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DESCRIPTIVE SETS IN UNIFORM SPACES

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This paper summarizes recent work of the author and others on selected topics in the theory of descriptive sets. In some sense it is a natural continuation of the line of thought introduced by Z. Frolík at the Third Prague Symposium ([Fr]$_1$). The present work attempts to show that uniform spaces are a natural setting for the study of descriptive phenomena. First, many classical separation and reduction theorema may be immediately generalized to the class of all uniform spaces (§2). Second, topological and measurable phenomena may be simultaneously considered (see application after 2.2 and §4). Third, the concept of a semi-compact paving has a natural formulation in uniform spaces (2.3). Fourth, uniform spaces are the natural setting for the study of uniform discreteness, an idea that has provided the impetus for recent work in non-separable descriptive theory (§3). Finally, uniform ideas allow the natural extension of classical results to arbitrary products of metric spaces (see [T]).

§1 Notation

If X is a uniform space, Z(X) (resp. coz (X)) denotes the family of zero-sets (resp. cozero sets) of the members from U(X) , the family of real-valued uniformly continuous functions, and Ba(X) denotes the σ-field of Baire sets generated by Z(X). The hyper-Baire sets (hyperBaire (X)) is the smallest σ-field containing Z(X) that is closed under the formation of uniformly discrete unions.

Let $S \subset P(X)$. $\mathcal{A}(S)$ denotes the family of sets Souslin derived from S , i.e. the sets of the form $\bigcup_t \bigcap_{n=1}^{\infty} F(t_1,t_2,\ldots,t_n)$, where $t = (t_1,t_2,\ldots)$ is a sequence of natural numbers and each $F(t_1,\ldots t_n) \in S$. Co-$\mathcal{A}(S)$ is the family of complements of members from $\mathcal{A}(S)$ and bi-$\mathcal{A}(S)$ = $\mathcal{A}(S) \cap$ Co-$\mathcal{A}(S)$. If S is finitely multiplicative and $\{\emptyset,X\} \subset S$, then $\mathcal{A}(\mathcal{A}(S)) = \mathcal{A}(S)$ and $\mathcal{A}(S)$ is countably multiplicative and countably additive, hence bi-$\mathcal{A}(S)$ is a

σ-field.

If X is a uniform space, Souslin (X) = $\mathcal{A}(Z(X))$ is the family of Souslin sets, Co-Souslin (X) = Co-$\mathcal{A}(Z(X))$, and bi-Souslin (X) = bi-$\mathcal{A}(Z(X))$. If X is a Tychonoff space, the above definitions apply, where X is viewed as a fine uniform space. In addition, if S is the family of compact G_δ-sets, let $\mathcal{S}(X)$ be the smallest family containing S that is finitely additive, countably multiplicative, and closed under differences, and define $\mathcal{S}(X)^{loc} = \{A \subset X \mid A \cap B \in \mathcal{S}(X)$ for each $B \in \mathcal{S}(X)\}$.

§2 Separation and Reduction Theorems

Theorem 2.1: (i) Let $\{A_n\}_{n=1}^{\infty} \subset$ Souslin (X). There exists a uniformly continuous mapping $X \xrightarrow{f} M$ onto a separable metric space M and a family $\{A_n'\}_{n=1}^{\infty}$ from Souslin (M) such that $f^{-1}(A_n') = A_n$, n=1,2,...

(ii) Let B be a Baire set in X of additive (resp. multiplicative) class α. There exists a uniformly continuous mapping $X \xrightarrow{f} M$ onto a separable metric space M and a Baire set B' in M of additive (resp. multiplicative) class α such that $f^{-1}(B') = B$.

Comments

1. 2.1 (ii) is essentially found in ([Fr]$_5$, Lemma 2).
2. One easily sees that 2.1 (i) is valid if Souslin is replaced by bi-Souslin (if $\{B_n\} \subset$ bi-Souslin (X), apply 2.1 (i) to the family $\{B_n, X-B_n\}$).
3. 2.1 is not valid for hyperBaire sets regardless of the metric space M chosen (see the example following 3.4, noting that each hyperBaire set in a metric space is a Souslin set).
4. 2.1 may be applied to obtain the following well known results. Each compact Souslin set is a zero set. Each Souslin set in a compact space is Lindelöf. Each Souslin set in a realcompact space is realcompact. More generally, one has the following result. Let P be a topological property inherited by zero sets and assume $X \times M \in P$ for each $X \in P$ and separable metric space M. Then Souslin $(X) \subset P$ for each $X \in P$. (If $A \in$ Souslin (X), choose a continuous mapping $X \xrightarrow{f} M$ as in 2.1 and $A' \subset M$ such that

$f^{-1}(A') = A$. Then $G(f) = \{(x,f(x)): x \in X\}$ is a zero set in $X \times M$, so $G(f_{/A}) = G(f) \cap (X \times A')$ belongs to P . Finally, A is homeomorphic to $G(f_{/A})$, so $A \in P$.)

5. 2.1 establishes the following general result: each classical separation or reduction theorem which is valid for all separable metric spaces is valid for all uniform spaces. For example (see [K]), one obtains the following results.

<u>(Lusin Second Separation Theorem)</u>: Let $\{A_n\}_{n=1}^{\infty} \subset$ Souslin (X). There exists a disjoint family $\{C_n\}_{n=1}^{\infty} \subset$ Co-Souslin (X) such that $A_n - (\bigcup_{i \neq n} A_i) \subset C_n$, $n=1,2,\ldots$.

<u>(Kuratowski Reduction Theorem)</u>: Let $\{U_n\}_{n=1}^{\infty} \subset$ Co-Souslin (X). There exists a disjoint family $\{V_n\}_{n=1}^{\infty} \subset$ Co-Souslin (X) such that $V_n \subset U_n$, $n=1,2,\ldots$, and $\bigcup_{n=1}^{\infty} U_n = \bigcup_{n=1}^{\infty} V_n$.

<u>(Kuratowski Second General Separation Theorem)</u>: Let $\{A_n\}_{n=1}^{\infty} \subset$ $\subset$ Souslin (X). There exists a family $\{B_n\}_{n=1}^{\infty} \subset$ Co-Souslin (X) such that $A_n - (\bigcap_{m=1}^{\infty} A_m) \subset B_n$, $n=1,2,\ldots$, and $\bigcap_{m=1}^{\infty} B_m = \emptyset$.

It should be noted, however, that Lusin's First Separation Theorem cannot be extended to all separable metric spaces (see comments following 2.4 (ii)).

<u>Corollary 2.2</u>: Let X be a uniform space. Each pair of disjoint Souslin sets may be separated by a bi-Souslin set.

Proof: Apply the Kuratowski Reduction Theorem.

<u>Application</u>: Let Σ be a σ-field on the set X . Applying 2.2 to the uniformity generated by the countable Σ-partitions of X , one obtains that disjoint members of $\mathcal{A}(\Sigma)$ may be separated by a member of bi-$\mathcal{A}(\Sigma)$.

Frequently the statement of an abstract separation theorem uses the concept of a semi-compact paving ($C \subset P(X)$ is semi-compact if $\bigcap C' \neq \emptyset$ for each countable sub-family $C' \subset C$ with the finite intersection property). The uniform analogue of this idea is a precompact metric-fine space (for by [Ha]$_2$, X is precompact metric-fine if and only if Z(X) is a semi-compact paving); hence one immediately obtains the following result (from [C], 2.2).

Theorem 2.3: Each pair of disjoint Souslin sets in a precompact metric-fine space may be separated by a Baire set.

Comments

1. 2.3 may be directly established using the fact that each uniformly continuous metric image of a precompact metric-fine space is compact ([Ha]$_2$), 2.1, and Lusin's First Separation Theorem.
2. The fine uniformity on a pseudcompact space is precompact and metric-fine; more generally, the precompact metric-fine spaces are exactly the G_δ-dense subspaces of compact spaces ([Ha]$_2$).

Finally, we note the following two equalities between classes of descriptive sets.

Theorem 2.4: Let X be a complete metric space.
(i) ([Hl]$_1$) bi-Souslin (X) = hyperBaire (X).
(ii) ([R]$_2$) bi-Souslin (X) = Baire (X) if and only if X is the union of a separable and σ-discrete subspace.

2.4 (ii) shows that Lusin's First Separation Theorem (which is the separable case of 2.4 (i) in view of 2.2) must fail for some separable metric space. (This is undoubtedly a folklore result, but I have no reference.) Let B be a bi-Souslin, non-Baire set in any space X. By 2.1 (i), there exists a uniformly continuous mapping $X \xrightarrow{f} M$ onto a separable metric space M and a bi-Souslin set B′ in M such that $f^{-1}(B') = B$. Then B′ is not a Baire set. This example also shows that B′ is not the restriction to M of a bi-Souslin set in the completion of M.

§3 Uniformly Discrete Families

Theorem 3.1: Let X be a uniform space and let $P \supset \mathrm{coz}(X)$ be a finitely multiplicative family. If P is closed under the formation of uniformly discrete unions, then $\mathcal{A}(P)$ and bi-$\mathcal{A}(P)$ are closed under the formation of uniformly discrete unions.

Corollary 3.2: Let X be a locally fine uniform space. Then Souslin (X) and bi-Souslin (X) are closed under the formation of uniformly discrete unions.

Proof: X locally fine implies coz (X) is closed under the formation of uniformly discrete unions, so one may apply 3.1.

Corollary 3.3: Let X be a paracompact topological space. Then Souslin (X) and bi-Souslin (X) are closed under the formation of discrete unions; alternately, each locally Souslin (resp. locally bi-Souslin) set is a Souslin (resp. bi-Souslin) set.

Proof: Apply 3.2 to the fine uniformity on X and note that uniformly discrete and discrete coincide in a fine paracompact setting.

Theorem 3.4: Let X be a uniformly locally compact uniform space. Then hyperBaire (X) = $\delta(X)^{loc}$ = σ-u.d.Baire (X) and bi-Souslin (X) $\subset$ hyperBaire (X). (Here σ-u.d. denotes σ-uniformly discrete unions.)

Example: There exists a uniformly locally compact space X with bi-Souslin (X) $\neq$ hyperBaire (X). Let $X = \{0,1\}^{\Omega} \times [0,\Omega)$, where $[0,\Omega)$ has the discrete uniformity. For each $\alpha < \Omega$, define $V_\alpha = \{(p,\beta) \mid p_\beta = 0\}$. Then $\{V_\alpha\}$ is a uniformly discrete cozero family, but $V = \bigcup V_\alpha$ is not a Souslin set since it does not depend on countably many co-ordinates (see [T]).

Corollary 3.5: Let X be a locally compact paracompact space. Then bi-Souslin (X) = hyperBaire (X) = $\delta(X)^{loc}$ = σ-u.d. Baire (X).

Proof: Apply 3.3 and 3.4 to the fine uniformity on X .

§4 Measurable Functions and Completely Additive Families

Theorem 4.1: Let X be a precompact metric-fine space. Then each disjoint completely additive Baire family is countable;

hence each Baire measurable metric-image of X is a separable absolute Souslin set.

4.1 is essentially contained in 1.3 and 2.3 of [C], which together establish that every separable-metric valued Baire measurable image of X is a Souslin set. (For if $\{A_\alpha\}_{\alpha<\Omega}$ is an uncountable completely additive disjoint Baire family that covers X, let $A = \{a_\alpha\}_{\alpha<\Omega}$ be a non-Souslin subset of the real line $\mathbb{R}$. Then the mapping $X \longrightarrow \mathbb{R}$ defined by $A \longrightarrow a_\alpha$ is Baire measurable, which is impossible.) We remark that the same proof technique was earlier discovered and used in $[Fr]_3$.

<u>Corollary 4.2</u>: (i) Each disjoint completely additive Baire family of a pseudocompact space is countable.

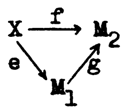
(ii) ($[Hl]_1$) Each disjoint completely additive Souslin family of a complete separable metric space is countable.

Proof: (i) Apply 4.1 to the fine uniformity on the space. (ii) (Baire case) follows from (i) since each complete separable metric space is the Baire measurable image of a compact metric space (see [Me]).

<u>Comment</u>: 4.2 (i) was first established for compact spaces in $[Fr]_3$ and 4.2 (ii) is also an immediate consequence of the results in $[Fr]_3$.

<u>Theorem 4.3</u>: ($[Fr]_2$, [P]) Let X be a complete metric space and let $X \xrightarrow{f} M$ be a metric-valued Baire measurable mapping. Then f is a Baire mapping of class α, for some $\alpha < \Omega$.

<u>Theorem 4.4</u>: Let X be a uniform space such that $m_* X$ (the measurable coreflection of X - see $[R]_3$) is proximally fine. If $X \xrightarrow{f} M$ is a Baire measurable mapping to a metric space M, there exists a complete metric space M_2 containing M (uniformly) and a complete metric space M_1 such that f may be factored as $g \circ e$, where g is Baire measurable and e is



uniformly continuous.

Corollary 4.5: Let X be a uniform space such that m_*X is proximally fine and let $X \xrightarrow{f} M$ be a metric-valued Baire measurable mapping.

(i) f is a Baire mapping of class α (for some $\alpha < \Omega$).

(ii) $G(f) = \{(x,f(x)) : x \in X\}$ is a Baire set of class α (for some $\alpha < \Omega$). Furthermore, if $X \xrightarrow{g} N$ is also a metric-valued Baire measurable mapping, then

(iii) $X \xrightarrow{f \times g} M \times N$ is Baire measurable.

Proof: Apply 4.3, 4.4, and the fact [K] that the graph of each metric-valued Baire measurable mapping of class α on a complete metric space is a Baire set of class α.

Comments

1. 4.3 is derived from the deeper result that m_*X is proximally fine for each complete metric space X (see $[Fr]_2$ and $[R]_{1,2}$ for details).
2. 4.4 and 4.5 (ii), (iii) have analogous statements, where m_* is replaced by the bi-analytic operator a_* defined in $[R]_1$ and Baire is replaced by bi-Souslin.
3. 4.5 (iii) is equivalent to the following property of Baire (X): if $\{A_s\}$ and $\{B_t\}$ are disjoint completely additive Baire (X) families, then $\{A_s \cap B_t\}$ is a completely additive Baire family (following $[Fr]_1$, a σ-field with this property is called proximally fine). It is an unsolved problem whether the Baire σ-field of each metric space is proximally fine, or whether the Lebesgue σ-field of the real line is proximally fine (without special set-theoretic assumptions).

Theorem 4.6: Let Σ be a σ-field on the set X.

(i) If $|X| \leq \aleph_1$, then Σ is proximally fine.
(MA) If $|X| \leq c$, then Σ is proximally fine.

(ii) (MA) If X is a uniform space and $|X| \leq c$, then the graph of each metric-valued Baire (resp. bi-Souslin) measurable mapping on X is a Baire (resp. bi-Souslin) set.

(iii) If M is a locally complete metric space, then bi-Souslin (M) is proximally fine. If M is also locally separable, then Baire (M) is proximally fine.

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