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CONTINUOUS FUNCTIONS ON PRODUCTS WITH STRONG TOPOLOGIES

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This paper is organized as follows. First, the necessary definitions, notation and terminology are introduced; next, our result is stated and proved; and last, various special cases of the theorem are mentioned together with references to the literature where these appear. This indicates the extent to which we are indebted to other authors both for ideas and motivation.

Throughout, we assume that we are given a nonvoid family $\{X_i\}_{i\in I}$ of nonvoid (completely regular, Hausdorff) spaces. We write $X_J = \prod_{i\in J} X_i$ for each nonvoid subset J of I, so that in particular $X_I = \prod_{i\in I} X_i$. We set

$$\mathscr{P}^*(I) = \{J \subset I : J \neq \emptyset\},$$

and

$$\mathscr{P}_\varkappa(I) = \{J \in \mathscr{P}^*(I) : |J| < \varkappa\}$$

for each cardinal number $\varkappa$.

When $\varkappa$ is infinite and $J \in \mathscr{P}^*(I)$, the $\varkappa$-box topology on X_J is that topology which has as base all sets of the form $U = \prod_{i\in J} U_i$ with U_i open in X_i for each i in J and with $|R(U)| < \varkappa$ (where we have set $R(U) = \{i \in I : U_i \subsetneqq X_i\}$). Thus the ω-box topology on X_J is the usual product topology on X_J. The set X_J with the $\varkappa$-box topology is denoted $(X_J)_\varkappa$. When $\varkappa$ is regular the space $(X_J)_\varkappa$ has the property that each intersection of fewer than $\varkappa$ of its open sets is open if and only if each of the spaces X_i $(i \in J)$ has this property.

When $J \in \mathscr{P}^*(I)$ the map π_J is the projection from X_I onto X_J. It is easily verified that for each $\varkappa$ with $\omega \leqq \varkappa$, the map π_J when considered as a map from $(X_I)_\varkappa$ to $(X_J)_\varkappa$ is continuous. For x in X_I, we write x_J for $\pi_J(x)$.

For $\omega \leqq \varkappa \leqq \alpha$, a space is said to be pseudo-$(\alpha, \varkappa)$-compact if, given any α of its open sets, each neighborhood of one of its points meets at least $\varkappa$. A pseudo-(α, ω)-compact is said, simply, to be pseudo-α-compact; and the pseudo-ω-compact spaces – i.e., those spaces with no infinite, locally finite family of open sets – are the familiar pseudocompact spaces.

Finally, if $Y \subset X_I$ and f is a function with domain Y, then f is said to depend on fewer than α coordinates if there exists $J \in \mathscr{P}^*_\alpha(I)$ for which $f(x) = f(y)$ whenever $x \in Y$ and $y \in Y$ and $x_J = y_J$.

Theorem. *Assume the following*: (1) $\omega \leqq \varkappa < \alpha$, α *is regular, and* $\beta^\lambda < \alpha$ *whenever* $\beta < \alpha$ *and* $\lambda < \varkappa$; (2) $Y \subset (X_I)_\varkappa$ *and* $\pi_J(Y) = X_J$ *for each* J *in* $\mathscr{P}^*_\alpha(I)$; *and* (3) $(X_J)_\varkappa$ *is pseudo-*$(\alpha, \varkappa)$*-compact for each* J *in* $\mathscr{P}^*_\varkappa(I)$. *Then* (a) Y *is pseudo-*$(\alpha, \varkappa)$*-compact*; (b) *each continuous function from* Y *to a metric space depends on fewer than* α *coordinates*; *and* (c) *each such function extends continuously over* $(X_I)_\varkappa$.

Proof. To prove (a) we will show that if $\mathscr{U}$ is a family of open subsets of $(X_I)_\varkappa$ with $|\mathscr{U}| = \alpha$, then there is a point p of Y each of whose neighborhoods meets at least $\varkappa$ elements of $\mathscr{U}$. (This will suffice. For, Y being dense in $(X_I)_\varkappa$ by (2), any pair of intersecting open subsets of $(X_I)_\varkappa$ have a point in common in Y.) We may suppose that each element U of $\mathscr{U}$ has the form $U = \prod_{i\in I} U_i$ with each U_i open in X_i and with $|R(U)| < \varkappa$. According to a combinatorial theorem of Erdös and Rado [9], there exist a subset $\mathscr{V}$ of $\mathscr{U}$ with $|\mathscr{V}| = \alpha$ and a (possibly empty) subset J of I for which $R(U) \cap R(V) = J$ whenever U and V are distinct elements of $\mathscr{V}$. If $J = \emptyset$ then for each point p of Y it is true that each neighborhood of p meets each member of $\mathscr{V}$, with fewer than $\varkappa$ exceptions. (Indeed, given a $\varkappa$-box basic neighborhood $W = \prod_{i\in I} W_i$ of p, each i in $R(W)$ belongs to $R(U)$ for at most one U in $\mathscr{V}$. Thus $R(W) \cap R(U) = \emptyset$ for all U in $\mathscr{V}$ with fewer than $\varkappa$ exceptions, and $W \cap U \neq \emptyset$ for each such U.) If $J \neq \emptyset$, i.e. if $J \in \mathscr{P}^*_\varkappa(I)$, then from hypothesis (3) and the fact that α is regular it follows that there is a point x of $(X_J)_\varkappa$ with the property that if W_J is a neighborhood in $(X_J)_\varkappa$ of x then there exists $\mathscr{V} \subset \mathscr{U}$ with $|\mathscr{V}| = \varkappa$ for which $W_J \cap \pi_J(U) \neq \emptyset$ whenever $U \in \mathscr{V}$. Then for p we choose any point of Y for which $\pi_J(p) = x$. (The existence of such a point p requires not all of hypothesis (2) but the weaker condition that $\pi_J(Y) = X_J$ whenever $J \in \mathscr{P}^*_\varkappa(I)$; (2) can be weakened to this, if only conclusion (a) is wanted.) Given a $\varkappa$-box neighborhood $W = W_J \times W_{I\setminus J}$ of p, where W_J and $W_{I\setminus J}$ are $\varkappa$-box open in $(X_J)_\varkappa$ and $(X_{I\setminus J})_\varkappa$ respectively, let $\mathscr{V} \subset \mathscr{U}$ with $|\mathscr{V}| = \varkappa$ and with $W_J \cap \pi_J(U) \neq \emptyset$ for each U in $\mathscr{V}$. Since $|R(W_{I\setminus J})| < \varkappa$ and each element of $R(W_{I\setminus J})$ lies in $R(U)$ for at most one element U of $\mathscr{V}$, there are $\varkappa$ elements U of $\mathscr{V}$ for which $R(W_{I\setminus J}) \cap R(U) = \emptyset$. Given such U, let $q_i \in W_J \cap \pi_J(U)$ if $i \in J$; let $q_i \in W_i$ if $i \in R(W_{I\setminus J})$; let $q_i \in U_i$ if $i \in R(U) \setminus J$; and let q_i be any point of X_i if $i \in I \setminus \setminus (J \cup R(W) \cup R(U))$. Thus $q \in W \cap U$, and it follows that W meets (at least) $\varkappa$ elements of $\mathscr{V}$. The proof of (a) is complete.

When (1) is given, statement (b) holds for any dense, pseudo-$(\alpha, \varkappa)$-compact subspace Y of $(X_I)_\varkappa$. It suffices to show that if f is continuous on Y to a metric space (M, ϱ), then for each $\varepsilon > 0$ there exists J in $\mathscr{P}^*_\alpha(I)$ for which $\varrho(f(x), f(y)) \leqq \varepsilon$ whenever x and y are in Y and $\pi_J(x) = \pi_J(y)$. Assuming the contrary, one argues by recursion to produce, for $0 \leqq \xi < \alpha$, points x^ξ and y^ξ of Y and neighborhoods U^ξ and V^ξ of x^ξ and y^ξ respectively, basic in $(X_I)_\varkappa$, for which:

(i) $R(U^\xi) = R(V^\xi)$;

(ii) $x^\xi_i = y^\xi_i$ whenever $i \in \bigcup_{\zeta<\xi} R(U^\zeta)$;

(iii) $\varrho(f(x), f(y)) > \varepsilon$ whenever $x \in U^\xi \cap Y$ and $y \in V^\xi \cap Y$;
(iv) $U_i^\xi \cap V_i^\xi = \emptyset$ whenever $U_i^\xi \neq V_i^\xi$.

This having been done, one uses the fact that Y is pseudo-$(\alpha, \varkappa)$-compact to find a point p in Y each of whose neighborhoods meets $\varkappa$ of the sets U^ξ. It then follows from the fact that for each i in I the relation $U_i^\xi = V_i^\xi$ is valid for all but at most one ξ that each neighborhood in $(X_I)_\varkappa$ of p meets (for some ξ) both $U^\xi \cap Y$ and $V^\xi \cap Y$. This contradicts the continuity of f at p and concludes the proof of (b).

To prove (c) we show that if (2) holds and $\omega \leqq \varkappa \leqq \alpha$, then one can extend continuously over $(X_I)_\varkappa$ any continuous function f from Y (to any space whatever) for which there exists J in $\mathscr{P}_\alpha^*(I)$ with $f(x) = f(y)$ whenever $x \in Y$, $y \in Y$, and $x_J = y_J$. Given such f and J, choose for each x in X a point $\bar{x}$ in Y for which $x_J = \bar{x}_J$ and define $g(x) = f(\bar{x})$. After checking that g is well-defined and extends f, one verifies the continuity of g at each point x of X as follows. Given a neighborhood W of $g(x)$, find neighborhoods U and V of $\bar{x}_J$ and $\bar{x}_{I \setminus J}$ in $(X_J)_\varkappa$ and $(X_{I\setminus J})_\varkappa$ respectively for which $f[(U \times V) \cap Y] \subset W$ and let z be any point in the neighborhood $\pi_J^{-1}(U)$ of x. Because $J \cup R(V) \in \mathscr{P}_\alpha^*(I)$ there exists z' in Y for which $z'_{J \cup R(V)} = z_{J \cup R(V)}$, and then $z' \in U \times V$ and $g(z) = f(z') \in W$.

We remark that in the presence of (1), hypothesis (3) can be replaced by the hypothesis that each of the spaces X_i has a dense subspace with fewer than α elements. For in this case, from (1), each $(X_J)_\varkappa$ with $J \in \mathscr{P}_\varkappa^*(I)$ also has such a subspace, hence is even pseudo-(α, α)-compact.

When $\varkappa = \omega$ and $Y = X$ hypotheses (1) and (2) are automatically fulfilled and (3) is the statement that each space X_F, with F a finite, nonvoid subset of I, is pseudo-α-compact. The argument we used for (b), a straightforward generalization of one presented by Glicksberg [10], shows that in this case there is for each $\varepsilon > 0$ and each continuous f from X_I to (M, ϱ) a finite subset F of I for which $\varrho(f(x), f(y)) \leqq \varepsilon$ whenever $x \in X_I$ and $y \in X_I$ and $x_F = y_F$; thus each such f depends on countably many coordinates. When each X_i is compact, this last assertion is due to Mibu [13] and Bishop [1]; when each is separable and metrizable, to Mazur [12] (with an additional cardinality hypothesis, but for sequentially continuous functions) and Corson and Isbell [5]; when each has Knaster's property (K), to Ross and Stone [16]; when each is separable, to Gleason (see Ross and Stone [16] or Isbell [11], pages 130–132); when each finite product of the X_i is a Lindelöf space, to Engelking [7]; and finally, when X (or each X_F with $F \in \mathscr{P}_\omega^*(I)$) is pseudo-$\omega^+$-compact, to Noble and Ulmer [15]. Some of these authors consider functions defined on subspaces of $\prod_{i \in I} X_i$, and some impose hypotheses weaker than metrizability on the range space.

The argument used to prove (a) was developed by Ulmer [17] to treat (in the case $\varkappa = \omega$) the case where Y is a Σ-space in the sense of Corson [4] or a Σ_α-space (the natural generalization); other applications are given by Noble and Ulmer (loc. cit.). More consequences and equivalents to the Erdös-Rado theorem appear in our work [2]. Results related to those of the present paper and [2], dealing with families of disjoint, open sets in various $\varkappa$-box topologies on a product, appear in Engelking-Karłowicz [8], Engelking [6], and Mostowski [14], Theorem 13.3.1.

The case $\varkappa = \omega$, $\alpha = \omega^+$, of the present theorem was announced by the authors in an abstract which appeared in the Notices of the American Mathematical Society *18* (1971), page 669. Detailed proofs and additional references will appear in [3].

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