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Some Remarks On the Terminal Value Problem In Hereditary Setting

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Abstract. The terminal value problem (TVP) is investigated. Conditions for its solvability are examined.

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1 Introductory remarks

In the last 50 years the theory of differential equations with delayed argument has been widely developped, mainly for the initial value problems. The boundary value problems for such differential equations were more or less omitted. In the paper [1] from 1975 I considered two boundary value problems:
For the differential equation

$$\dot{x} = f(t, x_t),\ x \in R_n,\ t \in [t_0, T), \tag{1}$$

1. given $X_1 \in R_n$ it is to prove the existence of the initial conditions (t_0, φ) such that the equation (1) has a solution existing on the interval $[t_0, T)$ with the property $\lim_{t \to T-} = X_1$;
2. let $X_0,\ X_1$ be two points of the space R_n. It is to prove the existence of a solution $x(t)$ of (1) such that $x(t_0) = X_0$, $\lim_{t \to T-} = X_1$.

I have assumed that:

This is the final form of the paper.

R_n is the n-dimensional euclidian space with the norm $|.|$. If

$$x(t) \ : \ [t_0 - h, T) \to R_n,$$

then

$$x(t) = x(t+s), \ s \in [-h, 0] \text{ for } t \in [t_0, T).$$

$C = C([-h,0], R_n)$ denotes the Banach space of all continuous functions $\varphi(t) : [-h, 0]$, R_n with the norm

$$\|\varphi(t)\| = \max_{t\in[-h,0]} |\varphi(t)|$$

and $C^0 = C^0([-h,0], R_n)$ denotes the Banach space of all $\varphi(t) \in C$ with $\varphi(0) = 0$. $B = B([t_0, T), R_n)$ is the Banach space of all functions $u(t)$ continuous and bounded with the norm

$$\|u(t)\|_B = \sup_{t\in[t_0,T)} |u(t)|.$$

B_0 is the Banach space of all functions $u(t) \in B$ such that $u(t_0) = 0$ and B_φ is the metric space of all functions $z(t)$ such that

$$z(t) = \varphi(t - t_0), \ t \in [t_0 - h, t_0], \ z(t) = u(t) \in B_0 \text{ for } t \in [t_0, T)$$

with the distance $\rho(z_1, z_2) = \|u_1 - u_2\|_B$, where

$$z_1(t) = u_1(t), \ z_2(t) = u_2(t) \text{ for t} \in [\mathrm{t}_0, \mathrm{T})$$

and

$$z_1(t) = z_2(t) = \varphi(t - t_0) \text{ for t} \in [\mathrm{t}_0 - \mathrm{h}, \mathrm{t}_0].$$

The following assumptions are made:

H$_1$ The function $f(t, \varphi)$ is defined and continuous on $[t_0, t) \times C$ and

$$\int_{t_0}^{T} |f(t, 0)| dt \ = \ K \ < \ \infty.$$

H$_2$ There exists a function $\beta(t)$ defined and continuous on $[t_0, T)$ such that

$$|f(t, \varphi_1) - f(t, \varphi_2)| \ < \ \beta(t)\|\varphi_1 - \varphi_2\| \text{ forall } \varphi_1, \varphi_2 \in \mathrm{C}$$

and $t \in [t_0, T)$ and $\int_{t_0}^{T} \beta(t) dt = k < 1$.

It is proven:

Lemma 1. Let H_1 and H_2 be satisfied. Let $X_1 \in R_n$ and $\varphi \in C_0$ be given. Then for a given $z \in B_\varphi$ there exists a unique $X_0 \in R_n$ such that

$$X_0 + \int_{t_0}^{T} f(s, X_0 + z_s) ds \ = \ X_1.$$

Theorem 1. Let H_1 and H_2 be satisfied and let be $K < 1/2$. Let $X_1 \in R_n$ and $\varphi \in C^0$ be given. Then there exists a unique $X_0^* \in R_n$ such that the solution $x(t, t_0, X^* + \varphi)$ of (1) exists on $[t_0, T)$ and

$$\lim_{t \to T-} x(t, t_0, X^* + \varphi) \;=\; X_1.$$

From this theorem follows immediately
Corollary 1. Let H_1 and H_2 with $k < 1/2$ be satisfied. Let $X_1 \in R_n$ be given. Then for each $\varphi \in C^0$ there exists one and only one point $X_0^*(\varphi)$ and the solution $x(t, t_0, X^* + \varphi)$ of (1) on $[t_0, T)$ such that

$$\lim_{t \to T-} x(t, t_0, X_0^* + \varphi) \;=\; X_1.$$

It is evident that for a given T the value T_0 may be chosen such that the requirement $k < 1/2$ from Theorem 1 will be satisfied.

Lemma 2. Let H_1 and H_2 be satisfied. Let $\varphi \in C^0$ and $X_0 \in R_n$ be given. Then for every $z \in B_\varphi$ there exists unique $X_1 \in R_1$ such that

$$X_0 + \int_{t_0}^{T} f(s, X_0 + z_s) ds \;=\; X_1.$$

Theorem 2. Let H_1 and H_2 be satisfied. Let be $\psi \in C$. Then the solution $x(t)$ of (1) given through the initial conditions (t_0, ψ) exists on $[t_0, T)$, is unique and $\lim_{t \to T-} x(t) \;=\; X_1 \in R_n$.
Lemma 3. Let H_1 and H_2 be satisfied. Let $\psi_1, \psi_2 \in C$ and $x(t, t_0, \psi_1)$, $x(t, t_0, \psi_2)$ be the corresponding solutions of (1). Then

$$\|x_t(t_0, \psi_1) - x_t(t_0, \psi_2)\| \leq exp[\int_{t_0}^{t} \beta(s) ds\ ] \|\psi_1 - \psi_2\|$$

for $t \in [t_0, T)$.

Other problem which is to solve is the following:

(P) Let be given X_0, X_1. It is to find $\varphi \in C^0$ such that there exists the solution $x(t) = x(t, t_0, X_0 + \varphi)$ of (1) satisfying the boundary conditions

$$x(t_0) \;=\; X_0, \;\; \lim_{t \to T-} x(t) \;=\; X_1.$$

The following two theorems give us some information about the solvability of the theorem (P).

Theorem 4. Let H_1 and H_2 be satisfied. Let $X_0 \in R_n$ and $|X_0| \; + \; K \;\neq\; 0$. Let $\varphi \in C^0$ be such that $\|\varphi\| < K\frac{a}{1-k}$, where $a \geq 1$. Then for the solution $x(t, t_0, X_0 + \varphi)$ of (1) holds

$$|x(t)| < [|X_0| + K]\frac{a}{1-k}, \; t \in [t_0, T).$$

From Theorem 4 we get

Theorem 5. Let the assumptions of Theorem 4 be satisfied. If $X_1 \in R_n$, $|X_1| > [|X_0| + K]\frac{a}{1-k}$, then there is no solution of the problem (P) for $\varphi \in C^0$, $\|\varphi\| < K\frac{a}{1-k}$.

It seems to be evident that the problem (P) can have, in general, no solution. The question which has not been answered till today is:
What is the weakest condition to add to H_1 and H_2, ($k < 1/2$) to guarantee the solvability of the problem (P)?

Remark: In my paper [1] mentioned at the beginning of this contribution the solutions are from space C^1. The authors Marcello Ragni and Paola Rubbioni from the University of Perugia in the paper [2] have considered this problem and have enlarged the class of the solutions to the AC (Absolute Continuous Functions). So, in the Carathéorody setting they give an existence theorem and have obtained the classical results on terminal value problems in the functional and also in the nonfunctional case.

I will give a short review of this paper:

Let be $a \in R^+ \cup \{+\infty\}$, $I = (-\infty, a)$, $t, s, \in I \cup \{a\}$, $I_s = (-\infty, s)$, $I_{ts} = [t, s)$, $t < s$.

Let $L^\infty_{loc}(I)$ be the space of essentially bounded measurable functions, $C(I)$ the subspace of continuous functions on I taking values on R^n.

Let w be a subset of $C(I)$ endowed with an inner operator $T : w \to w$ such that

$$Tx \cdot \chi_{I_{0a}} = x \cdot \chi_{I_{0a}}$$

and such that for every $x, \Phi \in W$ with $x(0) = \Phi(0)$ we get $Tx \cdot \chi_{I_0} = T\Phi \cdot \chi_{I_0}$; in other words, we assume that w is endowed with a map $\tau : w(0) \to w_{I_0}$ such that, put

$$Tx(t) = \begin{cases} x(t), \ t \in I_{0a} \\ \tau\left(x(0)\right)(t), \ t \in I_0 \end{cases} \text{ for every } x \in w ,$$

the element Tx belongs to w.

A recurring hypothesis will be:
(τ) $\quad \tau : w(0) \to w_{I_0}$ is a continuous map.

Remark: If (τ) holds, T is continuous.

Condition (C'): for every fixed $(y,x) \in R^n \times w$ the function $f(\bullet, y, x)$ is measurable in the set $\{t \in I \ : \ (t,y,x) \ \in \ D \times w\}$,

Condition (CV): C' holds and, for a. e. $t \in I$, for every sequence $(y_n)_n$ in $w(t)$ converging to $y_0 \in w(t)$, for every sequence $(x_n)_n$ in w and $x_0 \in w$ with $d_{I_t}(x_n, x_0) \ \to \ 0$ we have

$$f(t, y_n, x_n) \to f(t, y_0, x_0).$$

2 Statement of the problem (<u>Terminal Value Problem</u>)

Let be $f \ : \ D \times w \to R^n$. We have to solve the problem:
Fixed $y \in R^n$, determine an impact - time $p \in (0,a]$ and a function $x \in w$ (absolutely continuous on I_{0a}) such that

$$\text{(TVP)} \qquad \begin{cases} x'(t) = f(t, x(t), x), \ \text{a. e. } x \in I_{0p}, \\ \lim_{t \to p-} x(t) = y, \\ x|_{I_0} = \tau(x(0)). \end{cases}$$

The following theorem of the existence of solutions of problem (TVP) is founded on an admissibility condition on the mark y and the impact - time p with respect to the other data of the problem.

Definition: we will say that a couple $\bar{P} = (\bar{p}, \bar{y} \in (0,a] \times R^n$ is **admissible for problem (TVP)** if

(A) there exist two functions $\bar{u}, \bar{v} \in L^1(I_{0p}, R^n)$ such that

(A$_1$) $\bar{V}_{I_{o\bar{p}}} \subset w_{I_{o\bar{p}}}$,

(A$_2$) $-\bar{v}(t) \leq f(t, x(t), Tx) \leq -u(t)$ for a. e. $t \in I_{o\bar{p}}$ and for every $x \in \bar{V} \cap w$, where

$$\bar{V} = \{x \in C(I) \ : \ \bar{y} + \int_t^{\bar{p}} \bar{u}(s)\, ds \leq x(t) \leq \bar{y} + \int_t^{\bar{p}} \bar{v}(s)\, ds \text{ for every } t \in I_{0\bar{p}}\}.$$

Theorem 6: Suppose that $w \subset C(I)$ satisfies the property (τ) and let $f \ : \ D \times w \to R^n$ be a given function satisfying condition (CV). If $\bar{P} \equiv (\bar{p}, \bar{y}) \in (0,a] \times R^n$ is an admissible couple for (TVP), then there exists a solution of (TVP).

Proof: Put $M(t) = \int_t^{\bar{p}} \max\{|\bar{u}(s)|, |\bar{v}(s)|\}\, ds$ for every $t \in I_{o\bar{p}}$, and consider the set

$$G = \{x \in \bar{V}_{I_{o\bar{p}}} \ : \ w(x, I_{o\bar{p}}) \leq w(M, I_{o\bar{p}})\}.$$

The set is trivially nonempty and it is a compact and convex set in $C(I_{o\bar{p}})$. Moreover, $G \subset W_{I_{o\bar{p}}}$. Consider the functional $F \ : \ G \to C(I_{o\bar{p}})$ defined by

$$F\,x(t) = y - \int_t^{\bar{p}} f(s, x(s), Tx)\,ds \text{ for every } t \in I_{o\bar{p}}.$$

From (A) we get $F(G) \subset G$. Besides, F is continuous functional. Continuity follows from Lebesque dominated convergence theorem. So we can apply the Schauder-Tichonov fixed point theorem. Thus there exists $x \in w$ such that

$$\begin{aligned} x(t) &= \bar{y} - \textstyle\int_t^{\bar{p}} f(s, x(s), x)\,ds \text{ in } I_{o\bar{p}}, \\ x|_{I_o} &= \tau(x(0)) \text{ in } I_0. \end{aligned}$$

So $x(t)$ is the solution we were looking for.

Now it is to be answered:

- how many solutions there exist;
- which are the conditions to guarantee the unicity;
- if there is a set of all solutions, which properties has this set?

References

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