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UNIFORM ATOMS ON ω.

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Consider the lattice of all uniformities on the given set X. A uniformity is called a uniform atom on X, if it is non-discrete and if the discrete uniformity is the only uniformity finer than it. The aim of the present paper is to give some examples of uniform atoms. The proofs are shortened as much as possible; for the details, see [S].

0. Let us call a uniformity to be proximally discrete, if the induced proximity is discrete. Let $\mathcal{F}$ be a non-principal ultrafilter on an infinite set X, denote by $\mathcal{U}(\mathcal{F})$ the uniformity consisting of all covers $\mathcal{C}$ such that $\mathcal{C} \cap \mathcal{F} \neq \emptyset$.

The following facts can be found in [PR] (for 0.4, see also [V]):

0.1. Any proximally discrete atom on an infinite set X is uniformly finer than some $\mathcal{U}(\mathcal{F})$ for a suitable non-principal ultrafilter $\mathcal{F}$ on X.

0.2. If the set X is countable, then $\mathcal{U}(\mathcal{F})$ is a uniform atom if and only if $\mathcal{F}$ is selective.

0.3. (Ultrasum of atoms). Let X be an infinite set, let $\mathcal{F}$ be a non-principal ultrafilter on X. For each $x \in X$ let $\mathcal{U}_x$ be a uniform atom on a set Y_x with $|Y_x| \geq 2$.

Define a uniformity $\mathcal{F}\text{-}\sum \mathcal{U}_x$ on the set $\sum \{Y_x : x \in X\}$ ($= \{\langle x,y \rangle : x \in X,\ y \in Y_x\}$) by the following: A cover $\mathcal{C}$ belongs to $\mathcal{F}\text{-}\sum \mathcal{U}_x$ iff the set of all $x \in X$ such that the trace of $\mathcal{C}$ on Y_x belongs to $\mathcal{U}_x$ is a member of $\mathcal{F}$.

Then $\mathcal{F}\text{-}\sum \mathcal{U}_x$ is a uniform atom on $\sum \{Y_x : x \in X\}$. Moreover, any proximally non-discrete atom is uniformly isomorphic to some $\mathcal{F}\text{-}\sum \mathcal{U}_x$ with $\mathcal{F}$ non-principal and $|Y_x| = 2$ for each $x \in X$.

0.4. Every uniform atom on ω has a basis consisting of point-finite covers.

1. For the rest of the paper, let us restrict our attention to the case $X = \omega$. Theorems 0.2. and 0.3. give some examples of uniform atoms; according to 0.1., if we want to obtain an essentially new example, we need to search it below some $\mathcal{U}(\mathcal{F})$ with $\mathcal{F}$ non-selective and of minimal type in Rudin-Frolík's order. The main tool will be Lemma 1.3.

1.1. Definition. Let $\mathcal{R} = \{R_n : n < \omega\}$ be a partition of ω. We shall call $\mathcal{R}$ to be an <u>admissible partition</u> if each R_n is finite and $\sup \{ |R_n| : n<\omega\} = \omega$.

<u>1.2. Definition</u>. Let $\{\mathbb{P}(k) : k<\omega\}$ be a family of properties of finite subsets of ω. Call a family $\{\mathbb{P}(k) : k<\omega\}$ to be <u>nice</u>, if the following holds:

(i) There exists a $k<\omega$ such that the empty set has not $\mathbb{P}(k)$,

(ii) if $M \in \mathcal{P}_{fin}(\omega)$ satisfies $\mathbb{P}(k)$, then M satisfies $\mathbb{P}(k-1)$.

(iii) for every $k<\omega$, if $M, Q \in \mathcal{P}_{fin}(\omega)$ and $M \subset Q$, then Q has $\mathbb{P}(k)$ whenever M has $\mathbb{P}(k)$,

(iv) there exists a mapping $f:\omega \to \omega$ such that for each $k<\omega$ the following is true: If $M_1 \cup M_2 \in \mathcal{P}_{fin}(\omega)$ has $\mathbb{P}(f(k))$, then either M_1 or M_2 has $\mathbb{P}(k)$.

<u>1.3. Lemma</u>. Let $\mathcal{R}$ be an admissible partition of ω, $\mathcal{R} = \{R_n : n<\omega\}$, let $\{\mathbb{P}(k) : k<\omega\}$ be a nice family of properties of finite subsets of ω. Let $\mathbb{P}$ be the property of subsets of ω defined by

(v) $M \subset \omega$ has $\mathbb{P}$ iff for every $k<\omega$ and for every $n_0<\omega$ there is some $n>n_0$ such that $M \cap R_n$ has $\mathbb{P}(k)$.

Then the following holds:

A. If $\mathcal{F}$ is a filter on ω with a countable basis, if every $F \in \mathcal{F}$ has $\mathbb{P}$ and if M is a subset of ω, then there exists a filter $\mathcal{G}$ with a countable basis such that $\mathcal{G} \supset \mathcal{F}$, each member of $\mathcal{G}$ has $\mathbb{P}$ and either $M \in \mathcal{G}$ or $(\omega - M) \in \mathcal{G}$.

B. If $\mathcal{F}$ is a filter on ω with a countable basis and if every $F \in \mathcal{F}$ has $\mathbb{P}$, then there exists a subset $M \subset \omega$ such that $|M - F| < \omega$ and $M \cap F$ has $\mathbb{P}$ for each $F \in \mathcal{F}$.

C. Let $\{\mathfrak{S}_\alpha : \alpha < 2^\omega\}$ be a family of properties of subsets of ω. Suppose that for every filter $\mathcal{F}$ with a countable basis whose members satisfy $\mathbb{P}$ there exists a subset $M_\alpha \subset \omega$ such that M_α has $\mathfrak{S}_\alpha$ and $M_\alpha \cap F$ has $\mathbb{P}$ for every $F \in \mathcal{F}$.

Then, assuming (CH), there exists a P-ultrafilter q such that every $U \in q$ has $\mathbb{P}$ and for every $\alpha < 2^\omega$ there is a set $U_\alpha \in q$ satisfying $\mathfrak{S}_\alpha$.

The proof of C. goes by transfinite induction using A. on the successor steps and B. on the limit ones. The statements A. and B. are simple consequences of the definitions.

<u>1.4. Theorem</u>. Assume (CH), let L be a natural number. Then there exists a P-ultrafilter q on ω such that there are precisely L distinct uniform atoms below $\mathcal{U}(q)$.

We shall give a brief sketch of the proof. The special case $L = 0$ is stated in 0.2. In order to prove the theorem for $L > 0$ we want to use Lemma 1.3.C, thus we need to define an admissible partition $\mathcal{R}$, a nice family of properties $\{\mathbb{P}(k)\}$ and a family of properties $\{\mathbb{S}_\alpha : \alpha < 2^\omega\}$, and to verify all the assumptions.

Call a set $Q = A_1 \times A_2 \times \dots \times A_L$ to be an (L-dimensional) n-cube, if $|A_1| = |A_2| = |A_3| = \dots = |A_L| = n$. We may embed ω into ω^L onto a union $\bigcup \{Q_n : n < \omega\}$, where each Q_n is an n-cube and the family $\mathcal{R} = \{Q_n : n < \omega\}$ is pairwise disjoint. $\mathcal{R}$ is obviously an admissible partition of ω.

Let $M \in \mathcal{P}_{fin}(\omega^L)$, let $k < \omega$. Define: The set M has $\mathbb{P}(k)$ iff M contains some k-cube. In order to verify that $\{\mathbb{P}(k) : k < \omega\}$ is nice, we need the following combinatorial statement due to Erdös ([ES]):

(*) For each $L, n < \omega$ there exists an $N < \omega$ such that if $A_1 \times A_2 \times \dots \times A_L = C \cup D$, where $|A_1| = |A_2| = \dots = |A_L| = N$, then there are $B_i \subset A_i$, $|B_i| = n$ $(i = 1, 2, \dots, L)$ with $B_1 \times B_2 \times \dots \times B_L$ contained either in C or in D.

(*) implies 1.2.(iv), the other items from 1.2. are easy.

If Q is an L-dimensional cube, $Q = A_1 \times A_2 \times \dots \times A_L$, define $\mathcal{T}_i(Q) = \{ pr_i^{-1}(x) : x \in A_i \}$, where pr_i denotes the i´th projection. Next, define a partition $\mathcal{T}_i$ of ω by the rule $\mathcal{T}_i = \bigcup \{ \mathcal{T}_i(Q_n) : Q_n \in \mathcal{R} \}$.

Let $\mathcal{C}$ be a cover of ω. The set M is called $\mathcal{C}$-discrete iff for each $x \in M$, $st(x, \mathcal{C}) \cap M = \{x\}$. Denote by $st^2(x, \mathcal{C})$ the set $st(st(x, \mathcal{C}), \mathcal{C})$.

Let $\mathcal{C}$ be a cover of ω. Then, by definition, the set $M \subset \omega$ has a property $\mathbb{S}_{\mathcal{C}}$ iff either M is $\mathcal{C}$-discrete or there exists an $i \in \{1, 2, \dots, L\}$ and a sequence $\{x_T : T \in \mathcal{T}_i\}$, where $x_T \in T$, such that $st^2(x_T, T) \supset M \cap T$ for each $T \in \mathcal{T}_i$

The family $\{\mathbb{S}_{\mathcal{C}} : \mathcal{C}$ is a point-finite cover of $\omega\}$ is a family of cardinality at most 2^ω. This family satisfies the assumptions of 1.3.C. To see this, we need another combinatorial statement:

(**) For each $L, n < \omega$ there exists an $N < \omega$ such that the following is true:
If $Q = A_1 \times A_2 \times \dots \times A_L$ is an N-cube and if $\mathcal{C}$ is a cover of Q, then there exist $B_i \subset A_i$, $|B_i| = n$ $(i = 1, 2, \dots, L)$ such that the cube $Q' = B_1 \times B_2 \times \dots \times B_L$ is either $\mathcal{C}$-discrete or contained in $\bigcup \{st^2(x_T, \mathcal{C}) \cap T : T \in \mathcal{T}_j(Q),\ T \cap Q' \neq \emptyset\}$ for some $j \in \{1, 2, \dots, L\}$ and some suitable choice $x_T \in T$.

Using Lemma 1.3.C, we obtain some P-ultrafilter q, whose members contain arbitrarily large cubes. Therefore each uniformity $\mathcal{A}_i$ with a basis $\{\mathcal{T}_i \wedge \mathcal{V} : \mathcal{V} \in \mathcal{U}(q)\}$ (where $\mathcal{T}_i \wedge \mathcal{V} = \{T \cap V : T \in \mathcal{T}_i, V \in \mathcal{V}\}$) is not uniformly discrete and it remains to check that each $\mathcal{A}_i$ (i = 1,2,..., L) is an atom and that no other atom below $\mathcal{U}(q)$ exists. But this is a consequence of the fact that for each point-finite cover $\mathcal{C}$ of ω there is a member U of q satisfying $\mathfrak{S}_{\mathcal{C}}$.

One may ask whether each proximally discrete atom on ω can be obtained simply by adding a suitable partition to $\mathcal{U}(q)$, as it was the case in the theorem. This is not true in general, not even in the case of 0-dimensional atoms (i.e. atoms with a basis consisting of partitions), as may be seen from the following two theorems. The proofs are omitted, since both of them are the corollaries of Lemma 1.3. and the technique is again combinatorial and similar to the proof of 1.4.

<u>1.5. Theorem</u>. Assume (CH). Then there exists an ultrafilter q on ω such that there are 2^{2^ω} distinct 0-dimensional uniform atoms on ω below $\mathcal{U}(q)$.

<u>1.6. Theorem</u>. Assume (CH). Then there exists a P-ultrafilter q on ω and a 0-dimensional uniformity $\mathcal{W}$ on ω such that:

(a) $\mathcal{W}$ has a countable basis $\mathcal{B}$,

(b) for an arbitrary partition $\mathcal{T}$ of ω, a uniformity with a basis $\{\mathcal{T} \wedge \mathcal{V} : \mathcal{V} \in \mathcal{U}(q)\}$ is never a uniform atom on ω,

(c) there exists precisely one uniform atom below $\mathcal{U}(q)$ and its basis equals to $\{\mathcal{C} \wedge \mathcal{V} : \mathcal{C} \in \mathcal{B}, \mathcal{V} \in \mathcal{U}(q)\}$.

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