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DECOMPOSITIONS OF ADJOINT SITUATIONS

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As it is known, reflections and coreflections are special cases of adjunctions. The (adjoint) composition of reflections is again a reflection and similarly for coreflections. Clearly, the composition of a reflection and a coreflection (or in the converse order) need not be either reflection or coreflection. What are the adjunctions which can be expressed as such compositions?

Theorem 1. Any adjunction $(\eta, \varepsilon): F \dashv G: (\mathcal{A}, \mathcal{B})$ is a composition of a full monoreflection $(1_{\mathcal{B}}, \psi): J_2 \dashv C: (\mathcal{C}, \mathcal{B})$ and a full epireflection $(\varphi, 1_{\mathcal{A}}): R \dashv J_1: (\mathcal{A}, \mathcal{C})$. The ψ is epi iff ε is epi (i.e., iff G is faithful) and φ is mono iff η is mono (i.e., iff F is faithful).

An adjunction $(\eta, \varepsilon): F \dashv G$ is called normal, if ηG is iso (i.e., if $GF\eta = \eta GF$ or if $F\eta$ is iso or if $G\varepsilon F$ is iso, etc.).

Moreover, we can prove that (η, ε) is not normal iff no prolongation of $1_{\mathcal{B}} \xrightarrow{\eta} GF$ contains an iso (by a prolongation of the η we mean any left-continuous functor $\mathcal{H}$ on ordinal numbers into functors $\mathcal{B} \longrightarrow \mathcal{B}$ with $\mathcal{H}\langle \alpha, \alpha + 1 \rangle = (GF)^n \eta \mathcal{H}(\beta)$ for an $n < \omega_0$, $\beta = \alpha - n$); the same for ε.

Theorem 2. An adjunction is isomorphic to a composition of a full coreflection, an equivalence and a full reflection iff it is normal. The coreflection is mono iff the counit is mono, and the reflection is epi iff the unit is epi.

— As in the case of epireflective hulls one may now prove the existence of adjoint hulls if we restrict ourselves to adjunctions with the unit being epi (thus the adjunctions are normal).

Denote by FUNC the metacategory the objects of which are functors into categories with cointersections and terminal objects, and morphisms $G \longrightarrow G'$ are pairs $\langle K,H \rangle$ with: $HG = G'K$, if G is full and faithful in A from the left, then G' is full and faithfull in KA from the left, and $H[\text{epirefl}\, G[\mathcal{A}]] \subset \text{epirefl}\, G'[\mathcal{A}']$. (For the existence of epireflective hulls it suffices that the category has cointersections and is cowell-powered.) Denote by ADJ the full sub-metacategory of FUNC generated by those functors which are right adjoints (if $G_i : \mathcal{A}_i \longrightarrow \mathcal{B}_i$ are objects of ADJ, $\mathcal{C}_i$ are the corresponding categories epireflective in $\mathcal{B}_i$ and coreflective in $\mathcal{A}_i$, then $\langle K,H \rangle : G_1 \longrightarrow G_2$ iff $HG_1 = G_2K$ and $H/\mathcal{C}_1 = K/\mathcal{C}_1 : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$).

Theorem 3. ADJ is reflective in FUNC.

The condition on preserving epireflective hulls cannot be omitted (in that case the "reflection" has not the extension property); it is fulfilled if H is right continuous on complete categories. If we omit the condition on preserving "full and faithful left objects", then the "reflection" has the extension property but not unique. We may change the equality $HG = G'K$ to a transformation $\tau : HG \longrightarrow G'K$ (then morphisms are triples $\langle K, H, \tau \rangle$); in this case we also loss unicity.