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ON RINGS OF CONTINUOUS FUNCTIONS

Dedicated to Professor K. Morita, on his sixtieth birthday

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In the following discussions all topological spaces are at least Tychonoff, and all mappings are continuous. C(X) ($C^*(X)$) denotes the ring of all real-valued continuous functions (real-valued bounded continuous functions) on a Tychonoff space X.

As pointed out by late Professor Tamano, a remarkable property of rings of continuous functions is that they have infinite operations like infinite sum, infinite join etc., and thus it is desirable to study them together with infinite operations. For example, one cannot characterize very important topological properties like metrizability or paracompactness of X in terms of C(X) or $C^*(X)$ as long as they are regarded as ordinary rings with finite operations, but one can give nice characterizations of those properties once infinite operations are taken into consideration. From this point of view the author [7] characterized metrizability and paracompactness in terms of C(X) with operations $\cup$ and $\cap$ for infinitely many elements. H. Tamano [12], Z. Frolík [3] and J. Guthrie [4] also got interesting characterizations of paracompact spaces, Čech complete spaces and other spaces in terms of C(X) and $C^*(X)$ though they did not necessarily aim characterizations by purely <u>internal</u> properties of C(X) or $C^*(X)$. The purpose of this paper is to extend characterization to some generalizations of metric spaces and also to discuss relations between $C^*(X)$ and uniformities of X.

<u>Remark</u>. Only $C^*(X)$ will be used in the following though many results can be extended to C(X) with no or slight modification of their forms. For a (not necessarily finite) subset $\{f_\alpha \mid \alpha \in A\}$ of $C^*(X)$, $\bigcap_\alpha f_\alpha$ and $\bigcup_\alpha f_\alpha$ are defined as usual; namely $(\bigcap_\alpha f_\alpha)(x) = \inf\{f_\alpha(x) \mid \alpha \in A\}$ $(\bigcup_\alpha f_\alpha)(x) = \sup\{f_\alpha(x) \mid \alpha \in A\}$. In those theorems where $\bigcup_\alpha f_\alpha$ (or $\bigcap_\alpha f_\alpha$) is involved, it is implied that $\bigcup_\alpha f_\alpha$ (or $\bigcap_\alpha f_\alpha$) is bounded and continuous; also note that N, Q and R denote the natural numbers, the rational numbers and the real numbers, respectively.

As for standard symbols and terminologies of general topology, see [10].

Definition 1: A subset L_0 of $C^*(X)$ is called normal if $\bigcap_\alpha f_\alpha$ and $\bigcup_\alpha f_\alpha$ belong to L_0 for every subset $\{f_\alpha \mid \alpha \in A\}$ of L_0. A sequence $L_1, L_2, \ldots$ of normal subsets of $C^*(X)$ is called a normal sequence. A subset L of $C^*(X)$ is σ-normally generated by the normal sequence $\{L_i \mid i = 1,2,\ldots\}$ if $L = \{f \in C^*(X) \mid$ for every $\varepsilon > 0$ there are subsets $\{f_\beta \mid \beta \in B\}$ and $\{f_\gamma \mid \gamma \in C\}$ of $\bigcup_{i=1}^{\infty} L_i$ such that $\|\bigcap_\beta f_\beta - f\| < \varepsilon$ and $\|\bigcup_\gamma f_\gamma - f\| < \varepsilon\}$. (We may simply say that L is generated by $\{L_i\}$ when the latter is known to be a normal sequence.

In the following is a slight modification of an old theorem proved in [7].

Theorem 0. A Tychonoff space X is metrizable iff $C^*(X)$ is σ-normally generated by a normal sequence.

Proof. The "if" part of this theorem is implied by Corollary 8 of [7]. The proof of "only if" part is also not so difficult if we put $L_n = \{f \in C^*(X) \mid \|f\| \leq n, |f(x) - f(y)| \leq n\,\rho(x,y)$ for all $x, y \in X\}$. Some works are necessary to choose, for given $f \in C^*(X)$ and $\varepsilon > 0$, a subset $\{f_\beta \mid \beta \in B\}$ of $\bigcup L_n$ such that $\bigcap_\beta f_\beta \in C^*(X)$ and such that $\|\bigcap_\beta f_\beta - f\| < \varepsilon$, but the detail is left to the reader. (In view of Corollary 8 of [7] we know that a weaker condition is sufficient for the metrizability of X.
The author, however, needs the stronger condition for L as given in Definition 1 to characterize other spaces in the following, and he does not know if the condition can be weakened there or not.)

Among the various generalizations of metric spaces which are actively being studied M-space due to K. Morita [5] and p-space due to A.V. Archangelskii [1] are some of the most important ones. M and p coincide and are especially good if combined together with paracompactness. In fact,

Theorem (K. Morita - A.V. Archangelskii). The following conditions for X are equivalent:

(1) X is paracompact and M,
(2) X is paracompact and p,
(3) X is the pre-image of a metric space by a perfect mapping.

Thus our first aim is to characterize paracompact M-spaces in terms of $C^*(X)$.

<u>Definition</u>. A maximal ideal J in $C^*(X)$ is called <u>fixed</u> iff for every subset $\{f_\alpha \mid \alpha \in A\}$ of J such that $\bigcup_\alpha f_\alpha \in C^*(X)$, $\bigcup_\alpha f_\alpha \in J$ holds. A subset K of $C^*(X)$ is <u>fixed</u> iff there is a fixed maximal ideal which contains K; otherwise K is called <u>free</u>. A subset H of $C^*(X)$ is called <u>strongly free</u> iff there is a subset $\{f_\beta \mid \beta \in B\}$ of H such that $\bigcup_\beta f_\beta \in C^*(X)$ and $\bigcup_\beta f_\beta \geq \varepsilon$ for some positive number ε.

<u>Remark</u>. It is easy to see that $K \in C^*(X)$ is fixed iff there is $x \in X$ for which $f(x) = 0$ for all $f \in K$.

The following theorem suggests us what form of theorem we can expect to characterize paracompact M-spaces.

<u>Theorem 1</u>. <u>Let</u> f <u>be a map from</u> X <u>onto</u> Y. <u>Then</u> f <u>induces an imbedding of</u> $C^*(Y)$ <u>into</u> $C^*(X)$ <u>if</u> $g \in C^*(Y)$ <u>is associated with</u> $g \circ f \in C^*(X)$. <u>Then</u> f <u>is a perfect map iff the induced imbedding is such that for every free maximal ideal</u> J <u>in</u> $C^*(X)$, $J \cap C^*(Y)$ <u>is free in</u> $C^*(X)$.

To prove this theorem we need the following lemma whose proof is left to the reader.

<u>Lemma 1.</u> <u>Let</u> f <u>be a map from</u> X <u>onto</u> Y. <u>Then</u> f <u>is a perfect map iff for every free (= has no cluster point) maximal</u> z<u>-filter (= filter consisting of zero sets where we mean by a zero set the set of all zeros of a real-valued continuous function)</u> $\mathcal{F}$ <u>in</u> X, $f(\mathcal{F}) = \{f(F) \mid F \in \mathcal{F}\}$ <u>is free in</u> Y.

<u>Proof of Theorem 1.</u> The first half of the claim is obvious, so only the last half will be proved.

Assume that f is a perfect map and J is a given free maximal ideal in $C^*(X)$. For each $\phi \in C^*(X)$ and $\varepsilon > 0$, we put $Z_\varepsilon(\phi) = \{x \mid |\phi(x)| \leq \varepsilon\}$.

(This symbol will be used throughout the rest of the paper).

Further, let

$\mathcal{F}(J) = \{Z \mid Z$ is a zero set in X which contains $Z_\varepsilon(\phi)$ for some $\phi \in J$ and for some $\varepsilon > 0\}$.

Then $\mathcal{F}(J)$ is obviously a free z-filter.

Expand $\mathcal{F}(J)$ to a maximal z-filter $\mathcal{F}_0$. Then since f is perfect,

by Lemma 1 $f(\mathcal{F}_0)$ is free in Y. Let x be an arbitrary point of X, and let $f(x) = y$. Then there is $Z \in \mathcal{F}_0$ such that $y \notin f(Z)$.
Since $f(Z)$ is a closed set, there is $\phi \in C^*(Y)$ such that

$$\phi(y) > 0, \quad \phi(u) = 0 \text{ for all } u \in f(Z).$$

Then $\phi \circ f(x) > 0$, and $\phi \circ f \in C^*(Y)$, where $C^*(Y)$ is considered to be imbedded in $C^*(X)$.
To prove $\phi \circ f \in J$, let $\phi \circ f = \psi$. Then $J' = C^*(X)\psi + J$ is an ideal of $C^*(X)$ containing J. For each $\xi \in J$, and $\varepsilon > 0$, $Z_\varepsilon(\xi) \cap Z \neq \emptyset$, because these sets both belong to $\mathcal{F}_0$.
Since $\psi(Z) = 0$, this implies $|\alpha\psi + \xi| \leq \varepsilon$ for every $\alpha \in C^*(X)$ and at some point of X. Thus $J' \neq C^*(X)$, which implies $J' = J$ because J is maximal. Thus $\psi \in J$. Namely $\psi \in J \cap C^*(Y)$. Hence $J \cap C^*(Y)$ is free in $C^*(X)$.

Conversely, to prove the "if" part of the theorem, let $\mathcal{F}$ be a free maximal z-filter in X. Put

$$J = \{\psi \in C^*(X) \mid Z_\varepsilon(\psi) \in \mathcal{F} \text{ for all } \varepsilon > 0\}.$$

Then J is a free maximal ideal in $C^*(X)$. To see that J is maximal, let J' be an ideal such that $J \subsetneqq J'$. Select $\phi \in J' - J$; then $Z_\varepsilon(\phi) \notin \mathcal{F}$ for some $\varepsilon > 0$. Since $\mathcal{F}$ is maximal, this implies $Z_\varepsilon(\phi) \cap Z = \emptyset$ for some $Z \in \mathcal{F}$. Put $\psi = \min(0, |\phi| - \varepsilon)$; then $\psi \in J$, because $Z_\sigma(\psi) \supset Z$ for all $\sigma > 0$. Thus $\phi^2 + \psi^2 \in J'$ and $\phi^2 + \psi^2 \geq \frac{\varepsilon^2}{4}$, which imply $J' = C^*(X)$. Therefore J is maximal.
Now, we claim that $f(\mathcal{F})$ has no cluster point in Y. To see it, let $y \in Y$ be arbitrary and select $x \in f^{-1}(y)$. Since by the condition of the theorem $J \cap C^*(Y)$ is free in $C^*(X)$, there is $\phi \in C^*(Y)$ such that $\phi \circ f \in J$ and $\phi \circ f(x) > 0$. Let $\phi \circ f(x) = \varepsilon$. Then $Z_{\frac{\varepsilon}{2}}(\phi \circ f) \cap f^{-1}(y) = \emptyset$, which implies $y \notin f(Z_{\frac{\varepsilon}{2}}(\phi \circ f))$. On the other hand $Z_{\frac{\varepsilon}{2}}(\phi \circ f) \in \mathcal{F}$ follows from the definition of J. Since $f(Z_{\frac{\varepsilon}{2}}(\phi \circ f)) = Z_{\frac{\varepsilon}{2}}(\phi)$, $|\phi(u)| \leq \frac{\varepsilon}{2}$ holds for all $u \in f(Z_{\frac{\varepsilon}{2}}(\phi \circ f))$. Let $V = \{u \in Y \mid \phi(u) > \frac{\varepsilon}{2}\}$; then V is an open nbd of y which is disjoint from $f(Z_{\frac{\varepsilon}{2}}(\phi \circ f))$. Thus y is no cluster point of $f(\mathcal{F})$; namely $f(\mathcal{F})$ is free. Hence by Lemma 1, f is a perfect map.
Now, we can characterize paracompact M-spaces in terms of $C^*(X)$ as follows.

<u>Theorem 2</u>. <u>A Tychonoff space X is paracompact and M iff there is a</u> <u>σ-normally generated subring</u> L <u>of</u> $C^*(X)$ <u>such that for every free</u> <u>maximal ideal</u> J <u>in</u> $C^*(X)$, $J \cap L$ <u>is free</u>.

Proof of the "only if" part. Let X be paracompact and M; then by the previously mentioned Morita-Archangelskii's theorem there is a perfect map from X onto a metric space Y. By Theorem 1 this map induces an imbedding $C^*(Y) \cong L \subset C^*(X)$ satisfying the condition of this theorem. It easily follows from Theorem 0 that L is σ-normally generated in $C^*(X)$.

To prove the "if" part we need some lemmas.

Lemma 2. Let $\mathcal{V} = \{V_\alpha \mid \alpha < \tau\}$ be a well-ordered open cover of X such that $V_\alpha = \{x \mid f_\alpha(x) > 0\} \cap \{x \mid g_\alpha(x) > 0\}$, $\alpha < \tau$, where f_α, $g_\alpha \in C^*(X)$ for all $\alpha < \tau$, (α and τ denote ordinal numbers). If $\bigcup_{\beta \in B} f_\beta$ and $\bigcup_{\beta \in B} g_\beta$ belong to $C^*(X)$ for every subset B of $\{\alpha \mid 0 \leq \alpha < \tau\}$, then $\mathcal{V}$ has a σ-discrete open refinement consisting of cozero open sets (= complements of zero sets).

Proof. Note that $V_\alpha = \{x \mid h_\alpha(x) > 0\}$ for $h_\alpha = f_\alpha \cap g_\alpha$ and that $\bigcup_{\alpha < \beta} h_\alpha \in C^*(X)$ for every $\beta \leq \tau$ easily follows from the assumption of lemma. Let

$$V_{1\alpha} = \{x \mid h_\alpha(x) > \tfrac{1}{2}\},$$

$$V_{n\alpha} = \{x \mid h_\alpha(x) > \tfrac{1}{2} - \tfrac{1}{2^2} - \dots \tfrac{1}{2^n}\} \quad n = 2,3,\dots .$$

Then $V_{n\alpha} \subset V_{n+1\alpha} \subset V_\alpha$. Further, let

$$W_{n1} = V_{n1},$$

$$W_{n\alpha} = \{x \mid x \in V_{n\alpha},\ \bigcup_{\beta < \alpha} h_\beta(x) < \tfrac{1}{2} - \tfrac{1}{2^2} - \dots - \tfrac{1}{2^{n+1}}\}.$$

Then each $W_{n\alpha}$ is a cozero set. It is also obvious that $\{W_{n\alpha} \mid n = 1,2,\dots;\ \alpha < \tau\}$ covers X. Since $W_{n\alpha} \subset V_\alpha$, this cover refines $\mathcal{V}$. Thus it suffices to show that $\{W_{n\alpha} \mid \alpha < \tau\}$ is discrete for each fixed n.

Let $x \in X$ satisfy $x \in V_{n+1\alpha}$ and $x \notin V_{n+1\beta}$ for all $\beta < \alpha$, where $\alpha \leq \tau$. Then $h_\beta(x) \leq \frac{1}{2} - \frac{1}{2^2} - \dots - \frac{1}{2^n} - \frac{1}{2^{n+1}}$, $\beta < \alpha$.

Thus $\bigcup_{\beta < \alpha} h_\beta(x) \leq \frac{1}{2} - \frac{1}{2^2} - \dots - \frac{1}{2^{n+1}}$, and hence x has a nbd W on which $\bigcup_{\beta < \alpha} h_\beta(x') < \frac{1}{2} - \frac{1}{2^2} - \dots - \frac{1}{2^n}$ holds.

Hence $W \cap W_{n\beta} = \emptyset$ for all $\beta < \alpha$.

On the other hand if $\gamma > \alpha$, then $V_{n+1\alpha}$ is a nbd of x which is disjoint from $W_{n\gamma}$. Therefore $\{W_{n\alpha} \mid \alpha < \tau\}$ is discrete.

<u>Lemma 3. Every open cover $\mathcal{V}$ satisfying the condition of Lemma 2 is normal; namely there is a sequence $\mathcal{V}_1, \mathcal{V}_2, \dots$ of open covers such that</u> $\mathcal{V} > \mathcal{V}_1^* > \mathcal{V}_1 > \mathcal{V}_2^* > \dots$.

<u>Proof</u>. The proof directly follows from Lemma 2 and a known theorem (Proposition D) on page 254 of [10]).

<u>Proof of the "if" part of Theorem 2.</u> First of all we define some notations. Assume that L is (σ-normally) generated by the normal sequence $L_1, L_2, \dots$. Then

$$L_m^+ = \{f^+ \mid f \in L_m\}, \text{ where } f^+ = f \cup 0,$$

$$L_m^- = \{f^- \mid f \in L_m\}, \text{ where } f^- = f \cap 0,$$

$$K_m = L_m^+ \cup (-L_m^-).$$

For $x \in X$ and $n, m \in N$,

$$A_n^m(x) = \{f \mid f \in K_m, f(x) > \tfrac{2}{n}\},$$

$$U_n^m(x) = \{y \mid \cap\{f(y) \mid f \in A_n^m(x)\} > \tfrac{1}{3n} \ .$$

$$V_n^m(x) = \{y \mid \cap\{f(y) \mid f \in A_n^m(x)\} > \tfrac{1}{2n} \ .$$

$$W_n^m(x) = \{y \mid \cap\{f(y) \mid f \in A_n^m(x)\} > \tfrac{1}{n} \ .$$

Then $U_n^m(x)$, $V_m^m(x)$ and $W_n^m(x)$ are open sets satisfying

$$W_n^m(x) \subset V_n^m(x) \subset U_n^m(x),$$

because $\cap\{f \mid f \in A_n^m(x)\}$ is continuous.

The proof will be carried out in several steps.

<u>Claim 1.</u> Let $\mathcal{F}$ be a free maximal z-filter in X.
Then for each $x \in X$, there are $n, m \in N$ and $Z \in \mathcal{F}$ such that $U_n^m(x) \cap Z = \emptyset$. To prove it, let

$$J = \{\psi \in C^*(X) \mid Z_\varepsilon(\psi) \in \mathcal{F} \text{ for all } \varepsilon > 0\}.$$

Then as proved before for Theorem 1, J is a free maximal ideal in $C^*(X)$. Hence $J \cap L$ is free by the condition of the present theorem. Namely there is $f_0 \in J \cap L$ such that $f_0(x) > 0$ or $-f_0(x) > 0$.
Assume that the former is true; then there is $f \in \bigcup_{m=1}^{\infty} L_m$ such that $f(x) > 0$, $f \le f_0$ because L is generated by $\{L_m\}$. (Recall Definition 1). Thus there are $n, m \in N$ for which $f \in L_m$, $f^+ \in L_m^+$ and $f^+(x) > \frac{2}{n}$.
Then $Z_\sigma(f_0) \subset Z_\sigma(f_0^+) \subset Z_\sigma(f^+)$, for every $\sigma > 0$. Since $f_0 \in J$, $Z_\sigma(f_0) \in \mathcal{F}$ and accordingly $Z_\sigma(f^+) \in \mathcal{F}$ for every $\sigma > 0$, which implies $f^+ \in J$. On the other hand $f^+ \in A_n^m(x)$ follows from the above observation on f^+. Hence $U_n^m(x) \subset \{y \mid f^+(y) > \frac{1}{3n}\}$. This implies $U_n^m(x) \cap Z_{\frac{1}{3m}}(f^+) = \emptyset$. Since $Z_{\frac{1}{3m}}(f^+) \in \mathcal{F}$, our claim is proved. Even if

$-f_0(x) > 0$ is assumed, we can prove our claim in a similar way.

Claim 2. Let $Y \subset X$ to define

$$M_n^m(Y) = \mathrm{Int}\,[\cap\{U_n^m(x) \mid x \in Y\}] \cap \mathrm{Int}\,[\cap\{X - W_n^m(x) \mid x \in X - Y\}],$$

$\mathcal{M}_n^m = \{M_n^m(X) \mid Y \subset X\}$, where $m, n \in N$. Then each $\mathcal{M}_n^m$ is a normal open cover of X.

To prove it, define for m, $n \in N$ and $x' \in X$,

$$P_n^m(x') = \{y \mid \cup\{f(y) \mid f \in K_m,\ f(x') < \tfrac{2}{3n}\} < \tfrac{1}{n}\} \cap \{y \mid \cap\{f(y) \mid f \in K_m,\ f(x') > \tfrac{1}{2n}\} > \tfrac{1}{3n}\}.$$

Furthermore, define $\mathcal{P}_n^m = \{P_n^m(x') \mid x' \in X\}$.

Then by Lemma 3, $\mathcal{P}_n^m$ is a normal open cover, because each $P_n^m(x')$ satisfies the condition of V_α in Lemma 2, since L_m is a normal set. For each $x' \in X$, let $Y = \{x \mid \cap\{f(x') \mid f \in A_n^m(x)\} > \frac{1}{2n}\}$.

Then it is not difficult to prove that $P_n^m(x') \subset M_n^m(Y)$. Thus $\mathcal{P}_n^m < \mathcal{M}_n^m$, and hence $\mathcal{M}_n^m$ is also a normal open cover.

Claim 3. $S(x, \mathcal{M}_n^m) \subset U_n^m(x)$ at each point x of X.

To prove it, let $M_n^m(Y)$ be an arbitrary element of $\mathcal{M}_n^m$ which contains x. Then it follows from the definition of $M_n^m(Y)$ that $x \in Y$.

Thus the same definition implies $M_n^m(Y) \subset U_n^m(x)$. Therefore $S(x, \mathcal{M}_n^m) \subset U_n^m(x)$.

Now, we are in a position to complete our proof. Combine claim 1 and claim 3; then we see that for every free maximal z-filter $\mathcal{F}$ and for each $x \in X$ there are m, $n \in N$ and $Z \in \mathcal{F}$ such that $S(x, \mathcal{M}_n^m) \cap Z = \emptyset$. Since each $\mathcal{M}_n^m$ is normal by claim 2, there is a sequence $\mathcal{U}_1$, $\mathcal{U}_2, \ldots$ of open covers of X such that for each (m,n) and for some i, $\mathcal{U}_i < \mathcal{M}_n^m$ and such that $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \ldots$. Then for every free maximal z-filter $\mathcal{F}$ and for each $x \in X$, there are some i and some $Z \in \mathcal{F}$ such that $S(x, \mathcal{U}_i) \cap Z = \emptyset$. Assume that $F_1 \supset F_2 \supset \ldots$ is a decreasing sequence of nonempty closed sets in X such that for a fixed point x, $S(x, \mathcal{U}_i) \supset F_k$ holds for each i and for some k. Let $\mathcal{F}$ be a maximal z-filter which is obtained by expanding the collection $\{Z \mid Z$ is a zero set containing F_k for some $K\}$. Then $S(x, \mathcal{U}_i) \cap Z \neq \emptyset$ for every i and every $Z \in \mathcal{F}$. Hence by the above observation we know that $\mathcal{F}$ converges. Since $\cap\{F \mid F \in \mathcal{F}\} \subset \bigcap_{k=1}^{\infty} F_k$ follows from complete regularity of X, we have $\bigcap_{k=1}^{\infty} F_k \neq \emptyset$, which proves that X is an M-space.

Let $C_x = \bigcap_{i=1}^{\infty} S(x, \mathcal{U}_i)$; then as shown in [5], there is a closed map g from X onto a metric space Y such that for each $y \in Y$, $g^{-1}(y) =$

$= C_x$ for some $x \in X$. To prove compactness of the closed set C_x, let $\mathcal{F}_0$ be a collection of closed subsets of C_x with finite intersection property. Let $\mathcal{F}'$ be a maximal z-filter which is obtained by expanding the collection $\{Z \mid Z$ is a zero set containing some element of $\mathcal{F}_0\}$. Then obviously $S(x, \mathcal{U}_i) \cap Z \neq \emptyset$ for every i and every $Z \in \mathcal{F}'$. Thus $\mathcal{F}'$ converges, and hence $\bigcap \{F \mid F \in \mathcal{F}_0\} \neq \emptyset$. Therefore C_x is compact, i.e. g is a perfect map. This proves that X is paracompact, and now the proof of Theorem 2 is complete.

Now, let us turn to another generalization of (complete) metric spaces. Paracompact, Čech complete spaces are characterized as follows.

<u>Theorem (Z. Frolík [21]).</u> X <u>is a paracompact, Čech complete space iff it is the pre-image of a complete metric space by a perfect map</u>.

This theorem is in its appearance similar to the previously mentioned Morita-Archangelskii's theorem and indicates that all paracompact Čech complete spaces are paracompact M. In fact the latter theorem is a sort of generalization of the former. Thus it is natural to try to characterize paracompact Čech complete spaces in a similar way as we did for paracompact M-spaces. As a result we obtain the following theorem.

<u>Theorem 3.</u> <u>A Tychonoff space</u> X <u>is paracompact and Čech complete iff there is a normal sequence</u> $L_1, L_2, \dots$ <u>of subsets of</u> $C^*(X)$ <u>such that for every free maximal ideal</u> J <u>in</u> $C^*(X)$, $J \cap L_n$ <u>is strongly free for some</u> n.

<u>Proof.</u> To prove the "only if" part, let X be paracompact and Čech complete. Then by Frolík's theorem there is a perfect map f from X onto a complete metric space Y. Let

$$L_n = \{\phi \circ f \mid \phi \in C^*(Y),\ \|\phi\| \leq n,\ |\phi(y) - \phi(z)| \leq n\rho(y,z) \text{ for every } y,z \in Y\},$$

where we assume ρ is a metric of Y such that $\rho \leq 1$.
Then each L_n is a normal subset of $C^*(X)$.
Let $\mathcal{F}$ be a maximal z-filter in X which contains $Z_\varepsilon(\psi)$ for all $\psi \in J$ and for all $\varepsilon > 0$. Since J is free, so is $\mathcal{F}$. Since f is perfect, by Lemma 1 $f(\mathcal{F})$ is free. Since Y is a complete metric space, there is $\varepsilon > 0$ such that $S_\varepsilon(y) \not\supset f(Z)$ for all $y \in Y$ and for all $Z \in \mathcal{F}$. For each $y \in Y$ define $\phi_y \in C^*(Y)$ by

$$\phi_y(z) = \rho(z, Y - S_\varepsilon(y)).$$

Let $\psi_y = \phi_y \circ f$; then $\psi_y \in L_1$. Since $Z(\psi_y) \cap Z_\varepsilon(\xi) \neq \emptyset$ for all $\varepsilon > 0$ and all $\xi \in J$, $\psi_y \in J$ follows from maximality of J, where

$$Z(\psi_y) = \{ x \in X \mid \psi_y(x) = 0 \}.$$

(See the proof of Theorem 1.) Thus $\psi_y \in J \cap L_1$ for every $y \in Y$. On the other hand $\bigcup_{\psi \in Y} \psi_y \geq \varepsilon$ is obvious, and hence $J \cap L_1$ is strongly free.

To prove the "if" part, first note that by Theorem 2, X is at least paracompact and M. Thus it suffices to prove that X is Čech complete. For each $x \in X$ and $n, m \in N$ we define $A_n^m(x)$ and $U_n^m(x)$ exactly in the same way as in the proof of Theorem 2. Now, let $\mathcal{F}$ be an arbitrary free maximal z-filter in X; then we shall prove that there are $n, m \in N$ such that $X - U_n^m(x) \in \mathcal{F}$ for all $x \in X$. This would prove Čech completeness of X by N.A. Shanin's theorem [11]: X is Čech complete iff there is a sequence $\{\mathcal{G}_i \mid i = 1,2,\dots\}$ of collections of zero-sets with finite intersection property such that i) $\bigcap \{ G \mid G \in \mathcal{G}_i \} = \emptyset$, ii) for every free maximal z-filter $\mathcal{F}$, there is i for which $\mathcal{G}_i \subset \subset \mathcal{F}$. For this end, let

$$J = \{ \psi \in C^*(x) \mid Z_\varepsilon(\psi) \in \mathcal{F} \quad \text{for all} \quad \varepsilon > 0 \}.$$

Then as proved before for Theorem 1, J is a free maximal ideal. Hence $J \cap L_m$ is strongly free for some m. Namely there is a subset $\{ \phi_\alpha \mid \alpha \in A \}$ of $J \cap L_m$ such that $\bigcup_{\alpha \in A} \phi_\alpha \geq \varepsilon$ for some positive number ε. Choose $n \in N$ for which $\frac{\varepsilon}{2} > \frac{2}{n}$. Then for each $x \in X$ there is $\alpha \in A$ such that $\phi_\alpha(x) \geq \frac{\varepsilon}{2}\ \frac{2}{n}$. Thus $\phi_\alpha^+ \in A_n^m(x)$.

Since $0 \leq \phi_\alpha^+ \leq \frac{1}{3n}$ on $Z_{\frac{1}{3n}}(\phi_\alpha)$,

$$U_n^m(x) \subset \{ y \mid \phi_\alpha^+(y) > \tfrac{1}{3n} \} \subset X - Z_{\frac{1}{3n}}(\phi_\alpha).$$

Thus $X - U_n^m(x) \supset Z_{\frac{1}{3n}}(\phi_\alpha) \in \mathcal{F}$. (Note that $\phi_\alpha \in J$).

This proves that $X - U_n^m(x) \in \mathcal{F}$ for every x, and accordingly X is Čech complete.

Next, let us turn to a class of generalized metric spaces which contains all paracompact M-spaces as a proper subclass.

Definition 3. A Tychonoff space X is called a G_δ-space iff it is homeomorphic to a G_δ-set in the product of a metric space and a compact T_2-space.

G_δ-space was defined in [9] as a natural generalization of pa-

racompact M-spaces, because in [8] a paracompact (T_2) M-space was characterized as a closed G_δ-set in the product of a metric space and a compact T_2-space.
In [9] the author gave the following characterizations to G_δ-spaces.

Definition 4. Let f be a continuous map from X onto Y.
Then f is called a complete map if there is a sequence $\mathcal{U}_1$, $\mathcal{U}_2,\dots$ of covers of X by cozero sets such that for every free maximal z-filter $\mathcal{F}$ in X satisfying $\mathcal{G}_n = \{X - U \mid U \in \mathcal{U}_n\} \not\subset \mathcal{F}$, $n = 1,2,\dots$, $f(\mathcal{F})$ is free in Y.

Theorem A. X is a G_δ-space iff it is the pre-image of a metric space by a complete mapping.

Theorem B. X is a G_δ-space iff there are sequences $\{\mathcal{W}_i \mid i = 1, 2,\dots\}$ and $\{\mathcal{U}_i \mid i = 1,2,\dots\}$ of open covers of X such that

(1) $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \dots$.

(2) if $\mathcal{F}$ is a maximal closed filter such that $F_i \subset W_i \cap S(x, \mathcal{U}_i)$, $i = 1,2,\dots$ for some $F_i \in \mathcal{F}$, $W_i \in \mathcal{W}_i$ and a fixed point x of X, then $\mathcal{F}$ converges.

Remark. As for Theorem A a somewhat different (and more complicated) form of condition was considered for the map f in [9], but it is easy to prove that the original condition is equivalent with completeness of f as long as X and Y are Tychonoff.

This theorem should be compared with the previously mentioned theorem of Morita-Archangelskii on paracompact M-spaces. Definition 4 should be compared with Lemma 1 to recognize that complete map is a natural generalization of a perfect map. Thus a complete map may be defined more generally for topological spaces X and Y while replacing cozero sets and zero sets in the present definition with open sets and closed sets, respectively.

The following diagram is to clarify relations between generalized metric spaces being discussed in the present paper.

paracompact and Čech complete	$\Longrightarrow$	paracompact and M (or p)	$\Longrightarrow$	G_δ	$\Longrightarrow$	p
$\|$		$\|$		$\|$		
perfect pre-image of a complete metric space		perfect pre-image of a metric space		complete pre-image of a metric space		
$\|$		$\|$		$\|$		
closed G_δ-set in the product of a complete metric space and a compact T_2-space		closed G_δ-set in the product of a metric space and a compact T_2-space		G_δ-set in the product of a metric space and a compact T_2-space.		

It was proved in [9] that an M-space X is a G_δ-space iff it is a p-space but it is not known if the same is true without the assumption that X is an M-space though a negative answer is supposed. Namely

__Problem__. Give an example of a p-space which is no G_δ-space.
As suggested by Theorems A and B we can easily characterize the G_δ-spaces in terms of $C^*(X)$ in a similar way as we did for two other spaces in Theorems 2 and 3.

__Theorem 4__. __A Tychonoff space__ X __is a__ G_δ__-space iff there is a__ σ__-normally generated subring__ L __of__ $C^*(X)$ __and a sequence__ $G_1, G_2, \dots$ __of free subsets of__ $C^*(X)$ __such that for every free maximal ideal__ J __in__ $C^*(X)$ __satisfying__ $G_n \not\subset J$, $n = 1,2,\dots$, $J \cap L$ __is free in__ $C^*(X)$.

__Proof__. The proof is similar to that of Theorem 2, so only a sketch will be given in the following. Let X be a G_δ-space; then by Theorem A there is a metric space Y and a map f from X onto Y, which is complete with respect to open covers $\mathcal{U}_i$, $i = 1,2,\dots$ of X. Put $G_n = \{\varphi \mid \varphi \in C^*(X),\ X - Z(\varphi) \in \mathcal{U}_n\}$. Then each G_n is a free subset of $C^*(X)$. Now, suppose that J is a given free maximal ideal in $C^*(X)$ such that $G_n \not\subset J$, $n = 1,2,\dots$. Then let $\mathcal{F}$ be a maximal z-filter containing $Z_\varepsilon(\varphi)$ for all $\varphi \in J$ and $\varepsilon > 0$. Then we claim that $\mathcal{G}_n = \{X - U \mid U \in \mathcal{U}_n\} \not\subset \mathcal{F}$, $n = 1,2,\dots$. Since $G_n \not\subset J$, there is $\varphi \in G_n - J$. Then $Z(\varphi) \in \mathcal{G}_n$. $Z(\varphi) \notin \mathcal{F}$ follows from maximality of J, because otherwise $J \subsetneq C^*(X)\varphi + J \neq C^*(X)$ would hold. Thus $\mathcal{F}$ is a free maximal z-filter satisfying $\mathcal{G}_n \not\subset \mathcal{F}$, $n = 1,2,\dots$. Since f is a complete map, this implies that $f(\mathcal{F})$ is free in Y.
Thus we can use an argument like the one in the proof of Theorem 1

to conclude that $J \cap L$ is free in $C^*(X)$, where L is the isomorphic image of $C^*(Y)$ in $C^*(X)$ induced by the map f. Since L is σ-normally generated, necessity of the condition is proved.

Conversely assume that $C^*(X)$ satisfies the condition of the theorem. To prove that X is a G_δ-space, define a normal sequence $\mathcal{U}_1 > \mathcal{U}_2^* > \mathcal{U}_2 > \mathcal{U}_3^* > \dots$ of open covers of X in the same way as in the last part of the proof of Theorem 2. Further we define $\mathcal{W}_n = \{X - Z_\varepsilon(\psi) \mid \psi \in G_n,\ \varepsilon > 0\}$; then $\mathcal{W}_n$ is an open cover of X. Assume that $\mathcal{F}$ is a given free maximal closed filter in X such that for every n there is $W \in \mathcal{W}_n$ and $F \in \mathcal{F}$ satisfying $F \subset W$. Put $J = \{\psi \mid \psi \in C^*(X),\ Z_\varepsilon(\psi) \in \mathcal{F}$ for all $\varepsilon > 0\}$. Then as in the proof of Theorem 1, we can prove that J is a free maximal ideal. Moreover we can show that $G_n \not\subset J$, $n = 1,2,\dots$. Because $W = X - Z_\varepsilon(\psi) \supset F \in \mathcal{F}$ for some $\psi \in G_n$ and $\varepsilon > 0$. Hence $Z_\varepsilon(\psi) \notin \mathcal{F}$ proving that $\psi \notin J$. Therefore $J \cap L$ is free in $C^*(X)$. Thus in a similar way as in the proof of Theorem 2 we can prove that for each $x \in X$ there are $Z \in \mathcal{F}$ and i such that $S(x, \mathcal{U}_i) \cap Z = \emptyset$. This means that $F \not\subset S(x, \mathcal{U}_i)$ holds for all $F \in \mathcal{F}$. Hence by Theorem B X is a G_δ-space.

It would be easy to characterize (general) Čech complete spaces and perhaps general M-spaces, too, in terms of $C^*(X)$ by use of a similar method. How about p-spaces? There is another group of generalized metric spaces which can be characterized as <u>images</u> of metric spaces by certain types of maps, e.g. Lašnev space (= closed continuous image of metric space), stratifiable space, σ-space, etc. Is it possible to characterize them by simple properties of $C^*(X)$ as we have done for pre-images of metric spaces? In any case one may need a new technique which is different from the one we have used.

Now, let us turn to an extension of Theorem 0 to another direction. If X is metrizable, then by that theorem $C^*(X)$ is generated by a normal sequence. Then what is the relation between various normal sequences generating $C^*(X)$ and metric uniformities of X ? We shall see in the following that they correspond to each other in certain manner.

<u>Theorem 5.</u> <u>Let</u> X <u>be a metric space; then there is a normal sequence</u> $\{L_n \mid n = 1,2,\dots\}$ <u>generating</u> $C^*(X)$ <u>such that</u> $A(\{L_n\}) = \{f \in C^*(X) \mid$ <u>for every</u> $\varepsilon > 0$ <u>there is</u> $g \in \bigcup_{n=1}^{\infty} L_n$ <u>for which</u> $\|f - g\| < \varepsilon\}$ <u>is equal to the set</u> $V^*(X)$ <u>of all bounded uniformly continuous (real-va-</u>

<u>lued) functions on</u> X. <u>Moreover we can select</u> $\{L_n\}$ <u>satisfying the following condition</u>.

(A) $L_1 \subset L_2 \subset \dots,$

$f \in L_n$ <u>implies</u> $f \cup 0,\ f \cap 0 \in L_n$,

$f \in L_n$ <u>and</u> $\alpha \in R$ <u>imply</u> $f + \alpha,\ \alpha f \in L_m$

<u>for some</u> $m = m(n,\alpha)$.

<u>Proof</u>. Let $L_n = \{ f \in C^*(X) \mid \|f\| \leq n,\ |f(x) - f(y)| \leq n\,\rho(x,y)$ for all $x,y \in X\}$. Then $\{L_n \mid n = 1,2,\dots\}$ is a normal sequence satisfying the required conditions.
$A(\{L_n\}) \subset V^*(X)$ is obvious because each element of $\bigcup_{n=1}^{\infty} L_n$ is bounded and uniformly continuous. To prove $V^*(X) \subset A(\{L_n\})$ assume that the metric ρ of X is bounded and also let $f \in V^*(X)$ and $\varepsilon > 0$. Further suppose $\|f\| \leq A$. Select $k \in N$ such that $\rho(x,y) < \frac{1}{k}$ implies $\rho(f(x).f(y)) < \varepsilon$. Then put $F_n = \{x \mid n\varepsilon < f(x) \leq (n + 1)\varepsilon\}$, $n = 0, \pm 1, \pm 2, \dots$. (We define F_n only for such n that satisfies $[n\varepsilon,\ (n + 1)\varepsilon] \cap [-A,A] \neq \emptyset$.
For each $n \in N$, let $p_n \in N$ be such that

$$p_n - 1 < k(A - n\varepsilon) \leq p_n.$$

Put $f_n(x) = p_n \rho(F_n,x) + n\varepsilon$; then $f_n \in L_{q_n}$ for some $q_n \in N$, $f_n(x) = n\varepsilon$ for $x \in F_n$ and $f_n(x) \geq A$ for $x \notin F_{n-1} \cup F_n \cup F_{n+1}$. Thus $(n - 1)\varepsilon \leq \bigcap_n f_n(x) \leq n\varepsilon$ holds for each $x \in F_n$.
Therefore $\|f - \bigcap_n f_n\| \leq 2\varepsilon$. Note that $\bigcap_n f_n \in L_m$ for some m (= the largest n for which F_n is defined). This proves $V^*(X) \subset A(\{L_n\})$ and eventual coincidence of these two sets.

<u>Theorem 6</u>. <u>Let</u> $\{L_n \mid n = 1,2,\dots\}$ <u>be a normal sequence generating</u> $C^*(X)$ <u>and satisfying the condition (A) of the previous theorem. Then there is a metric uniformity (agreeing with the topology) of X for which</u> $V^*(X) = A(\{L_n\})$, <u>where the symbols</u> V^* <u>and</u> A <u>are defined in the same way as in the previous theorem</u>.

<u>Proof</u>. 1. First note that X is metrizable.
For each $x \in X$, $n \in N$ and $v, v' \in Q$ (the rationals) such that $v < v'$, we define

$$B_{nvv'}(x) = \{f \mid f \in L_n,\ f(x) \geq \frac{v + v'}{2}\},$$

$$\tilde{B}_{nvv'}(x) = \{f \mid f \in L_n,\ f(x) \leq \frac{v + v'}{2}\},$$

$$U_{nvv'}(x) = \{y \in X \mid \bigcap\{f(y) \mid f \in B_{nvv'}(x)\} > v \text{ and}$$

$$\bigcup\{f(y) \mid f\in \widetilde{B}_{nvv'}(x)\} < v'\},$$

$$V_{nvv'}(x) = \{y\in X \mid \bigcap\{f(y) \mid f\in B_{nvv'}(x)\} > \frac{5v+v'}{6} \text{ and } \bigcup\{f(y) \mid f\in \widetilde{B}_{nvv'}(x)\} < \frac{v+5v'}{6}\},$$

$$S_{nvv'}(x) = \{y\in X \mid \bigcap\{f(y) \mid f\in B_{nvv'}(x)\} > \frac{2v+v'}{3} \text{ and } \bigcup\{f(y) \mid f\in \widetilde{B}_{nvv'}(x)\} < \frac{v+2v'}{3}\}.$$

Then $S_{nvv'}(x)\subset V_{nvv'}(x)\subset U_{nvv'}(x)$, and they are all open nbds of x. For any $(n,v,v')\in N\times Q\times Q$ with $v<v'$ and for any $Y\subset X$ we define

$$U_{nvv'}(Y) = \operatorname{Int}[\bigcap\{U_{nvv'}(x) \mid x\in Y\}\cap(\bigcap\{X - S_{nvv'}(x) \mid x\in X - Y\})],$$

$$\mathcal{U}_{nvv'} = \{U_{nvv'}(Y) \mid Y\subset X\}.$$

Then $\mathcal{U}_{nvv'}$ is obviously an open cover of X. Furthermore we claim:

(a) For every $(n,v,v')\in N\times Q\times Q$, there are (s,s') and $(t,t')\in Q\times Q$ such that

$$\{U_{nss'}(y)\cap U_{ntt'}(y) \mid y\in X\} < \mathcal{U}_{nvv'}.$$

To see it, let y be a fixed point of X.

Then for each $x\in X$, either $y\in V_{nvv'}(x)$ or $y\notin V_{nvv'}(x)$ holds. If $y\in V_{nvv'}(x)$, then it is easy to see that $U'(y) = U_{nss'}(y)\cap U_{ntt'}(y) \subset U_{nvv'}(x)$, where $s = \frac{v+5v'}{6} - \frac{v'-v}{12}$, $s' = \frac{v+5v'}{6} + \frac{v'-v}{12}$, $t = \frac{5v+v'}{6} - \frac{v'-v}{12}$, $t' = \frac{5v+v'}{6} + \frac{v'-v}{12}$.

If $y\notin V_{nvv'}(x)$, then either $\bigcap\{f(y) \mid f\in B_{nvv'}(x)\} \le \frac{5v+v'}{6}$ or $\bigcup\{f(y) \mid f\in \widetilde{B}_{nvv'}(x)\} \ge \frac{v+5v'}{6}$.

If the former is the case, then for every $\varepsilon > 0$ there is $f_\varepsilon \in B_{nvv'}(x)$ such that $f_\varepsilon(y) \le \frac{5v+v'}{6} + \varepsilon$. Let $f = \bigcap_{\varepsilon>0} f_\varepsilon$; then $f\in \widetilde{B}_{nvv'}(x)$, $f(y)\le \frac{5v+v'}{6}$. Hence $f\in \widetilde{B}_{ntt'}(y)$, and hence for each $u\in U'(y)$ $f(u)\le t'$. On the other hand, since $f\in B_{nvv'}(x)$, $f(w) > \frac{2v+v'}{3} > t'$ for each $w\in S_{nvv'}(x)$.

Thus $U'(y)\cap S_{nvv'}(x) = \emptyset$.

Even if the latter is the case, we can prove the same in a similar way. Thus we obtain

$U'(y)\subset U_{nvv'}(Y)$, where

$Y = \{x\in X \mid y\in V_{nvv'}(x)\}$.

This proves that $\{U_{nss'}(y)\cap U_{ntt'}(y) \mid y\in X\} < \mathcal{U}_{nvv'}$, as claimed in

(a). On the other hand the following relation is almost obvious:
(b) for every $(n,v,v')\in N\times Q\times Q$ and $x\in X$,

$$S(x,\mathcal{U}_{nvv'})\subset U_{nvv'}(x).$$

As easily seen, $\{U_{nvv'}(x)\mid (n,v,v')\in N\times Q\times Q,\ v<v'\}$ forms a nbd base at each $x\in X$, and therefore (b) implies that
$\{S(x,\mathcal{U}_{nvv'})\mid (n,v,v')\in N\times Q\times Q,\ v<v'\}$ is also a nbd base at x.

Now we can conclude this section of the proof with the following observation:

$$\mu=\{\bigwedge_{i=1}^{\ell}\mathcal{U}_{n_iv_iv_i'}\mid (n_i,v_i,v_i')\in N\times Q\times Q,\ v_i<v_i',\ i=1\dots\ell;\ \ell=1,2,\dots\}$$

is a countable (= metric) uniformity base agreeing with the topology of X.

Let $\mathcal{U}_{nvv'}\in\mu$ be given; then there are (s,s') and (t,t') satisfying (a). Put $\mathcal{U}=\mathcal{U}_{nss'}\wedge\mathcal{U}_{ntt'}$; then by (b) $\mathcal{U}^{\Delta}<\mathcal{U}_{nvv'}$ while $\mathcal{U}\in\mu$. This proves that μ is a uniformity base while we have seen before that this uniformity agrees with the topology of X.

2. From now on we regard X as a (metrizable) uniform space with the uniformity defined by μ.
The objective of the present section is to prove that $A(\{L_n\})\subset V^*(X)$. It suffices to show that every $f\in\bigcup_{n=1}^{\infty}L_n$ is uniformly continuous with respect to μ. Assume $f\in L_n$, $a\le f\le b$ and $a,b\in Q$. Given $\varepsilon>0$; then choose $k\in N$ for which $\frac{b-a}{k}<\varepsilon$. Put $a_i=a+\frac{b-a}{k}i$, $i=-1,0,1,\dots,k,k+1$. Assume that x and y are points of X satisfying

$$y\in U_{na_{-1}a_1}(x)\cap U_{na_0a_2}(x)\cap U_{na_1a_3}(x)\cap\dots\cap U_{na_{k-2}a_k}(x)\cap U_{na_{k-1}a_{k+1}}(x).$$

Then assume that $a_i\le f(x)<a_{i+1}$; then

$$f\in B_{na_{i-1}a_{i+1}}(x)\cap\tilde{B}_{na_ia_{i+2}}(x),\ \text{and hence}$$

$$U_{na_{i-1}a_{i+1}}(x)\subset\{y\mid f(y)>a_{i-1}\},\ \text{i.e. } f(y)>a_{i-1}.$$

Similarly we can show $f(y)<a_{i+2}$.
Thus $|f(x)-f(y)|<2\varepsilon$ proving that f is uniformly continuous.

3. Finally we are going to prove $V^*(x)\subset A(\{L_n\})$.
Note that condition (A) will be fully used for the first time in this section. Let us begin with simple remarks, of which only the last one is given a proof.

(i) Let $v_1<v_2<v_3<v_4$ be rationals satisfying

$\frac{v_1 + v_4}{2} = \frac{v_2 + v_3}{2}$; then $U_{nv_2v_3}(x) \subset U_{nv_1v_4}(x)$ for every $n \in N$ and $x \in X$.

(ii) Let $m < n$ be natural numbers; then

$U_{mvv'}(x) \subset U_{nvv'}(x)$ for every $v, v' \in Q$ and $x \in X$.

(iii) Let $v, v', \alpha \in Q$ and $n \in N$; then there is $m \in N$ (independent from x) such that

$U_{m,v-\alpha,v'-\alpha}(x) \subset U_{nvv'}(x)$ for all $x \in X$.

To see it, let $m = m(n, -\alpha)$ in the condition (A), i.e. $f \in L_n$ implies $f - \alpha \in L_m$. Let $y \in U_{m,v-\alpha,v'-\alpha}(x)$; then

$\bigcap \{f(y) \mid f \in B_{m,v-\alpha,v'-\alpha}(x)\} \geq v - \alpha + \varepsilon > v - \alpha$ and

$\bigcup \{f(y) \mid f \in \tilde{B}_{m,v-\alpha,v'-\alpha}(x)\} \leq v' - \alpha - \varepsilon < v' - \alpha$ for some $\varepsilon > 0$.

Let $f \in B_{nvv'}(x)$; then

$f - \alpha \in B_{m,v-\alpha,v'-\alpha}(x)$, and hence

$f(y) - \alpha \geq v - \alpha + \varepsilon$, i.e. $f(y) \geq v + \varepsilon$.

Thus $\bigcap \{f(y) \mid f \in B_{nvv'}(x)\} \geq v + \varepsilon > v$.

Similarly $\bigcup \{f(y) \mid f \in \tilde{B}_{nvv'}(x)\} \leq v' - \varepsilon < v'$.

Hence $y \in U_{nvv'}(x)$, proving $U_{m,v-\alpha,v'-\alpha}(x) \subset U_{nvv'}(x)$.

Combining (i),(ii) and (iii) we can conclude that for every $\mathcal{U} \in \mathcal{E}\mu$, there is $(n, v, v') \in N \times Q \times Q$ such that $U_{nvv'}(x) \subset S(x, \mathcal{U})$ for all $x \in X$.

Now we are in a position to prove that for every $f \in V^*(x)$ and for every $\varepsilon > 0$, there is $\psi \in \bigcup_{n=1}^{\infty} L_n$ such that $\| f - \psi \| < \varepsilon$. Assume $\| f \| \leq K$. Since f is uniformly continuous, there is $(m, v, v') \in N \times Q \times Q$ for which

$\{U_{mvv'}(x) \mid x \in X\} < \{f^{-1}((n\varepsilon, (n+2)\varepsilon)) \mid n = 0, \pm 1, \dots\}$.

Let x be a given point of X, and suppose that $n\varepsilon \leq f(x) < (n+1)\varepsilon$. Then

$U_{mvv'}(x) \subset f^{-1}(((n-1)\varepsilon, (n+2)\varepsilon))$.

Thus for each $y \notin f^{-1}(((n-1)\varepsilon, (n+2)\varepsilon))$, $y \notin U_{mvv'}(x)$.

Namely either there is $f_{xy} \in L_m$ satisfying $f_{xy}(x) \leq \frac{v + v'}{2}$, $f_{xy}(y) \geq v'$ or else there is $f_{xy} \in L_m$ satisfying $f_{xy}(x) \geq \frac{v + v'}{2}$, $f_{xy}(y) \leq v$.

Hence there are $\alpha, \beta \in R$ such that $g_{xy} = 0 \cup (\alpha f_{xy} + \beta)$ satisfies $g_{xy} \geq 0$, $g_{xy}(x) = 0$, $g_{xy}(y) \geq K - n\varepsilon$, and $g_{xy} \in L_p$ for some p independent from y. Let $h_{xy} = g_{xy} + n\varepsilon \in L_q$, where q is independent from y. Thus $\Phi_x = \bigcup \{h_{xy} \mid y \notin U_{mvv'}(x)\}$ satisfies

$\Phi_x \in L_q$, $\Phi_x \geq n\varepsilon$, $\Phi_x(x) = n\varepsilon$, $\Phi_x(y) \geq K$ for all $y \notin U_{mvv'}(x)$.

Observe that q may be assumed to be common to all $x \in f^{-1}([n\varepsilon, (n+1)\varepsilon))$. Thus

$$\psi_n = \bigcap \{ \phi_x \mid x \in f^{-1}([n\varepsilon, (n+1)\varepsilon)) \} \text{ satisfies}$$

$$\psi_n \in L_q, \ \psi_n(x) = n\varepsilon \ \text{ for all } x \in f^{-1}([n\varepsilon, (n+1)\varepsilon)),$$

$$\psi_n \geq n\varepsilon, \text{ and } \ \psi_n(y) \geq K \text{ for all } y \notin f^{-1}(((n-1)\varepsilon, (n+2)\varepsilon)).$$

Finally put $\psi = \bigcap_n \psi_n$; then $\psi \in L_\ell$ for some $\ell \in N$. Moreover it is easy to see that $\| f - \psi \| \leq 2\varepsilon$. Therefore $V^*(x) \subset A(\{L_n\})$, which completes the proof of the theorem.

As proved in [6] (Lemma 2), $V^*(X)$ determines a metric uniformity of a metrizable space X, and hence Theorems 5 and 6 indicate that normal sequences generating $C^*(X)$ and satisfying (A) and metric uniformities of X are corresponding to each other though the correspondence is not one-to-one, because different normal sequences can induce the same $V^*(X)$.

References

1. A.V. Archangelskii, On a class of spaces containing all metric and all locally bicompact spaces, Soviet Math. 4(1963), 751-754.
2. Z. Frolík, On the topological product of paracompact spaces, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 8(1960), 747-750.
3. Z. Frolík, A characterization of topologically complete spaces in the sense of E. Čech in terms of convergence of functions, Czechoslovak Math. J. 13(1963), 148-151.
4. J.A. Guthrie, Metrization and paracompactness in terms of real functions, Bull. Amer. Math. Soc. 80(1974), 720-721.
5. K. Morita, Products of normal spaces with metric spaces, Math. Ann. 154(1964), 365-382.
6. J. Nagata, On lattices of functions on topological spaces and of functions on uniform spaces, Osaka J. Math. 1(1949), 166-181.
7. J. Nagata, On coverings and continuous functions, J. Inst. Polytech. Osaka City Univ. 7(1956), 29-38.
8. J. Nagata, A note on M-space and topologically complete space, Proc. Japan Acad. 45(1969), 541-543.
9. J. Nagata, On G_δ-sets in the product of a metric space and a compact space I, II, Proc. Japan Acad. 49(1973), 179-186.
10. J. Nagata, Modern General Topology, Amsterdam-London-Groningen, 1974.

11. N.A. Shanin, On the theory of bicompact extensions of topological spaces, Dokl. Akad. Nauk SSSR 38(1943), 154-156.
12. H. Tamano, On rings of real valued continuous functions, Proc. Japan Acad. 34(1958), 361-366.