[p_\xi](0) \leq M_0$ independently on $\xi$. Then there are constants $M > 0$ and $M_1 > 0$ such that*

$$sup\{\int_\Omega |\nabla G(p_\xi)| dx \mid t \in \langle 0, T \rangle\} \leq M \tag{68}$$

*and, for $0 \leq t_1 < t_2$,*

$$\int_{t_1}^{t_2} \int_\Omega |\partial_t G(p_\xi)| dx dt \leq M_1 (t_2 - t_1)^{0.5} \quad . \tag{69}$$

*Proof.* We have shown that

$$E_\xi[p_\xi](t) \leq M_0 + C_1 \quad ,$$

on $\langle 0, T \rangle$. We write

$$\begin{aligned} E_\xi[p](t) = \int_\Omega [\xi \frac{1}{2} |\nabla p|_E^2 + \frac{1}{\xi} w_0(p)] dx \geq \\ \geq \int_\Omega \sqrt{2} |\nabla p_\xi| \sqrt{w_0(p_\xi)} dx = \sqrt{2} \int_\Omega |\nabla G(p_\xi)|_E dx \quad , \end{aligned} \tag{70}$$

which shows (68) by setting $M = \frac{1}{\sqrt{2}}(M_0 + C_1)$. Furthermore, if

$$\begin{aligned} \int_{t_1}^{t_2} dt \int_\Omega dx |\partial_t G(p_\xi)| = \int_{t_1}^{t_2} dt \int_\Omega dx |\dot{p}_\xi| |G'(p_\xi)| \leq \\ \leq \left( \int_{t_1}^{t_2} dt \int_\Omega dx |\dot{p}_\xi|^2 \right)^{\frac{1}{2}} \left( \int_{t_1}^{t_2} dt \int_\Omega dx |G'(p_\xi)|^2 \right)^{\frac{1}{2}} \leq \\ \leq (\frac{2}{\alpha}(C_1 + M_0)^2)^{\frac{1}{2}} (t_2 - t_1)^{\frac{1}{2}} \quad , \end{aligned} \tag{71}$$

then (69) is shown, if setting $M_1 = \sqrt{\frac{2}{\alpha}}(C_1 + M_0)$. □

The previous statement leads to the existence of a step function as expected.

**Theorem 13.** *Let $u_\xi, p_\xi$ is the solution of the problem (34) with the initial data satisfying $E_\xi[p_\xi](0) < M_0$ and $u_\xi, p_\xi \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1(\Omega)$, and let*

$$\int_\Omega |p_\xi(0, \mathbf{x}) - v_0(\mathbf{x})| d\mathbf{x} \to 0, \quad \|u_\xi(0)\|_{\mathrm{L}_2(\Omega)} \leq C_2 \quad ,$$

*as $\xi \to 0$, for a function $v_0 \in \mathrm{L}_1(\Omega)$. Then for any sequence tending to 0 there is a subsequence $\xi_{n'}$ such that*

$$\lim_{\xi_{n'} \to 0} p_{\xi_{n'}}(t,\mathbf{x}) = v(t,\mathbf{x}), \quad u_{\xi'_n}(t,\mathbf{x}) \rightharpoonup u(t,\mathbf{x}) \ \textit{in}\ \mathrm{L}_2((0,T)\times\Omega),$$

*and $u$, $v$ are defined a.e. in $(0,T)\times\Omega$. The function $v$ reaches values 0 and 1, and satisfies*

$$\int_\Omega |v(t_1,\mathbf{x}) - v(t_2,\mathbf{x})| dx \le C|t_2 - t_1|^{\frac{1}{2}} \quad ,$$

*where $C > 0$ is a constant, and*

$$\sup_{t\in\langle 0,T\rangle} \int_\Omega |\nabla v|_E d\mathbf{x} \le C_1 \quad ,$$

*in the sense of $BV(\Omega)$, where $C_1 > 0$ is a constant. The initial condition is*

$$\lim_{t\to 0} v(t,\mathbf{x}) = v_0(\mathbf{x}) \quad ,$$

*a.e.*

*Proof.* The proof follows steps presented in [4]. We find that

$$G(s) = 2s^2 - \frac{4}{3}s^3 \quad \text{for} \quad s \in \langle 0,1\rangle \quad ,$$
$$G(s) = \frac{4}{3}s^3 - 2s^2 + \frac{4}{3} \quad \text{for} \quad s \in (1,+\infty) \quad .$$

Consequently, a direct computation justifies that

$$|G(s)| \le \frac{4}{3} + [1 - (1-2s)^2]^2 \quad .$$

Then, we are able to obtain the upper bounds for the function $G$ and its spatiotemporal gradient. According to (64), we have

$$\int_0^T \int_\Omega w_0(p) d\mathbf{x} dt \le M_2 \xi \quad . \tag{72}$$

Putting (72), (68) and (69) together, we conclude, that $G(p_\xi)$ is in $BV((0,T)\times\Omega)$ regardless the value $\xi > 0$. Following [6],

$$BV((0,T)\times\Omega) \subset\!\!\!\!_{>}\subset\!\!\!\!_{>}\ \mathrm{L}_1((0,T)\times\Omega) \quad .$$

Consequently, there is a sequence $G(p_{\xi_n})$ converging to an element $G^*$ in the space $\mathrm{L}_1((0,T)\times\Omega)$. According to [9], there is a further subsequence $G(p_{\xi_{n'}})$ converging to $G^*$ almost everywhere in $(0,T)\times\Omega$.

The function $G :< 0,+\infty) \to < 0,+\infty)$ is monotone, which implies existence of the unique function $v$ such that

$$G^* = G(v) \quad ,$$

and

$$p_{\xi_{n'}} \to v \quad \text{a.e. in } (0,T)\times\Omega$$

According to (72) and by the Fatou lemma, we obtain

$$\int_0^T \int_\Omega w_0(v) d\mathbf{x} dt = 0 \quad , \tag{73}$$

from which follows that the function $p$ takes only the values 0, 1.

Now, we prove that $G$ is Hölder-continuous in the time variable. The function $p_{\xi_{n'}}$ satisfies

$$|G(p_{\xi_{n'}}(t_1,\mathbf{x})) - G(p_{\xi_{n'}}(t_2,\mathbf{x}))| \le \int_{t_1}^{t_2} |\partial_t G(p_{\xi_{n'}}(t,\mathbf{x}))dt \quad ,$$

for $0 \le t_1 \le t_2 \le T$. Integrating over $\Omega$,

$$\int_\Omega |G(p_{\xi_{n'}}(t_1,\mathbf{x})) - G(p_{\xi_{n'}}(t_2,\mathbf{x}))| d\mathbf{x} \le M_1 |t_1 - t_2|^{0.5} \quad , \tag{74}$$

according to the Lemma 12. Passing to the limit for $n' \to \infty$, we find

$$\int_\Omega |G^*(t_1,\mathbf{x}) - G^*(t_2,\mathbf{x})| d\mathbf{x} \le M_1 |t_1 - t_2|^{0.5} \quad ,$$

for almost all $t_1, t_2 \in (0,T)$. The statement of theorem is obtained by the fact that

$$|G^*(t_1,\mathbf{x}) - G^*(t_2,\mathbf{x})| = (G(1) - G(0))|v(t_1,\mathbf{x}) - v(t_2,\mathbf{x})| \quad .$$

The a.e. argument makes from the function $v$ a continuous map from $\langle 0,T\rangle$ to $L_1(\Omega)$. Taking $t_1 = 0$ in (74) and according to the assumption

$$\int_\Omega |p_\xi(0,\mathbf{x}) - v_0(\mathbf{x})| d\mathbf{x} \to 0 \quad ,$$

as $\xi \to 0$, (similarly for G-valued function) we have

$$\int_\Omega |G(v_0(\mathbf{x})) - G(v(t_2,\mathbf{x}))| d\mathbf{x} \le M_1 t_2^{0.5} \quad ,$$

from which

$$\int_\Omega |v_0(\mathbf{x}) - v(t_2,\mathbf{x})| d\mathbf{x} \le \frac{M_1}{G(1) - G(0)} t_2^{0.5} \quad .$$

This concludes the proof. It remains to show the boundedness of the total variation of $v$ ([6]). The lower semicontinuity of the total variation in $L_1$-space together with the Lemma 12 yields

$$\operatorname*{ess\,sup}_{0<t<T} \int_{\Omega\times\{t\}} |\nabla G^*| d\mathbf{x} \le M_0 \quad ,$$

and by the continuity of $v$ in time, we have

$$\sup_{0<t<T} \int_{\Omega\times\{t\}} |\nabla v| d\mathbf{x} \leq \frac{M_0}{G(1)-G(0)} \quad .$$

It remains to show the convergence of $u_\xi$. The relation (64) implies that $u_\xi$ is bounded in $\mathrm{L}_2((0,T)\times\Omega)$. Then, using the subsequence argument, $u_{\xi_{n'}}$ converges weakly to an element $u \in \mathrm{L}_2((0,T)\times\Omega)$. This completes the proof. □

## 5 Conclusion

The purpose of the paper was to show the convergence property of the semi-discrete scheme based on the method of lines. The compactness technique allowed to prove existence and uniqueness of the original weak solution. The phase function depending on the small parameter $\xi$ is bounded in the BV sense. Consequently, it converges to a step-wise function indicating different phases. The technique of the recovery of the sharp-interface relation can also be applied to the presented problem. The presented approach is applicable even in case of different modifications of the model.

**Acknowledgement.**
The author was partially supported by the grant No. 3097486 of the Czech Technical University.

## References

[1] N. D. Alikakos and P. W. Bates. On the singular limit in a phase field model of phase transitions. *Ann. Inst. Henri Poincaré*, 5:141–178, 1988.
[2] M. Beneš. *Phase-Field Model of Microstructure Growth in Solidification of Pure Substances*. PhD thesis, Faculty of Nuclear Sciences and Physical Engineering, Czech Technical University, 1997.
[3] D. Brochet, X. Chen, and D. Hilhorst. Finite dimensional exponential attractor for the phase-field model. *J. Appl. Anal.*, 49:197–212, 1993.
[4] L. Bronsard and R. Kohn. Motion by mean curvature as the singular limit of Ginzburg-Landau dynamics. *J. Differential Equations*, 90:211–237, 1991.
[5] G. Caginalp. An analysis of a phase field model of a free boundary. *Arch. Rational Mech. Anal.*, 92:205–245, 1986.
[6] E. Giusti. *Minimal Surfaces and Functions of Bounded Variation*. Birkhäuser, Basel, 1984.
[7] M. Gurtin. On the two-phase Stefan problem with interfacial energy and entropy. *Arch. Rational Mech. Anal.*, 96:200–240, 1986.
[8] R. Kobayashi. Modeling and numerical simulations of dendritic crystal growth. *Physica D*, 63:410–423, 1993.
[9] A. Kufner, O. John, and S. Fučík. *Function Spaces*. Academia, Prague, 1977.
[10] P. Laurençot. Solutions to a Penrose-Fife model of phase-field type. *J. Math. Anal. Appl.*, 185:262–274, 1994.

[11] J. L. Lions. *Quelques Méthodes aux Résolution des Problèmes Nonlinéaires.* Dunod Gauthiers-Villars, Paris, 1969.

[12] K. Mikula and J. Kačur. Evolution of convex plane curves describing anisotropic motions of phase interfaces. *SIAM J. Scientific Computing*, 17:1302–1327, 1996.

[13] O. Penrose and P. C. Fife. Thermodynamically consistent models of phase-field type for the kinetics of phase transitions. *Physica D*, 43:44–62, 1990.

[14] P. A. Raviart. Sur la résolution de certaines équations paraboliques non linéaires. *J. Func. Anal.*, 5:299–328, 1970.

[15] A. A. Samarskii. *Theory of Difference Schemes.* Nauka, Moscow, 1977.

[16] A. A. Wheeler, B. T. Murray, and R. J. Schaefer. Computation of dendrites using a phase field model. *Physica D*, 66:243–262, 1993.