SEMIGROUPS AND NEAR-RINGS OF CONTINUOUS FUNCTIONS

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1. Introduction

Let X and G be topological spaces and let $\mathscr{S}(X, G)$ denote the family of all continuous functions from X into G. It has long been recognized that if G has an algebraic structure with which the topological structure is compatible, then one can provide $\mathscr{S}(X, G)$ with an algebraic structure be defining pointwise operations. However, even in the absence of any algebraic structure on G one can, in a natural way, provide $\mathscr{S}(X, G)$ with an algebraic structure. In fact, each continuous function α from G into X induces an associative binary operation on $\mathscr{S}(X, G)$. Specifically, one can define the product fg of any two functions f and g of $\mathscr{S}(X, G)$ by $fg = f \circ \alpha \circ g$, that is, fg is just the composition of the functions f, α and g. We will denote the resulting semigroup by $\mathscr{S}(X, G, \alpha)$. Such semigroups were introduced and first investigated in [2] and [3]. However, it was assumed in the latter papers that α mapped G onto X. We will not generally make that assumption here.

In Section 2 of this paper, a result is proved which gives the form of an isomorphism between two semigroups $\mathscr{S}(X, G, \alpha)$ and $\mathscr{S}(Y, H, \beta)$. In Section 3, we take G to be an additive topological group. This allows us to define point-wise addition on the continuous functions from X into G and the result, with multiplication defined as before, is a near-ring which we denote by $\mathscr{N}(X, G, \alpha)$. If $G = X$ and α is the identity map, then $\mathscr{N}(X, G, \alpha)$ becomes the near-ring of all continuous selfmaps of G under point-wise addition and ordinary composition. In this case, we use the simpler notation $\mathscr{N}(G)$. The isomorphism theorem for semigroups has an analogue for near-rings which is given in Section 3 and this result is then applied to get further results in the case when G is the additive topological group of one of the N-dimensional real number spaces.

2. Semigroups of continuous functions

The following result has not appeared before although most of the basic techniques needed to prove it were actually developed in [2] and [3]. We will make use of various results in those papers. In the statement of the theorem, $\mathscr{R}(\alpha)$ and $\mathscr{R}(\beta)$ denote the ranges of the functions α and β respectively.

Theorem 2.1. *Let α and β be nonconstant continuous functions from G and H into completely regular Hausdorff spaces X and Y respectively. Suppose that each of the subspaces $\mathscr{R}(\alpha)$ and $\mathscr{R}(\beta)$ contains a compact subspace with nonempty interior and suppose also that both G and H are connected, locally arcwise connected metric spaces. Then for each isomorphism φ from $\mathscr{S}(X, G, \alpha)$ onto $\mathscr{S}(Y, H, \beta)$ there exists a unique homeomorphism h from $\mathscr{R}(\alpha)$ onto $\mathscr{R}(\beta)$ and a unique homeomorphism t from G onto H such that the following diagram commutes for each $f \in \mathscr{S}(X, G, \alpha)$.*

$$\begin{array}{ccccc}
\mathscr{R}(\alpha) & \xrightarrow{\;f\;} & G & \xrightarrow{\;\alpha\;} & \mathscr{R}(\alpha) \\
\Big\downarrow h & & \Big\downarrow t & & \Big\downarrow h \\
\mathscr{R}(\beta) & \xrightarrow{\;\varphi(f)\;} & H & \xrightarrow{\;\beta\;} & \mathscr{R}(\beta)
\end{array}$$

Proof. The existence and uniqueness of the bijections h and t and the fact that the diagram commutes all follow immediately from Theorem (2.3) of [2, p. 83]. We must show that h and t are, in fact, homeomorphisms and we consider h first. For each $p \in X$ and $f \in \mathscr{S}(X, G, \alpha)$, let

$$A(p, f) = \{x \in X : \alpha(f(x)) = p\}.$$

Similarly, for $q \in Y$ and $g \in \mathscr{S}(Y, H, \beta)$, let

$$B(q, g) = \{y \in Y : \beta(g(y)) = q\}.$$

Using the fact that the diagram commutes, one shows with some minor calculations that

$$h[\mathscr{R}(\alpha) \cap A(p, f)] = \mathscr{R}(\beta) \cap B(h(p), \varphi(f))$$

and also that

$$h^{-1}[\mathscr{R}(\beta) \cap B(q, g)] = \mathscr{R}(\alpha) \cap A(h^{-1}(q), \varphi^{-1}(g)).$$

Therefore, in order to conclude that h is a homeomorphism, it is sufficient to show that

$$\{A(p, f) : p \in X \text{ and } f \in \mathscr{S}(X, G, \alpha)\}$$

is a basis for the closed subsets of X. Since α is nonconstant, we may choose two distinct points $a, b \in \mathscr{R}(\alpha)$ and then choose any two points $v, w \in G$ such that $\alpha(v) = a$ and $\alpha(w) = b$. Since G is both connected and locally arcwise connected, it must also be arcwise connected so we let k be any homeomorphism from the closed unit interval I into G such that $k(0) = v$ and $k(1) = w$. Now let W be any closed subset of X. Since X is completely regular and Hausdorff, there exists, for each $z \in X - W$, a continuous function f_z from X into I such that $f_z(z) = 0$ and $f_z(x) = 1$ for $x \in W$. Now let $k_z = k \circ f_z$. Then $k_z \in \mathscr{S}(X, G, \alpha)$ and one readily shows that

$$W = \bigcap\{A(b, k_z) : z \in X - W\}.$$

It follows from all this that h is a homeomorphism.

Now we show that t is a homeomorphism. Since both G and H are k-spaces, it will be sufficient to show that $t(K)$ is compact for each compact subset K of G and that $t^{-1}(K)$ is compact for each compact subset K of H. In fact, it will be sufficient to show the former since the latter follows in the same manner. So let K be a compact subset of G. We will verify the existence of a continuous function k from X into G such that

$$K \subset k(\mathscr{R}(\alpha)) \tag{2.1.1}$$

and

$$\mathscr{R}(\alpha) \cap k^{-1}(K) \text{ is compact}. \tag{2.1.2}$$

We will first dispose of the case where $K = G$. Then G is a Peano continuum and since α is nonconstant, $\mathscr{R}(\alpha)$ contains two distinct points a and b. Let f be any continuous function from X into the closed unit interval I such that $f(a) = 0$ and $f(b) = 1$. Then let g be any continuous mapping from I *onto* G and let $k = g \circ f$. Since $\mathscr{R}(\alpha)$ is connected, it follows that (2.1.1) is satisfied and (2.1.2) is satisfied since $\mathscr{R}(\alpha)$ is compact.

Now we consider the case where $K \neq G$ and we choose $a \in G - K$. By Theorem 5 of [1, p. 253], there exists a Peano continuum K^* such that

$$K \cup \{a\} \subset K^* \subset G.$$

By hypothesis, there is a point $b \in \mathscr{R}(\alpha)$, an open subset A of $\mathscr{R}(\alpha)$ and a compact subset W such that

$$b \in A \subset W \subset \mathscr{R}(\alpha).$$

Since $\mathscr{R}(\alpha)$ is a connected space with more than one point, it follows that there exists a point $c \in A - \{b\}$. Let $B = A - \{c\}$ and let B^* be an open subset of X such that $B = B^* \cap \mathscr{R}(\alpha)$. Now let f be any continuous function from X into I such that $f(b) = 0$ and $f(x) = 1$ for $x \in X - B^*$. Since K^* is a Peano continuum, there exists a continuous function g from I *onto* K^* such that $g(1) = a$. Then $k = g \circ f$ belongs to $\mathscr{S}(X, G, \alpha)$ and since $f(b) = 0$ and $f(c) = 1$ and $\mathscr{R}(\alpha)$ is connected, it readily follows that (2.1.1) holds. Furthermore one can verify that $\mathscr{R}(\alpha) \cap k^{-1}(K) \subset W$ which implies that (2.1.2) also holds.

Now we are in a position to show that $t(K)$ is compact. Because h is a homeomorphism, if follows from (2.1.2) that $h(\mathscr{R}(\alpha) \cap k^{-1}(K))$ is compact. Consequently,

$\varphi(k)\,[h(\mathscr{R}(\alpha) \cap k^{-1}(K))]$ is compact, but it follows from (2.1.1) (and the fact that the diagram commutes) that this latter set is just $t(K)$. Since t^{-1} behaves in a similar manner, it follows that t is a homeomorphism.

3. Near-rings of continuous functions

The near-ring analogue of Theorem 2.1 follows very quickly. The only additional thing one must do is show that t is, in this case, also a group isomorphism. For $a \in G$, let $\langle a \rangle$ denote the constant function which maps all of X into the point a. Then $\varphi\langle a \rangle = \langle t(a) \rangle$ for all $a \in G$ and for any $a, b \in G$ we have

$$\begin{aligned} \langle t(a + b) \rangle &= \varphi\langle a + b \rangle = \varphi(\langle a \rangle + \langle b \rangle) = \\ &= \varphi\langle a \rangle + \varphi\langle b \rangle = \langle t(a) \rangle + \langle t(b) \rangle = \langle t(a) + t(b) \rangle \end{aligned}$$

which implies that $t(a + b) = t(a) + t(b)$. Thus, t is a group isomorphism and we have the following

Theorem 3.1. *Let G and H be connected, locally arcwise connected metrizable topological groups and let X and Y be completely regular Hausdorff spaces. Let α and β be nonconstant continuous functions from G into X and H into Y, respectively, such that both $\mathscr{R}(\alpha)$ and $\mathscr{R}(\beta)$ contain compact subspaces with nonempty interiors. Then for each isomorphism φ from the near-ring $\mathscr{N}(X, G, \alpha)$ onto the near-ring $\mathscr{N}(Y, H, \beta)$, there exists a unique homeomorphism h from $\mathscr{R}(\alpha)$ onto $\mathscr{R}(\beta)$ and a unique topological isomorphism t from the group G onto the group H such that the following diagram commutes for each $f \in \mathscr{N}(X, G, \alpha)$.*

$$\begin{array}{ccccc} \mathscr{R}(\alpha) & \xrightarrow{\ f\ } & G & \xrightarrow{\ \alpha\ } & \mathscr{R}(\alpha) \\ {\scriptstyle h}\downarrow & & {\scriptstyle t}\downarrow & & {\scriptstyle h}\downarrow \\ \mathscr{R}(\beta) & \xrightarrow{\ \varphi(f)\ } & H & \xrightarrow{\ \beta\ } & \mathscr{R}(\beta) \end{array}$$

Now let R^N denote the additive topological group of the N-dimensional real number space. We use the latter theorem to get information about the automorphisms of the near-rings $\mathscr{N}(X, R^N, \alpha)$. We will find, among other things, that the existence of a certain type of automorphism on $\mathscr{N}(X, R^N, \alpha)$ has a considerable effect on the behavior of the function α.

Theorem 3.2. *Let X be a completely regular Hausdorff space and let α be a quotient map from R^N into X which is injective on some neighborhood of zero. Suppose also that $\mathscr{R}(\alpha)$ contains a compact subspace with nonempty interior.*

Then for each automorphism φ of the near-ring $\mathcal{N}(X, R^N, \alpha)$ there exists a unique homeomorphism h from $\mathcal{R}(\alpha)$ onto $\mathcal{R}(\alpha)$ and a unique linear automorphism t of the vector space R^N such that the following diagram commutes for each $f \in \mathcal{N}(X, R^N, \alpha)$.

$$\begin{array}{ccccc} \mathcal{R}(\alpha) & \xrightarrow{\ f\ } & R^N & \xrightarrow{\ \alpha\ } & \mathcal{R}(\alpha) \\ {\scriptstyle h}\big\downarrow & & {\scriptstyle t}\big\downarrow & & {\scriptstyle h}\big\downarrow \\ \mathcal{R}(\alpha) & \xrightarrow{\ \varphi(f)\ } & R^N & \xrightarrow{\ \alpha\ } & \mathcal{R}(\alpha) \end{array}$$

Moreover, if $\max \{ \sum_{j=1}^{N} |a_{ij}| \}_{i=1}^{N} < 1$ *where (a_{ij}) is the matrix of t with respect to the canonical basis, then α is a homeomorphism. If, in addition to this, $\mathcal{R}(\alpha) = X$, then $\mathcal{N}(X, R^N, \alpha)$ is isomorphic to $\mathcal{N}(R^N)$ and its automorphism group is isomorphic to $GL(N, R)$, the full linear group of all real $N \times N$ nonsingular matrices.*

Proof. Let φ be an automorphism of $\mathcal{N}(X, R^N, \alpha)$. According to the previous theorem, there exists a unique homeomorphism h and a unique topological group isomorphism t such that the diagram above commutes. Since t is additive, it readily follows that $t(rx) = rt(x)$ for every rational number r and since t is continuous, it follows from this that $t(ax) = at(x)$ for every real number a. Thus, t is a linear automorphism of the vector space R^N.

Now let $M = \max \{ \sum_{j=1}^{N} |a_{ij}| \}_{i=1}^{N}$ and suppose that $M < 1$. We must show that α is a homeomorphism. In view of the fact that it is a quotient map, it is sufficient to show that it is injective so we assume that $\alpha(v) = \alpha(w)$ and we show that $v = w$. First, we take the norm of an element $x = (x_1, x_2, \ldots, x_N) \in R^N$ to be $\max \{ |x_i| \}_{i=1}^{N}$. Then, if $\|x\| \leqq 1$, it readily follows that

$$\|t(x)\| = \max \{ | \sum_{j=1}^{N} x_j a_{ij} | \}_{i=1}^{N} \leqq M .$$

Thus, $\|t\| < 1$ where $\|t\|$ denotes the norm of the operator t.

Next, let φ^n denote the composition of φ with itself n times. One readily shows that the unique homeomorphism associated with φ^n is h^n and that the unique linear automorphism associated with φ^n is t^n. Since the corresponding diagram commutes, it follows that

$$\alpha(t^n(v)) = h^n(\alpha(v)) = h^n(\alpha(w)) = \alpha(t^n(w)) .$$

However, $\|t^n(v)\| \leqq \|t\|^n \|v\|$ and $\|t^n(w)\| \leqq \|t\|^n \|w\|$ and since $\lim \|t\|^n = 0$, we can choose n so large that both $t^n(v)$ and $t^n(w)$ belong to the neighborhood on which α is injective. Consequently, for such an n, we have $t^n(v) = t^n(w)$ and since t^n is injective, it follows that $v = w$. Thus, α is a homeomorphism. If, in addition to this, $\mathcal{R}(\alpha) = X$, one easily verifies that the mapping which sends $f \in \mathcal{N}(X, R^N, \alpha)$ into $f \circ \alpha$ is an isomorphism from $\mathcal{N}(X, R^N, \alpha)$ onto $\mathcal{N}(R^N)$. To complete the proof of the theorem, we need only verify that the automorphism group of $\mathcal{N}(R^N)$ is isomorphic to $GL(N, R)$. As a matter of fact, it follows from our previous considerations that for each automorphism θ of $\mathcal{N}(R^N)$ there exists a unique linear automorphism s such that $\theta(f) = s \circ f \circ s^{-1}$ for each $f \in \mathcal{N}(R^N)$. One can easily verify that the mapping which sends θ into the matrix of s is an isomorphism from the automorphism group of $\mathcal{N}(R^N)$ onto the full linear group $GL(N, R)$.

References

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